Orbits of the class $\mathcal{O}_6$ of lines external with respect to the twisted cubic in PG$(3, q)$ *

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Abstract

In the projective space PG$(3, q)$, we consider orbits of lines under the stabilizer group of the twisted cubic. In the literature, lines of PG$(3, q)$ are partitioned into classes, each of which is a union of line orbits. We propose an approach to obtain orbits of the class named $\mathcal{O}_6$, whose complete classification is an open problem. For all even and odd q we describe a family of orbits of $\mathcal{O}_6$ and their stabilizer groups. The orbits of this family include an essential part of all $\mathcal{O}_6$ orbits.

1 Introduction

In the three-dimensional projective space PG$(3, q)$ over a Galois field $\mathbb{F}_q$ with q elements, the normal rational curve $\mathscr{C}$, named twisted cubic, has as many as $q + 1$ points. Up to a change of the projective frame of PG$(3, q)$, these points are $P_t = (t^3, t^2, t, 1)$, $t \in \mathbb{F}_q$, together with $P_\infty = (1, 0, 0, 0)$. In particular, they form a complete $(q + 1)$-arc in PG$(3, q)$. This is a well known, relevant property which has already proved to be useful not only from a theoretical point of view but also for practical applications in cryptography; see for instance [3, 5–8, 10, 17, 21, 23]. A novel application of twisted cubic aimed at the construction of covering codes has been the motivation for the study of

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certain submatrices of the point-plane incidence matrix of $PG(3,q)$ arising from the action of the stabilizer group $G_q \cong \mathrm{PGL}(2,q)$ of $\mathscr{C}$ in $\mathrm{PG}(3,q)$. The investigation, based on the known classification of the point and plane orbits of G_q given in [19], was initiated by D. Bartoli and the present authors in 2020 [1] and produced optimal multiple covering codes. The results in [1] were also an important ingredient to classify the cosets of the $[q+1,q-3,5]_q3$ generalized doubly-extended Reed-Solomon code of codimension 4 by means of their weight distributions [11].

For the study of the plane-line and the point-line incidence matrices, an explicit description of line orbits is useful. In [19], a partition of the lines in $\mathrm{PG}(3,q)$ into classes is given, each of which is a union of line orbits under G_q; see also Section 2. Apart from one class denoted by $\mathcal{O}_6$, the number and the structure of the orbits forming those unions are independently considered by distinct methods in [12–14] (for all $q \geq 2$), [2, Section 7] (for all $q \geq 23$), and [18] (for finite fields of characteristic > 3); see also the references therein. The results on the line orbits from [2, 12, 18] are in accordance with each other. They (together with ones from [5, 19]) are collected in [16, Section 2.2, Table 1]. In [16] the relations between [12, 15, 16] and [18] are described.

The class $\mathcal{O}_6$ contains lines external to the twisted cubic such that they are not chords of the cubic and do not lie in its osculating planes. The complete classification of the line orbits in $\mathcal{O}_6$ constitutes an open problem.

Using the representation of the line orbits in [12], for all $q \geq 2$ and apart from the lines in class $\mathcal{O}_6$, the *plane-line* incidence matrix of $\mathrm{PG}(3,q)$ is given in [15] and the *point-line* incidence matrix of $\mathrm{PG}(3,q)$ is obtained in [16]. For $\mathcal{O}_6$, in [15, 16], the corresponding average and cumulative values are calculated.

In [18], apart from the lines in class $\mathcal{O}_6$, for odd $q \not\equiv 0 \pmod 3$ the numbers of distinct planes through distinct lines (called "the plane orbit distribution of a line") and the numbers of distinct points lying on distinct lines (called "the point orbit distribution of a line") in of $\mathrm{PG}(3,q)$ are obtained. For finite fields of characteristic > 3, the results of [18] on "the plane orbit distribution of a line" and "the point orbit distribution of a line" are in accordance with those from [15, 16] on the plane-line and the point-line incidence matrices.

In $\mathrm{PG}(3,q)$, for even $q = 2^n$, $n \geq 3$ the $(q+1)$-arc $\mathcal{A} = \{(1,t,t^{2^h},t^{2^h+1}) | t \in \mathbb{F}_q\} \cup (0,0,0,1)$ with $gcd(n,h) = 1$ (twisted cubic for $h = 1$), has been considered in a recent paper [9], where it is shown that the orbits of points and of planes under the projective stabilizer G_h of $\mathcal{A}$ are similar to those under G_1 described in [19]; moreover, the point-plane incidence matrix with respect to G_h-orbits mirrors the case h=1 described in [1]. In [9], it is also proved that for even q, $q \equiv \xi \pmod 3$, $\xi \in \{1,-1\}$, G_h has $2q+7+\xi$ orbits on lines, providing a proof of a conjecture of ours [12, 14, Conjecture 8.2] in the case even q.

In this paper, we propose an approach to obtain orbits of the class $\mathcal{O}_6$. Our approach is based on the analysis of the stabilizer of a line from $\mathcal{O}_6$. For all even and odd q we describe a family of orbits of $\mathcal{O}_6$ and their stabilizer groups. The orbits of this family include an essential part of all $\mathcal{O}_6$ orbits.

The paper is organized as follows. Section 2 contains background and preliminaries. In particular, in Theorem 2.5, the known computer results on line orbits of the class $\mathcal{O}_6$ are given.

In Section 3, the stabilizer group and the orbit of the line through the points $\mathbf{P}(1,0,0,1)$, $\mathbf{P}(0,0,1,0)$ is described for all the values of q for which such a line belongs to $\mathcal{O}_6$.

In Section 4, the family of lines $\{\ell_\mu \mid \mu \in \mathbb{F}_q\}$ such that the points $\mathbf{P}(0,\mu,0,1), \mathbf{P}(1,0,1,0) \in \ell_\mu$ is defined, and for all q it is proven when $\ell_\mu \in \mathcal{O}_6$. The stabilizer groups and the orbits of the lines ℓ_μ are analyzed in the following sections.

In Section 5, for even q, the stabilizer groups, and the size and the number of orbits of ℓ_μ-lines is established. Also it is stated that for even q no ℓ_μ-line belongs to the orbit described in Section 3.

In Section 6, for $q \equiv 0 \pmod 3$, the stabilizer groups and the size of orbits of ℓ_μ-lines is established. The exact number of orbits is found for $q \equiv -1 \pmod 4$; for $q \equiv 1 \pmod 4$ a lower and an upper bound are given.

In Section 7, for q odd, $q \not\equiv 0 \pmod 3$, the stabilizer groups and the size of orbits of ℓ_μ-lines is established. Also it is stated when an ℓ_μ-line belongs to the orbit described in Section 3.

2 Preliminaries

In this section, in the beginning we cite some results from [19, 20] useful in this paper. Then, in Theorem 2.5 we give computational results from [12].

Let $\boldsymbol{\pi}(c_0, c_1, c_2, c_3)$ be the plane of $\mathrm{PG}(3,q)$ with equation $c_0x_0+c_1x_1+c_2x_2+c_3x_3 = 0$, $c_i \in \mathbb{F}_q$. We denote $\mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$, $\mathbb{F}_q^+ = \mathbb{F}_q \cup \{\infty\}$.

Let $\mathbf{P}(x_0, x_1, x_2, x_3)$ be a point of $\mathrm{PG}(3,q)$ with homogeneous coordinates $x_i \in \mathbb{F}_q$.

Let $P(t)$ be a point of $\in \mathrm{PG}(3,q)$ with

$$t \in \mathbb{F}_q^+,\ P(t) = \mathbf{P}(t^3, t^2, t, 1) \text{ if } t \in \mathbb{F}_q,\ P(\infty) = \mathbf{P}(1,0,0,0). \tag{2.1}$$

Let $\mathscr{C} \subset \mathrm{PG}(3,q)$ be the *twisted cubic* consisting of $q+1$ points no four of which are coplanar. We consider $\mathscr{C}$ in the canonical form

$$\mathscr{C} = \{P(t) \,|\, t \in \mathbb{F}_q^+\}. \tag{2.2}$$

A *chord* of $\mathscr{C}$ through the points $P(t_1)$ and $P(t_2)$ is a line joining either a pair of real points of $\mathscr{C}$ or a pair of complex conjugate points. In the last case, it is an *imaginary chord*. If the real points are distinct, it is a *real chord*; if they coincide with each other, it is a *tangent.* The coordinate vector of a chord has the form

$$L_{\rm ch} = (a_2^2, a_1a_2, a_1^2 - a_2, a_2, -a_1, 1) \tag{2.3}$$

where $a_1 = t_1 + t_2$, $a_2 = t_1t_2$. If $x^2 - a_1x + a_2$ has 2, 1, or 0 roots in $\mathbb{F}_q$ then (2.3) gives, respectively, a real chord, a tangent, or an imaginary chord.

The *osculating plane* $\pi_{\rm osc}(t)$ in the point $P(t) \in \mathscr{C}$ has the form

$$\pi_{\rm osc}(t) = \boldsymbol{\pi}(1, -3t, 3t^2, -t^3) \text{ if } t \in \mathbb{F}_q;\ \pi_{\rm osc}(\infty) = \boldsymbol{\pi}(0, 0, 0, 1). \tag{2.4}$$

The $q + 1$ osculating planes form the osculating developable Γ to $\mathscr{C}$, that is a *pencil of planes* for $q \equiv 0 \pmod 3$ or a *cubic developable* for $q \not\equiv 0 \pmod 3$.

A line is an *axis*, intersection of a pair of real or complex conjugate planes of Γ, say $\pi_{\rm osc}(t_1)$ and $\pi_{\rm osc}(t_2)$, if its coordinate vector has the form

$$L_{\rm ax} = (\beta_2^2, \beta_1\beta_2, 3\beta_2, (\beta_1^2 - \beta_2)/3, -\beta_1, 1) \tag{2.5}$$

where $\beta_1 = t_1 + t_2$, $\beta_2 = t_1t_2$. We call the line a *real axis*, a *generator (tangent)* or an *imaginary axis* of Γ as $x^2 - \beta_1x + \beta_2$ has 2, 1, or 0 roots in $\mathbb{F}_q$.

The null polarity $\mathfrak{A}$ [20, Sections 2.1.5, 5.3], [19, Theorem 21.1.2] is given by

$$\mathbf{P}(x_0, x_1, x_2, x_3)\mathfrak{A} = \boldsymbol{\pi}(x_3, -3x_2, 3x_1, -x_0),\ q \not\equiv 0 \pmod 3. \tag{2.6}$$

It interchanges $\mathscr{C}$ and Γ, and their corresponding chords and axes.

Notation 2.1. Throughout the paper, we consider $q \equiv \xi \pmod 3$ with $\xi \in \{-1, 0, 1\}$. Many values depend of ξ or make sense only for specific ξ. If it is not clear by the context, we note this by remarks. The following notation is used.

G_q	the group of projectivities in PG$(3, q)$ fixing $\mathscr{C}$;
tr	the sign of transposition;
$\#S$	the cardinality of a set S;
$\overline{AB}$	the line through the points A and B;
$\triangleq$	the sign "equality by definition";
EnΓ-line	a line, external respect to the cubic $\mathscr{C}$, not in osculating planes, and other than a chord and an axis;
$\mathcal{O}_6 = \mathcal{O}_{\rm En\Gamma}$	the union (class) of all orbits of EnΓ-lines under G_q.

Remark 2.2. The words "and an axis" are included to the definition of EnΓ-line by context of [19, Lemma 21.1.4].

Theorem 2.3. [19, Chapter 21] *The following properties of the twisted cubic $\mathscr{C}$ of* (2.2) *hold:*

(i) *The group G_q acts triply transitively on $\mathscr{C}$; $G_q \cong PGL(2,q)$ for $q \geq 5$. A matrix* $\mathbf{M}$ *corresponding to a projectivity of G_q has the general form*

$$\mathbf{M} = \begin{bmatrix} a^3 & a^2c & ac^2 & c^3 \\ 3a^2b & a^2d + 2abc & bc^2 + 2acd & 3c^2d \\ 3ab^2 & b^2c + 2abd & ad^2 + 2bcd & 3cd^2 \\ b^3 & b^2d & bd^2 & d^3 \end{bmatrix}, a, b, c, d \in \mathbb{F}_q, ad - bc \neq 0. \quad (2.7)$$

(ii) *The lines of* $\mathrm{PG}(3,q)$ *can be partitioned into classes called $\mathcal{O}_i$ and $\mathcal{O}'_i = \mathcal{O}_i\mathfrak{A}$, each of which is a union of orbits under G_q, see* [19, Lemma 21.1.4] *for details. In particular, all orbits of* $\mathrm{En}\Gamma$*-lines form the class $\mathcal{O}_6$ (we call it also $\mathcal{O}_{\mathrm{En}\Gamma}$) of the size* $\#\mathcal{O}_6 = \#\mathcal{O}_{\mathrm{En}\Gamma} = (q^2-q)(q^2-1)$. *If $q \not\equiv 0 \pmod 3$, $q \geq 5$, then $\mathcal{O}'_6 = \mathcal{O}_6\mathfrak{A} = \mathcal{O}_6$.*

Lemma 2.4. *In $\mathbb{F}_q$, q odd, the equation $3x^2 + 1 = 0$ has: no roots, if $q \equiv -1 \pmod 3$, or 2 distinct roots, if $q \equiv 1 \pmod 3$.*

Proof. By [20, Section 1.5(xi)(xii)], -3 is a non-zero square (resp. a non-square) in $\mathbb{F}_q$ if $q \equiv 1 \pmod 3$ (resp. $q \equiv -1 \pmod 3$). $\square$

The following theorem states computational results regarding the orbits of $\mathrm{En}\Gamma$-lines for $5 \leq q \leq 37$ and $q = 64$.

Theorem 2.5. [12, Section 8] *Let $q \equiv \xi \pmod 3$, $\xi \in \{1, -1, 0\}$.*

(i) *Let $5 \leq q \leq 37$ and $q = 64$. Then*

(**b**) *The total number of line orbits in* $\mathrm{PG}(3,q)$ *is $2q + 7 + \xi$.*

(**a**) *For the total number $L_{\mathrm{En}\Gamma\Sigma}$ of orbits of* $\mathrm{En}\Gamma$*-lines we have* $L_{\mathrm{En}\Gamma\Sigma} = 2q - 3 + \xi$ *for q odd,* $L_{\mathrm{En}\Gamma\Sigma} = 2q - 2 + \xi$ *for q even.*

(ii) *Let q be odd, $5 \leq q \leq 37$. Then under G_q, for* $\mathrm{En}\Gamma$*-lines, there are*
$(q-\xi)/3$ *orbits of length* $q^3 - q$,
$q - 1$ *orbits of length* $(q^3 - q)/2$,
$n_q^{(\xi)}$ *orbits of length* $(q^3 - q)/4$,
where $n_q^{(1)} = (2q-11)/3$, $n_q^{(-1)} = (2q-10)/3$, $n_q^{(0)} = (2q-6)/3$.
In addition, for $q \in \{7, 13, 19, 25, 31, 37\}$ where $q \equiv 1 \pmod 3$, there are
1 *orbit of length* $(q^3 - q)/12$,
2 *orbits of length* $(q^3 - q)/3$.

(iii) *Let $q = 8, 16, 32, 64$. Then under G_q, for* $\mathrm{En}\Gamma$*-lines, there are*
$2 + \xi$ *orbits of length* $(q^3 - q)/(2+\xi)$; $2q - 4$ *orbits of length* $(q^3 - q)/2$.

Conjecture 2.6. [12] *The results of Theorem 2.5 hold for all $q \geq 5$ with the corresponding parity and ξ value.*

For odd $q \not\equiv 0 \pmod 3$, the conjecture on the case (i) of Theorem 2.5 is given also in [18].

3 An orbit $\mathscr{O}_{\mathcal{L}}$ of the class $\mathcal{O}_6 = \mathcal{O}_{\mathrm{En}\Gamma}$ for $q \not\equiv 0 \pmod 3$

In this section $q \not\equiv 0 \pmod 3$.

Let Q_β and Q_∞ be the points such that $Q_\beta = \mathbf{P}(1,0,\beta,1)$, $\beta \in \mathbb{F}_q$; $Q_\infty = \mathbf{P}(0,0,1,0)$. We consider the line $\mathcal{L} = \overline{Q_0Q_\infty}$ through the points Q_0 and Q_∞. We have

$$\mathcal{L} = \overline{\mathbf{P}(1,0,0,1)\mathbf{P}(0,0,1,0)} = \{\mathbf{P}(0,0,1,0), \mathbf{P}(1,0,\beta,1) | \beta \in \mathbb{F}_q\}. \tag{3.1}$$

Lemma 3.1. *The line $\mathcal{L}$ is an* EnΓ*-line.*

Proof. For every q, the points Q_∞ and Q_β, $\beta \in \mathbb{F}_q$, do not belong to the cubic $\mathscr{C}$, so $\mathcal{L}$ is an external line.

We show that no osculating plane $\pi_{\mathrm{osc}}(t)$ of (2.4) contains two distinct points of $\mathcal{L}$. We have $Q_\infty \in \pi_{\mathrm{osc}}(\infty)$, $Q_\infty \in \pi_{\mathrm{osc}}(0)$, $Q_\beta \notin \pi_{\mathrm{osc}}(\infty) \cup \pi_{\mathrm{osc}}(0)$, $\beta \in \mathbb{F}_q$. Let $t \in \mathbb{F}_q^*$. Obviously, $Q_\infty \notin \pi_{\mathrm{osc}}(t)$. Points Q_i, Q_j simultaneously belong to $\pi_{\mathrm{osc}}(t)$ if and only if $1 + 3it^2 - t^3 = 1 + 3jt^2 - t^3 = 0$ that implies $it^2 = jt^2$. As $t \neq 0$ we have $i = j$.

Finally, the coordinate vector of $\mathcal{L}$ is $(0,-1,0,0,0,1)$. Comparing it with (2.3) and (2.5), we see that $\mathcal{L}$ cannot be an imaginary chord or an imaginary axis. $\square$

Note that for $q \equiv 0 \pmod 3$, $\mathcal{L}$ is not an EnΓ-line. In fact, $\mathcal{L}$ is contained in the plane of equation $x_0 - x_3 = 0$ that, if $q \equiv 0 \pmod 3$, is the osculating plane $\pi_{\mathrm{osc}}(1)$, see (2.4).

We denote by G_q^∞ the subgroup of G_q fixing the point $Q_\infty = \mathbf{P}(0,0,1,0)$. Let $\mathbf{M}^\infty$ be a matrix corresponding to a projectivity of G_q^∞.

Lemma 3.2. *The general form of the matrix $\mathbf{M}^\infty$ is as follows:*

$$\mathbf{M}^\infty = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & d^2 & 0 \\ 0 & 0 & 0 & d^3 \end{bmatrix}, \ d \in \mathbb{F}_q^*. \tag{3.2}$$

Proof. We find the version of the matrix $\mathbf{M}$ of (2.7) fixing the point $\mathbf{P}(0,0,1,0)$. For $\delta \in \mathbb{F}_q^*$, $\mathbf{P}(0,0,1,0)$ and $\mathbf{P}(0,0,\delta,0)$ represent the same point. We have $[0,0,1,0]\mathbf{M} = 0,0,\delta,0]$, $\delta \in \mathbb{F}_q^*$, that implies $3ab^2 = 0$, $b^2c + 2abd = 0$, $ad^2 + 2bcd = \delta$, $3cd^2 = 0$. If $a = b = 0$ then $\delta = ad^2 + 2bcd = 0$, contradiction. If $a = 0$, $b \neq 0$ then $b^2c = 0$ and $\delta = 2bcd = 0$, contradiction. So, $a \neq 0$, $b = 0$. As $\mathbf{M}$ is defined up to a factor of proportionality, we can put $a = 1$. Now we have, $d^2 = \delta \neq 0$, $3cd^2 = 0$, so $c = 0$ and the assertion follows from (2.7) . $\square$

We want to determine the subgroup $G_q^{\mathcal{L}}$ of G_q fixing $\mathcal{L}$ and its orbit $\mathscr{O}_{\mathcal{L}}$ under G_q. Let $\mathbf{M}^{\mathcal{L}}$ be a matrix corresponding to a projectivity of $G_q^{\mathcal{L}}$.

Lemma 3.3. *Let q be even or let $-1/2$ be a non-cube in $\mathbb{F}_q$. Then the general form of a matrix $\mathbf{M}^{\mathcal{L}}$ corresponding to a projectivity of $G_q^{\mathcal{L}}$ is as follows:*

$$\mathbf{M}^{\mathcal{L}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & d^2 & 0 \\ 0 & 0 & 0 & d^3 \end{bmatrix}, \ d \in \mathbb{F}_q^*, \ d \textit{ is a cubic root of unity.} \tag{3.3}$$

Proof. Let a projectivity $\psi \in G_q^{\mathcal{L}}$. We consider the case $Q_\infty\psi = Q_\beta$ for some $\beta \in \mathbb{F}_q$. The general form of a matrix $\mathbf{M}$ corresponding to ψ is given by (2.7). We have $[0,0,1,0]\mathbf{M} = [1,0,\beta,1]$ that implies $b^2c + 2abd = 0$, $ab^2 = cd^2$, and $a,b,c,d \neq 0$. If q is even, we have also $b^2c = 0$, contradiction. Now consider the case q odd. As $\mathbf{M}$ is defined up to a factor of proportionality, we can put $b = 1$. From $a = cd^2$ and $c + 2ad = 0$ we obtain $d^3 = -1/2$, contradiction if $-1/2$ is not a cube in $\mathbb{F}_q$.

Thus, $Q_\infty\psi \neq Q_\beta$ with $\beta \in \mathbb{F}_q$. The only possible case is $Q_\infty\psi = Q_\infty$, see Lemma 3.2. The matrix $\mathbf{M}^{\mathcal{L}}$ must be the same form as $\mathbf{M}^\infty$ (3.2) but the set of possible values of d can be a proper subset of $\mathbb{F}_q^*$. We should provide $Q_0\psi = Q_\beta$ for some $\beta \in \mathbb{F}_q$. As $[1,0,0,1]\mathbf{M}^\infty = [1,0,0,d^3]$, it can happen only if $d^3 = 1$. □

Remind that over $\mathbb{F}_q$, the equation $x^3 = c$ has a unique solution, if $q \equiv -1 \pmod 3$, and 3 or 0 solutions, if $q \equiv 1 \pmod 3$, see [20, Section 1.5(iv),(v)].

Lemma 3.4. (i) *Let $q \equiv -1 \pmod 3$, q odd. Then $G_q^{\mathcal{L}}$ has order 2 and a matrix $\mathbf{M}^{\mathcal{L}}$ corresponding to the non-trivial projectivity of $G_q^{\mathcal{L}}$ has the form (2.7) with*

$$a = \sqrt[3]{1/2}, \ b = 1, \ c = \sqrt[3]{2}, \ d = -\sqrt[3]{1/2}.$$

(ii) *Let $q \equiv 1 \pmod 3$, q odd and let $-1/2$ be a cube in $\mathbb{F}_q$. Then $G_q^{\mathcal{L}}$ has order 12 and is isomorphic to the group A_4. A matrix $\mathbf{M}^{\mathcal{L}}$ of $G_q^{\mathcal{L}}$ has either the form (3.3) or the form (2.7) with*

$$a = \textit{a cubic root of } 1/2, \ b = 1, \ c = -d/a^2, \ d = \textit{a cubic root of } -1/2. \tag{3.4}$$

Proof. **(i)** Let a projectivity $\psi \in G_q^{\mathcal{L}}$ and let $\mathbf{M}^{\mathcal{L}}$ be a matrix corresponding to ψ.

(a) Let $Q_\infty\psi = Q_\infty$. We have $\mathbf{M}^{\mathcal{L}} = \mathbf{M}^\infty$, see Lemma 3.2 and (3.2). In (3.2) we have $d = 1$, so ψ is the identity projectivity.

(b) Let $Q_\infty\psi = Q_\beta$, $\beta \in \mathbb{F}_q$. The general form of $\mathbf{M}$ is given by (2.7). We have $[0,0,1,0]\mathbf{M} = [1,0,\beta,1]$ that implies $b^2c + 2abd = 0$, $ab^2 = cd^2$, and $a,b,c,d \neq 0$. As $\mathbf{M}$ is defined up to a factor of proportionality, we can put $b = 1$. From $c + 2ad = 0$, $a = cd^2$, we obtain $d^3 = -1/2$.

Now for $b = 1$ and $d^3 = -1/2$, we consider $Q_0\psi$. The following holds

$$[1,0,0,1]\mathbf{M} = [a^3 + 1,\ a^2c + d,\ ac^2 + d^2,\ c^3 - 1/2].$$

(b1) Let $Q_0\psi = Q_\infty$. Then $a^3+1 = 0$, $a^2c+d = 0$, $ad-bc = ad-(-d/a^2) = 0$, contradiction.

(b2) Let $Q_0\psi = Q_\beta$, $\beta \in \mathbb{F}_q$. Then $a^3 + 1 = c^3 - 1/2$ and $a^2c + d = 0$ that implies $c = -d/a^2$, $2a^9 + 3a^6 - 1 = 0$. Putting $t = a^3$ we obtain $(t+1)^2(2t-1) = 0$.

(b21) Let $t + 1 = 0$. Then $a^3 = -1$ So, $c = ad$ and $ad - bc = 0$, contradiction.

(b22) Let $2t - 1 = 0$. Then $t = 1/2$ and $a = \sqrt[3]{1/2}$. So, $c = \sqrt[3]{2}$ and $ad - bc = -1 - \sqrt[3]{2} \neq 0$.

(ii) The assertions can be proved similarly to case (i) taking into account that if $-1/2$ is a cube then $1/2$ also is a cube. By direct computation using the computer algebra system Magma [4], a non trivial matrix of the form (3.3) has order three, whereas of the nine matrices of the form (3.4), three have order two and the other six have order three. The only group of order 12 having three elements of order two and eight elements of order three is A_4, see [22]. □

Theorem 3.5. *Let the* EnΓ*-line* $\mathcal{L}$ *be as in* (3.1), $q \equiv \xi \pmod 3$. *Let* $G_q^{\mathcal{L}}$ *be the subgroup of* G_q *fixing* $\mathcal{L}$ *and let* $\mathscr{O}_{\mathcal{L}}$ *be the orbit of* $\mathcal{L}$ *under* G_q. *Then the sizes of* $G_q^{\mathcal{L}}$ *and* $\mathscr{O}_{\mathcal{L}}$ *are as follows:*

(i) $\xi = 1$, *q is even or* $-\dfrac{1}{2}$ *is a non-cube in* $\mathbb{F}_q$. $\quad \#G_q^{\mathcal{L}} = 3,\ \#\mathscr{O}_{\mathcal{L}} = \dfrac{1}{3}(q^3 - q)$.

(ii) $\xi = 1$, *q is odd and* $-\dfrac{1}{2}$ *is a cube in* $\mathbb{F}_q$. $\quad \#G_q^{\mathcal{L}} = 12,\ \#\mathscr{O}_{\mathcal{L}} = \dfrac{1}{12}(q^3 - q)$, $\#G_q^{\mathcal{L}} \cong A_4$.

(iii) $\xi = -1$, *q is even.* $\quad \#G_q^{\mathcal{L}} = 1,\ \#\mathscr{O}_{\mathcal{L}} = q^3 - q$.

(iv) $\xi = -1$, *q is odd.* $\quad \#G_q^{\mathcal{L}} = 2,\ \#\mathscr{O}_{\mathcal{L}} = \dfrac{1}{2}(q^3 - q)$.

Proof. The sizes of $G_q^{\mathcal{L}}$ follow from Lemmas 3.3, 3.4 and the results of [20, Section 1.5 (ii),(iii)]. By [20, Lemma 2.44(ii)], the size of $\#\mathscr{O}_{\mathcal{L}}$ is $\#G_q/\#G_q^{\mathcal{L}}$. □

4 A family of $\mathrm{En}\Gamma$-lines ℓ_μ, $\mu \in \mathbb{F}_q^* \setminus \{1, 1/9\}$

Let $\mu \in \mathbb{F}_q$. Let $R_{\mu,\gamma}$ be the point such that

$$R_{\mu,\gamma} = \mathbf{P}(\gamma, \mu, \gamma, 1),\ \gamma \in \mathbb{F}_q^+;\ R_{\mu,\infty} = \mathbf{P}(1, 0, 1, 0). \tag{4.1}$$

We consider the line $\ell_\mu = \overline{R_{\mu,0}R_{\mu,\infty}}$ through $R_{\mu,0}$ and $R_{\mu,\infty}$.

$$\ell_\mu = \overline{\mathbf{P}(0, \mu, 0, 1)\mathbf{P}(1, 0, 1, 0)} = \{\mathbf{P}(\gamma, \mu, \gamma, 1) | \gamma \in \mathbb{F}_q^+,\ \mu \text{ is fixed},\ \mu \in \mathbb{F}_q\}. \tag{4.2}$$

The points of the line ℓ_μ satisfy the equations

$$x_0 = x_2,\ x_1 = \mu x_3,\ \mu \text{ is fixed},\ \mu \in \mathbb{F}_q. \tag{4.3}$$

By (4.2), the coordinate vector L_μ of ℓ_μ is

$$L_\mu = (\mu, 0, 1, -\mu, 0, 1). \tag{4.4}$$

Lemma 4.1. *Let $q \geq 5$. The line ℓ_0 is an unisecant of the cubic $\mathscr{C}$ not in an osculating plane. The line ℓ_1 is a tangent if q is even, a real chord if q is odd.*

Proof. The assertions follow from (2.1), (2.4), (4.2). □

Lemma 4.2. *Let $\mu \in \mathbb{F}_q^* \setminus \{1\}$. For all $q \geq 5$, the line ℓ_μ (4.2) has the following properties:*

(i) *The line ℓ_μ is an external line to the twisted cubic $\mathscr{C}$.*

(ii) *The line ℓ_μ is not a chord of the twisted cubic $\mathscr{C}$.*

(iii) *The line ℓ_μ does not lie in the osculating planes $\pi_{osc}(\infty)$, $\pi_{osc}(0)$. Moreover,*

$$R_{\mu,\gamma} \notin \pi_{osc}(\infty),\ \gamma \in \mathbb{F}_q;\ R_{\mu,\infty} \notin \pi_{osc}(0),\ R_{\mu,\infty} \in \pi_{osc}(\infty). \tag{4.5}$$

(iv) *Let $R_{\mu,\infty} \notin \pi_{osc}(t)$, $\forall t \in \mathbb{F}_q$. Then the line ℓ_μ is not contained in any osculating plane.*

(v) *We have $R_{\mu,\infty} \in \pi_{osc}(t)$, $t \in \mathbb{F}_q$, if and only if $3t^2 + 1 = 0$.*

Proof. **(i)** By (2.1) and (4.2), no point of ℓ_μ belongs to $\mathscr{C}$.

(ii) Comparing the coordinate vectors (2.3) and (4.4) one sees $a_2^2 = \mu$ and $a_2 = -\mu$ that implies $\mu = \mu^2$, contradiction as $\mu \in \mathbb{F}_q^* \setminus \{1\}$.

(iii),(v) The assertions follow from (2.4), (4.1).

(iv) Together with (4.5), the hypothesis means that for any osculating plane there is a point of ℓ_μ not belonging to the plane. The assertion follows. $\square$

Lemma 4.3. *Let $q \not\equiv 0 \pmod 3$. Then for the line ℓ_μ (4.2) the following holds:*

(i) *A line ℓ_μ is an imaginary axis if and only if q is odd, $q \equiv -1 \pmod 3$, $\mu = 1/9$;*

(ii) *A line ℓ_μ is a real axis if and only if q is odd, $q \equiv 1 \pmod 3$, $\mu = 1/9$.*

Proof. Considering the vectors (2.5) and (4.4), we obtain $\beta_2^2 = \mu$, $\beta_2 = 1/3$, $\beta_1 = 0$. This implies $\mu = 1/9$. By Lemma 2.4, the equation $x^2 - \beta_1 x + \beta_2 = x^2 + 1/3 = 0$ has 2 distinct roots if q is odd, $q \equiv 1 \pmod 3$, or 0 roots if q is odd, $q \equiv -1 \pmod 3$. The assertions follow. $\square$

Lemma 4.4. *Let $\mu \in \mathbb{F}_q^* \setminus \{1\}$. Let $q \geq 5$. The line ℓ_μ (4.2) is an* EnΓ*-line in the following cases:*

(i) *q is even;*

(ii) *$q \equiv 0 \pmod 3$;*

(iii) *$q \not\equiv 0 \pmod 3$, q is odd, $\mu \neq 1/9$.*

Proof. We prove some properties of ℓ_μ that are not included in Lemma 4.2. By Lemma 4.2, it is sufficient to consider only the case $3t^2 + 1 = 0$.

(i) For even q, the equality $3t^2 + 1 = 0$ gives $t = 1$. Assume that $R_{\mu,\gamma} \in \pi_{\text{osc}}(1)$. Then, for even q, by (2.4), (4.1), $1 + \gamma + \mu + \gamma = 0$ that implies $\mu = 1$, contradiction.

(ii) The equality $3t^2 + 1 = 0$ comes to $0 = 1$, contradiction. The axis of Γ has equations $x_0 = 0, x_3 = 0$, so by (4.3) it is not a line of type ℓ_μ and it intersects no ℓ_μ line.

(iii) We apply Lemma 2.4. If $q \equiv -1 \pmod 3$, then $3t^2 + 1 \neq 0$, $\forall t \in \mathbb{F}_q$. However, by Lemma 4.3, $\ell_{1/9}$ is an imaginary axis.

If $q \equiv 1 \pmod 3$, let $3t^2 + 1 = 0$. Then $t^2 = -1/3 \neq 0$. Assume $R_{\mu,\gamma} \in \pi_{\text{osc}}(t)$, $t \in \mathbb{F}_q^*$, see Lemma 4.2(iii). Then, by (2.4), (4.1), $t^3 - 3\gamma t^2 + 3\mu t - \gamma = 0$ that implies $t^3 + 3\mu t = 0$, $\mu = 1/9$, contradiction. $\square$

Let $G_q^{\ell_\mu}$ be the subgroup of G_q fixing ℓ_μ. For q odd, $G_q^{\ell_\mu}$ has order at least two.

Lemma 4.5. *Let $q \geq 5$ be odd. Let a projectivity $\psi \in G_q^{\ell_\mu}$ fix $R_{\mu,\infty}$. Then ψ has a matrix of the form*

$$\mathbf{M}(\psi) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & d^2 & 0 \\ 0 & 0 & 0 & d^3 \end{bmatrix}, \ d \in \{1, -1\}. \tag{4.6}$$

Moreover, $(\psi)^2 = \psi$ and $R_{\mu,\gamma}\psi = R_{\mu,d\gamma}$, $\gamma \in \mathbb{F}_q$.

Proof. The matrix $\mathbf{M}(\psi)$ is a version of $\mathbf{M}$ of (2.7). By hypothesis, $R_{\mu,\infty}\psi = R_{\mu,\infty}$ and $R_{\mu,0}\psi = R_{\mu,\gamma}$, $\gamma \in \mathbb{F}_q$, i.e. $[1,0,1,0]\mathbf{M} = [1,0,1,0]$ and $[0,\mu,0,1]\mathbf{M} = [\gamma,\mu,\gamma,1]$, $\gamma \in \mathbb{F}_q$, that implies

$$a^2c + b^2c + 2abd = 0,\ c^3 + 3cd^2 = 0,\ a^3 + 3ab^2 = ac^2 + ad^2 + 2bcd; \tag{4.7}$$
$$3\mu a^2b + b^3 = \mu(bc^2 + 2acd) + bd^2,\ \mu(a^2d + 2abc) + b^2d = 3\mu^2c^2d + \mu d^3. \tag{4.8}$$

(a) Let $c = 0$. As $ad - bc = ad \neq 0$, $a, d \neq 0$. From the first relation of (4.7) we obtain $b = 0$. Then the last relation of (4.7) becomes $a(a-d)(a+d) = 0$. As $\mathbf{M}$ is defined up to proportionality, we can fix $a = 1$, then $d = \pm 1$. For $c = b = 0$, $a = 1$, $d = \pm 1$, (4.8) is also satisfied and (2.7) becomes (4.6).

(b) Let $c \neq 0$. The second relation of (4.7) becomes $c^2 + 3d^2 = 0$.

(b1) Let $q \equiv 0 \pmod 3$. Then $c^2 = 0$, contradiction.

(b2) Let $q \not\equiv 0 \pmod 3$. Then $d \neq 0$, otherwise again $c^2 = 0$, contradiction. As $\mathbf{M}$ is defined up to proportionality, we can fix $d = 1$. Then $q \equiv 1 \pmod 3$, as $c = \pm\sqrt{-3}$. From the first relation of (4.7) we obtain $a = -3b/c$ or $a = b/c$.

(b21) Let $a = -3b/c$. Then $ad - bc = 0$, contradiction.

(b22) Let $a = b/c$. The third relation of (4.7) becomes $8b(b^2+3)/c^3 = 0$ that implies $b = 0$ or $b^2 = -3$, $q \equiv 1 \pmod 3$, $b = \pm\sqrt{-3}$.
If $b = 0$ also $a = 0$ and $ad - bc = 0$, contradiction.
If $b^2 = -3$, the first relation of (4.8) implies $\mu = 1$, contradiction.

The last assertion follows by direct computation. □

5 The $q-2$ distinct orbits of ℓ_μ-lines, $\mu \in \mathbb{F}_q^* \setminus \{1\}$, for even q

In this section we consider the orbits of the lines ℓ_μ of type (4.2), $\mu \in \mathbb{F}_q^* \setminus \{1\}$, for even $q \geq 8$. In this case the matrix of (2.7) corresponding to a projectivity of G_q reduces to

$$\mathbf{M} = \begin{bmatrix} a^3 & a^2c & ac^2 & c^3 \\ a^2b & a^2d & bc^2 & c^2d \\ ab^2 & b^2c & ad^2 & cd^2 \\ b^3 & b^2d & bd^2 & d^3 \end{bmatrix},\ a,b,c,d \in \mathbb{F}_q,\ ad - bc \neq 0. \tag{5.1}$$

Lemma 5.1. *Let $q \geq 8$, q even. Let $\psi \in G_q^{\ell_\mu}$ fix $R_{\mu,\infty}$. Then $\psi = I$, the identity element of $G_q^{\ell_\mu}$.*

Proof. A matrix corresponding to ψ is a version of $\mathbf{M}$ of (5.1). By hypothesis, $R_{\mu,\infty}\psi = R_{\mu,\infty}$ and $R_{\mu,0}\psi = R_{\mu,\gamma}$, $\gamma \in \mathbb{F}_q$, i.e. $[1,0,1,0]\mathbf{M} = [1,0,1,0]$ and $[0,\mu,0,1]\mathbf{M} = [\gamma,\mu,\gamma,1]$, that implies

$$a^2c + b^2c = 0,\ c^3 + cd^2 = 0,\ a^3 + ab^2 = ac^2 + ad^2; \tag{5.2}$$

$$\mu a^2b + b^3 = \mu bc^2 + bd^2,\ \mu a^2d + b^2d = \mu,\ \mu c^2d + d^3 \neq 0. \tag{5.3}$$

(a) Let $c \neq 0$.

(a1) Let $a = 0$. By (5.2), $b^2 = 0$ that implies $ad - bc = 0$, contradiction.

(a2) Let $a \neq 0$. By (5.2), $(a+b)^2 = 0,\ (c+d)^2 = 0$, that implies, $a = b$, $c = d$, and $ad - bc = 0$, contradiction.

(b) Let $c = 0$. As $ad - bc = ad \neq 0$, $a, d \neq 0$ and (5.2) and (5.3) become, respectively,

$$a^2 + b^2 = d^2; \tag{5.4}$$

$$\mu ba^2 + b^3 = bd^2, \mu a^2d + b^2d = \mu. \tag{5.5}$$

(b1) Let $b = 0$. By (5.4), $a^2 + d^2 = (a+d)^2 = 0$, i.e. $a = d$. Then from (5.5) $\mu d^3 = \mu$, so $d^3 = 1$ i.e. d is a cubic root of 1 and by (5.1) $\mathbf{M}$ is the identity matrix. Therefore $\psi = I$, the identity element of $G_q^{\ell_\mu}$.

(b2) Let $b \neq 0$. From (5.5) we obtain $\mu a^2 + b^2 = d^2$, and from (5.4) $\mu a^2 + a^2 + d^2 = d^2$, so $\mu a^2 + a^2 = 0$ that implies $\mu = 1$, contradiction. □

Theorem 5.2. *Let q be even. Then the size of the subgroup $G_q^{\ell_\mu}$ of G_q fixing the* EnΓ*-line ℓ_μ is $\#G_q^{\ell_\mu} = 2$. The size of the orbit $\mathscr{O}_\mu$ of ℓ_μ under G_q is equal to $\#\mathscr{O}_\mu = (q^3 - q)/2$. The non-trivial element ψ of $G_q^{\ell_\mu}$ has a matrix of the form:*

$$\mathbf{M}^{\ell_\mu} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \sqrt{\mu} & 0 \\ 0 & \mu & 0 & 0 \\ \sqrt{\mu^3} & 0 & 0 & 0 \end{bmatrix}. \tag{5.6}$$

Moreover $R_{\mu,\infty}\psi = R_{\mu,0}$ and vice versa $R_{\mu,0}\psi = R_{\mu,\infty}$.

Proof. Let $\psi \in G_q^{\ell_\mu}$, $\psi \neq I$, the identity element of $G_q^{\ell_\mu}$ and let $\mathbf{M}^{\ell_\mu}$ be a matrix corresponding to ψ. We find it as is a version of $\mathbf{M}$ of (5.1). By Lemma 5.1, $R_{\mu,\infty}\psi = R_{\mu,\gamma}$, $\gamma \in \mathbb{F}_q$, i.e. $[1,0,1,0]\mathbf{M} = [\gamma,\mu,\gamma,1]$ that implies

$$a^3 + ab^2 = ac^2 + ad^2,\ \ a^2c + b^2c = \mu,\ \ c^3 + cd^2 = 1. \tag{5.7}$$

The last relation of (5.7) gives $c \neq 0$. As $\mathbf{M}$ is defined up to proportionality, we can fix $c = 1$.

(a) Let $a = 0$. By (5.7), $b^2 = \mu$, $d = 0$. As q is even, every element of $\mathbb{F}_q$ is a square and has exactly one square root, so $\mathbf{M}$ has the form (5.6) and $R_{\mu,\infty}\psi = R_{\mu,0}$. By direct computation, $\mathbf{M}$ has order 2. This implies $R_{\mu,0}\psi = R_{\mu,\infty}(\psi)^2 = R_{\mu,0}$, so $\ell_\mu\psi = \ell_\mu$. For the size of the orbit we apply [20, Lemma 2.44(ii)].

(b) Let $a \neq 0$. Then (5.7) becomes $a^2 + b^2 = 1 + d^2 = 1,\ a^2 + b^2 = \mu$ that implies $\mu = 1$, contradiction. $\square$

Theorem 5.3. *Let q be even. Then any two lines $\ell_{\mu'}$, $\ell_{\mu''}$ of type* (4.2)*, $\mu', \mu'' \in \mathbb{F}_q^* \setminus \{1\}, \mu' \neq \mu''$, belong to different orbits of G_q. No ℓ_μ-line of type* (4.2) *belongs to the orbit $\mathscr{O}_{\mathcal{L}}$ of the line $\mathcal{L}$.*

Proof. Let $\ell_{\mu'}\psi = \ell_{\mu''}$, $\psi \in G_q$. Then by Lemma 5.1, $R_{\mu',\infty}\psi \neq R_{\mu',\infty} = R_{\mu'',\infty}$. Reasoning as in the proof of Theorem 5.2, we have that $R_{\mu',\infty}\psi = R_{\mu'',0}$ and a matrix corresponding to ψ is

$$\mathbf{M}(\psi) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \sqrt{\mu''} & 0 \\ 0 & \mu'' & 0 & 0 \\ \sqrt{\mu''^3} & 0 & 0 & 0 \end{bmatrix}.$$

Then $[0, \mu', 0, 1]\mathbf{M}(\psi) = [\sqrt{\mu''^3}, 0, \mu'\sqrt{\mu''}, 0]$ that is equal to $R_{\mu'',\infty}$. This implies $\sqrt{\mu''^3} = \mu'\sqrt{\mu''}$, so $\mu' = \sqrt{\mu''^2}$. As q is even, by [20, Lemma 2.44(viii)]. $\mu' = \mu''$, contradiction. Finally, by Theorems 3.5 and 5.2, the lines $\mathcal{L}$ and ℓ_μ have different stabilizer groups. $\square$

Remark 5.4. By Theorems 3.5(i)(iii), 5.2, and 5.3, for even q, we obtained sizes and structures of $q - 1$ orbits, which contain in total $(q-2)(q^3-q)/2 + \#\mathscr{O}_{\mathcal{L}}$ EnΓ-lines where $\#\mathscr{O}_{\mathcal{L}} = (q^3 - q)/3$ if $q \equiv 1 \pmod 3$ and $\#\mathscr{O}_{\mathcal{L}} = q^3 - q$ if $q \equiv -1 \pmod 3$. This is approximately half of all EnΓ-lines, see $\#\mathcal{O}_6 = \#\mathcal{O}_{\mathrm{En}\Gamma} = (q^2 - q)(q^2 - 1)$ in Theorem 2.3(ii).

6 On the orbits of lines ℓ_μ, $\mu \in \mathbb{F}_q^* \setminus \{1\}$, $q \equiv 0 \pmod 3$

For $q \equiv 0 \pmod 3$, a matrix $\mathbf{M}$ corresponding to a projectivity of G_q has the general form

$$\mathbf{M} = \begin{bmatrix} a^3 & a^2c & ac^2 & c^3 \\ 0 & a^2d - abc & bc^2 - acd & 0 \\ 0 & b^2c - abd & ad^2 - bcd & 0 \\ b^3 & b^2d & bd^2 & d^3 \end{bmatrix},\ a, b, c, d \in \mathbb{F}_q,\ ad - bc \neq 0. \tag{6.1}$$

By (6.1), (4.1), we have

$$R_{\mu,\infty}\mathbf{M} = [1, 0, 1, 0]\mathbf{M} = [a^3, a^2c + b^2c - abd, ac^2 + ad^2 - bcd, c^3]; \tag{6.2}$$

$$R_{\mu,0}\mathbf{M} = [0,\mu,0,1]\mathbf{M} = [b^3, \mu(a^2d - abc) + b^2d, \mu(bc^2 - acd) + bd^2, d^3]. \tag{6.3}$$

Lemma 6.1. *Let $q \equiv 0 \pmod 3$. Let $\psi \in G_q$ fix $R_{\mu,\infty}$. Then a matrix $\mathbf{M}(\psi)$ corresponding to ψ has the general form*

$$\mathbf{M}(\psi) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & d^2 & 0 \\ 0 & 0 & 0 & d^3 \end{bmatrix}, \ d \in \{1, -1\}. \tag{6.4}$$

Moreover, for any point $P \in \mathrm{PG}(3,q)$, we have $P\psi = P$ if $d = 1$ and $P(\psi)^2 = P$ if $d = -1$. Also, $R_{\mu,\gamma}\psi = R_{\mu,d\gamma}$, $\gamma \in \mathbb{F}_q$.

Proof. Let $\mathbf{M}$ be given by (6.1). By hypothesis, $[1,0,1,0]\mathbf{M} = [1,0,1,0]$. Comparing components of the vectors $[1,0,1,0]\mathbf{M}$ and $[1,0,1,0]$ we consequently obtain $c = 0$, $a \neq 0$, $abd = 0$, $ad^2 \neq 0$, $d \neq 0$, $b = 0$, $a^3 = ad^2$, $d^2 = a^2$. As $\mathbf{M}$ is defined by the factor of proportionality, we can put $a = 1$. Then $d = \sqrt{1} \in \{1, -1\}$, and from $\mathbf{M}$ we obtain $\mathbf{M}(\psi)$.

The remaining assertions are obvious. □

Lemma 6.2. *Let $q \equiv 0 \pmod 3$. Let $\varphi \in G_q^{\ell_\mu}$ be such that $R_{\mu,\infty}\varphi = R_{\mu,\gamma}$, $\gamma \in \mathbb{F}_q$. If μ is not a square in $\mathbb{F}_q$ then φ does not exist. If μ is a square in $\mathbb{F}_q$ then a matrix $\mathbf{M}(\varphi)$ corresponding to φ has the general form*

$$\mathbf{M}(\varphi) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & b & 0 \\ 0 & b^2 & 0 & 0 \\ b^3 & 0 & 0 & 0 \end{bmatrix}, \ b \in \{\sqrt{\mu}, -\sqrt{\mu}\}. \tag{6.5}$$

Moreover, for any point $P \in \mathrm{PG}(3,q)$, we have $P(\varphi)^2 = P$. Also,

$$R_{\mu,\infty}\mathbf{M}(\varphi) = R_{\mu,0},\ R_{\mu,0}\mathbf{M}(\varphi) = R_{\mu,\infty},\ R_{\mu,\gamma}\mathbf{M}(\varphi) = R_{\mu,\gamma'},\ \gamma,\gamma' \in \mathbb{F}_q^*,\ \gamma' = \frac{\pm\sqrt{\mu^3}}{\gamma}.$$

Proof. Let $\mathbf{M}$ be given by (2.7). By hypothesis, there exist $\gamma, \zeta \in \mathbb{F}_q$ such that $[1,0,1,0]\mathbf{M} = [\gamma,\mu,\gamma,1]$ and $[\zeta,\mu,\zeta,1]\mathbf{M} = [1,0,1,0]$. The first equality implies $c^3 \neq 0$, see (6.2). As $\mathbf{M}$ is defined up to a factor of proportionality, we can put $c = 1$. Now, comparing components of the vectors $[1,0,1,0]\mathbf{M}$ with $[1,\gamma,1,\gamma]$ and $[\zeta,\mu,\zeta,1]\mathbf{M}$ with $[1,0,1,0]$, we obtain

$$a^3 = a + ad^2 - bd; \tag{6.6}$$
$$\mu = a^2 + b^2 - abd. \tag{6.7}$$
$$\zeta a^3 + b^3 = -\mu ad + \mu b + \zeta a + \zeta ad^2 - \zeta bd + bd^2 \tag{6.8}$$
$$\mu a^2 d - \mu ab + \zeta a^2 - \zeta abd + \zeta b^2 + b^2 d = 0; \tag{6.9}$$

$$\zeta + d^3 = 0. \tag{6.10}$$

(a) Let $\zeta = 0$. Then, by (6.10), $d = 0$. By (6.6), $a(a-1)(a+1) = 0$.

- **(a1)** Let $a = 0$. By (6.7), μ is a square and $b = \pm\sqrt{\mu}$. In this case, i.e. for $\zeta = a = 0$, $b = \pm\sqrt{\mu}$, also (6.8), (6.9), are satisfied.
- **(a2)** Let $a = \pm 1$. Then (6.9) becomes $\mp\mu b = 0$ that implies $b = 0$ and $ad - bc = 0$, contradiction.

(b) Let $\zeta \neq 0$. By (6.10), $\zeta = -d^3$, $d = \sqrt[3]{-\zeta} \neq 0$. From (6.6), $b = (a + ad^2 - a^3)/d$ that implies $ad - bc = (a^3 - a)/d$. If $a = 0, \pm 1$ then $ad - bc = 0$, contradiction. For the obtained b, by (6.7), $\mu = a^2((a^2 - d^2)(a^2 + 1) + 1)/d^2$. If $a = \pm d$ then $\mu = 1$, contradiction. For the obtained ζ, b, and μ, by (6.8) and (6.9), we have

$$-a^2(a+1)(a-1)(a+d)(a-d)(a^2+d^2+1)/d = 0;$$
$$a^2(a+1)(a-1)(a+d)(a-d)(a^4+a^2-d^4+d^2+1)/d^3 = 0.$$

By above, $a \neq 0, \pm 1, \pm d$. Therefore $a^2 + d^2 + 1 = 0$ and $a^4 + a^2 - d^4 + d^2 + 1 = 0$ that implies $a^4 - d^4 = (a+d)(a-d)(a^2+d^2) = 0$, $a^2 + d^2 = 0$, contradiction.

The remaining assertions can be obtained by direct computation. □

Theorem 6.3. *Let $q \equiv 0 \pmod 3$.*

(i) *Let μ be a non-square in $\mathbb{F}_q$. Then the size of the subgroup $G_q^{\ell_\mu}$ of G_q fixing the* EnΓ-*line ℓ_μ is $\#G_q^{\ell_\mu} = 2$. The length of the orbit of ℓ_μ under G_q is equal to $(q^3 - q)/2$. The elements of $G_q^{\ell_\mu}$ have the matrix form* (6.4)*.*

(ii) *Let μ be a square in $\mathbb{F}_q$. Then the subgroup $G_q^{\ell_\mu}$ of G_q fixing the* EnΓ-*line ℓ_μ is isomorphic to $C_2 \times C_2$. The length of the orbit of ℓ_μ under G_q is equal to $(q^3 - q)/4$. Two elements of $G_q^{\ell_\mu}$ have the matrix form* (6.4) *and the other two ones are given by* (6.5)*.*

Proof. The assertions follow from Lemmas 6.1, 6.2 and [20, Lemma 2.44(ii)]. □

Lemma 6.4. *Let $q \equiv 0 \pmod 3$, $q \geq 9$.*

(i) *Two lines ℓ_μ and $\ell_{\mu'}$, $\mu, \mu' \in \mathbb{F}_q^* \setminus \{1\}$, $\mu \neq \mu'$ belong to the same orbit of G_q if and only if $\mu = d^4$, $\mu' = d^4 + d^2 + 1$, d is such that $1 - d^2$ is a square, $d \in \mathbb{F}_q^* \setminus \{\pm 1\}$, and also $d \neq \pm\sqrt{-1}$ if $q \equiv 1 \pmod 4$.*

(ii) *All lines ℓ_μ with μ non-square in $\mathbb{F}_q$ belong to distinct orbits of G_q.*

(iii) *Let μ by a square in $\mathbb{F}_q$. If $q \equiv -1 \pmod 4$ then at most two ℓ_μ-lines belong to the same orbit of G_q; there are $(q-3)/8$ pairs of ℓ_μ-lines belonging to the same orbit. If $q \equiv 1 \pmod 4$ then at most three ℓ_μ-lines belong to the same orbit of G_q; there are $(q-9)/8$ pairs of ℓ_μ-lines belonging to the same orbit.*

Proof. **(i)** Let $\psi \in G_q$, $\ell_\mu \psi = \ell_{\mu'}$, $\mu \neq \mu'$, let $\mathbf{M}(\psi)$ be a matrix corresponding to ψ. We find it as is a version of $\mathbf{M}$ of (2.7). Note that $R_{\mu,\infty} = R_{\mu',\infty} = \mathbf{P}(1,0,1,0)$ and $R_{\mu,\infty}\psi$ is given by (6.2). By Lemma 6.1, if $R_{\mu,\infty}\psi = R_{\mu',\infty}$, then $R_{\mu,0}\psi = R_{\mu,0}$ that implies $\ell_\mu\psi = \ell_{\mu'}$.

(a) Let $\mathbf{P}(1,0,1,0)\psi = \mathbf{P}(0,\mu',0,1)$. Then $a = d = 0$, $b, c \neq 0$. As $\mathbf{M}$ is defined up to a factor of proportionality, we can put $c = 1$, obtaining $\mu' = b^2$. On the other hand, $\mathbf{P}(0,\mu,0,1)\psi = \mathbf{P}(b^3, 0, \mu b, 0)$ that implies $\mu = b^2$, so $\mu = \mu'$, contradiction.

(b) Let $\mathbf{P}(1,0,1,0)\psi = \mathbf{P}(\zeta,\mu',\zeta,1)$, $\zeta \in \mathbb{F}^*$. Then $a, c \neq 0$ and putting again $c = 1$ we obtain

$$a^3 - a - ad^2 + bd = 0; \tag{6.11}$$
$$\mu' = a^2 + b^2 - abd. \tag{6.12}$$

(b1) Let $\mathbf{P}(0,\mu,0,1)\psi = \mathbf{P}(1,0,1,0)$. By (6.3) with $c = 1$ and $a \neq 0$, we consistently obtain $d = 0$, $\mu ab = 0$, $b = 0$; so $ad - bc = 0$, contradiction.

(b2) Let $\mathbf{P}(0,\mu,0,1)\psi = \mathbf{P}(0,\mu',0,1)$. By (6.3) with $c = 1$ and $a \neq 0$, we consistently obtain $b = 0$, $\mu ad = 0$, $d = 0$; so $ad - bc = 0$, contradiction.

(b3) Let $\mathbf{P}(0,\mu,0,1)\psi = \mathbf{P}(\zeta',\mu',\zeta',1)$, $\zeta' \in \mathbb{F}^*, \zeta' \neq \zeta$. By (6.3) with $c = 1$ and (6.12), we have

$$\mu(b - ad) + bd^2 - b^3 = 0; \tag{6.13}$$
$$\mu(a^2d - ab) + b^2d - \mu'd^3 = (ad - b)(\mu a - ad^2 + bd^3 - bd) = 0. \tag{6.14}$$

As $c = 1$, $a \neq 0$, from $ad - bc \neq 0$ and (6.14) we obtain

$$\mu = (ad^2 - bd^3 + bd)/a. \tag{6.15}$$

Now consider $R_{\mu,\gamma}\psi = \mathbf{P}(\gamma,\mu,\gamma,1)\psi$, $\gamma \in \mathbb{F}^*$. By (6.1),

$$[\gamma,\mu,\gamma,1]\mathbf{M} = [\gamma a^3 + b^3,\ \mu a^2 d - \mu ab + \gamma a^2 - \gamma abd + \gamma b^2 + b^2 d, \tag{6.16}$$
$$-\mu ad + \mu b + \gamma a + \gamma ad^2 - \gamma bd + bd^2,\ \gamma + d^3].$$

As $R_{\mu,\infty}\psi, R_{\mu,0}\psi \neq R_{\mu',\infty}$, there exists $\gamma' \in \mathbb{F}^*$ such that $R_{\mu,\gamma'}\psi = R_{\mu',\infty}$. By (6.16) we obtain $\gamma' = -d^3$ and, using also (6.15), $(ad-b)^2(a^2d - ab - d^3 + d) = 0$ from which, as $ad - b \neq 0$ and $a \neq 0$, it follows

$$b = (a^2d - d^3 + d)/a. \tag{6.17}$$

From (6.11), substituting the value of b, we obtain $(a+d)(a-d)(a^2+d^2-1)=0$. If $a=\pm d$ then, by (6.17), $b=\pm 1$, that implies, by (6.15), $\mu=1$, contradiction. Therefore

$$a^2+d^2-1=0. \tag{6.18}$$

This implies $1-d^2$ is a square and from (6.17)

$$b=-ad. \tag{6.19}$$

From (6.15), (6.12), (6.18), and (6.19), we obtain

$$\mu=d^4,\ \mu'=d^4+d^2+1=(d+1)^2(d-1)^2. \tag{6.20}$$

For $c=1$, by (6.19), $ad-bc=-ad\neq 0$.

Now we consider forbidden values of d. As $\mu\neq 1$, $d^4\neq 1$, so $d\neq\pm 1$, $d\neq\pm\sqrt{-1}$. By [20, Section 1.5 (ix)] and [20, Section 1.5 (x)], -1 is a square if and only if $q\equiv 1 \pmod 4$, whereas -1 is not a square if and only if $q\equiv -1 \pmod 4$. Also $d\neq 0$, otherwise $\mu=0$, contradiction. Note that $\mu'=0$ implies $d=\pm 1$, whereas $\mu'=1$ implies $d=0$ or $d=\pm\sqrt{-1}$, so no other condition on d is obtained considering μ'.

(ii) The assertion follows from the case (i).

(iii) If ℓ_μ and $\ell_{\mu'}$ belong to the same orbit, then there exists a projectivity $\psi'\in G_q$ other than ψ of (i) such that $\ell_{\mu'}\psi'=\ell_\mu$. Repeating for ψ' the same argument as in the case (i) and taking in mind that $\sqrt{-1}$ does not exist for $q\equiv -1 \pmod 3$, we obtain

$$\mu'=d'^4,\ \mu=d'^4+d'^2+1=(d'+1)^2(d'-1)^2,\ d'\in\mathbb{F}_q^*\setminus\{\pm 1,\pm\sqrt{-1}\}. \tag{6.21}$$

From (6.20), (6.21) we have $\mu+\mu'=d^4+d'^4=d'^4+d'^2+1+d^4+d^2+1$ that implies

$$d^2+d'^2-1=0,\ d,d'\in\mathbb{F}_q^*\setminus\{\pm 1,\pm\sqrt{-1}\}. \tag{6.22}$$

We can see (6.22) as the affine equation of the irreducible conic $d^2+d'^2-z^2=0$, with respect to the infinite line $z=0$.

If $q\equiv -1 \pmod 4$, -1 is not a square, so the line $z=0$ is external to the conic, as $d^2\neq -d'^2$ if $d,d'\neq 0$. Therefore all the points of the conic are affine points, so there exist $q+1$ pairs (d,d') satisfying (6.22). However not all pairs are feasible: by the constraints on d, the 4 pairs: $(0,\pm 1)$, $(\pm 1,0)$ must be excluded.

If $q\equiv 1 \pmod 4$, -1 is a square, so the line $z=0$ has two common points with the conic, namely $\mathbf{P}(1,\pm\sqrt{-1},0)$. Therefore there exist $q-1$ pairs (d,d') satisfying

(6.22). However not all pairs are feasible: by the constraints on d, the 4 pairs $(0, \pm 1)$, $(\pm 1, 0)$ and the 4 pairs $(\pm\sqrt{-1}, \pm\sqrt{-1})$, $(\pm\sqrt{-1}, \mp\sqrt{-1})$ must be excluded.

By the symmetry of the equation, if the pair (d_1, d_2) satisfies (6.22), the eight pairs $\{(\pm d_i, \pm d_j), (\pm d_i, \mp d_j), i, j \in \{1, 2\}, i \neq j\}$ satisfy the same equation. All the eight pairs produce the same unordered pair of values $\{\mu_1 = d_1^4, \mu_2 = d_2^4\}$, so in total we have $(q-3)/8$ different pairs of lines $\ell_\mu, \ell_{\mu'}$ belonging to the same orbit for $q \equiv -1 \pmod 4$, whereas for $q \equiv 1 \pmod 4$ we have $(q-9)/8$ different pairs of lines $\ell_\mu, \ell_{\mu'}$ belonging to the same orbit.

However it can happen that more than 2 lines of type ℓ_μ belong to the same orbit. If the lines $\ell_\mu, \ell_{\mu'}, \ell_{\mu''}$, μ, μ', μ'' pairwise distinct, belong to the same orbit, then there exist d', d'' such that $\mu = d'^4 = d''^4$, $d'^4 + d'^2 + 1 = \mu'$, $d''^4 + d''^2 + 1 = \mu''$. As both d'^2 and d''^2 are square roots of μ, $d'^2 = -d''^2$, otherwise $\mu' = \mu''$. This implies -1 is a square that, by [20, Section 1.5 (ix)], is equivalent to say $q \equiv 1 \pmod 4$.

If there exists μ''' belonging to the same orbit $\ell_\mu, \ell_{\mu'}, \ell_{\mu''}$ belong to, then there exists d''' such that $\mu = d'''^4$, $d'''^4 + d'''^2 + 1 = \mu'''$. As d'''^2 is a square root of μ, $d'''^2 = d'^2$ or $d'''^2 = d''^2$, so $\mu''' = \mu'$ or $\mu''' = \mu''$. Therefore at most three lines of type ℓ_μ can belong to the same orbit. □

Theorem 6.5. *Let $q \equiv 0 \pmod 3$, $q \geq 9$. Let $\mathfrak{n}_q$ be the total number of distinct orbits of ℓ_μ-lines. Let $\mathfrak{S}_q$ be the total number of* EnΓ*-lines contained in these orbits. Recall that the total number of* EnΓ*-lines is $\#\mathcal{O}_6 = \#\mathrm{En}\Gamma = (q^2 - q)(q^2 - 1)$.*

(i) *Let $q \equiv -1 \pmod 4$. We have $(q-1)/2$ orbits of length $(q^3 - q)/2$ generated by ℓ_μ-lines, where μ is a non-square in $\mathbb{F}_q$, and $(3q-9)/8$ orbits of length $(q^3 - q)/4$ generated by ℓ_μ-lines, where μ is a square. This implies $\mathfrak{n}_q = (7q - 13)/8$, $\mathfrak{S}_q = (q^3 - q)(11q - 17)/32 \approx \frac{11}{32}\#\mathcal{O}_{\mathrm{En}\Gamma} \approx 0.343\#\mathcal{O}_{\mathrm{En}\Gamma}$*

(ii) *Let $q \equiv 1 \pmod 4$. We have $(q-1)/2$ orbits of length $(q^3-q)/2$ generated by ℓ_μ-lines, where μ is a non-square in $\mathbb{F}_q$, and $(3q-3)/8+2\mathfrak{t}_q$ orbits of length $(q^3-q)/4$ generated by ℓ_μ-lines, where μ is a square, $\mathfrak{t}_q$ is the number of triples of ℓ_μ-lines belonging to the same orbit, $0 \leq \mathfrak{t}_q \leq (q-9)/24$. This implies $(7q-7)/8 \leq \mathfrak{n}_q \leq (23q - 39)/24$, $(q^3 - q)(11q - 7)/32 \leq \mathfrak{S}_q \leq (q^3 - q)(35q - 45)/96$, $0.343\#\mathcal{O}_{\mathrm{En}\Gamma} \approx \frac{11}{32}\#\mathcal{O}_{\mathrm{En}\Gamma} \lesssim \mathfrak{S}_q \lesssim \frac{35}{96}\#\mathcal{O}_{\mathrm{En}\Gamma}$.*

Proof. The assertions follow from Theorem 6.3 and Lemma 6.4 and obvious direct computations. In the point (ii), the left inequalities correspond to the case when there are no triples of ℓ_μ-lines belonging to the same orbit, the right ones correspond to the case when all equivalent ℓ_μ-lines come in triples. □

Example 6.6. Using Magma, for $q \equiv 1 \pmod 4$, we obtain the following:

$$q = 3^{2m} \equiv 1 \pmod 4,\ m = 1, \dots 5,\ \mathfrak{t}_q = \frac{q - (-1)^m \sqrt{q} - 15}{48}. \tag{6.23}$$

This implies that for $q = 3^{2m},\ m = 1, \dots 5$, we have exactly $(10q - 24 - (-1)^m \sqrt{q})/24$ orbits of length $(q^3 - q)/4$ generated by ℓ_μ-lines, where μ is a square. Also, $\mathfrak{S}_q = (q^3 - q)(34q - 48 - (-1)^m \sqrt{q})/96 \approx \frac{17}{48} \#\mathcal{O}_{\mathrm{En}\Gamma} \approx 0.354 \#\mathcal{O}_{\mathrm{En}\Gamma}$.

Conjecture 6.7. *The results of Example* 6.6 *hold for all* $m \geq 1$.

7 On the orbits of lines ℓ_μ, $\mu \in \mathbb{F}_q^* \setminus \{1, 1/9\}$, q odd, $q \not\equiv 0 \pmod 3$

Lemma 7.1. *Let q be odd, $q \not\equiv 0 \pmod 3$. Let $\psi \in G_q^{\ell_\mu}$ be such that $R_{\mu,\infty}\psi = R_{\mu,\gamma}$, $\gamma \in \mathbb{F}_q$. If μ is not a square in $\mathbb{F}_q$ then ψ does not exist. If μ is a square in $\mathbb{F}_q$ and either $\mu \neq -1/3$, or $q \not\equiv 1 \pmod{12}$, or $-1/3$ is not a fourth power, then a matrix $\mathbf{M}(\psi)$ corresponding to ψ is as follows.*

$$\mathbf{M}(\psi) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & b & 0 \\ 0 & b^2 & 0 & 0 \\ b^3 & 0 & 0 & 0 \end{bmatrix},\ b \in \{\sqrt{\mu}, -\sqrt{\mu}\}. \tag{7.1}$$

Moreover, for any point $P \in \mathrm{PG}(3, q)$, we have $P(\mathbf{M}(\psi))^2 = P$. Also,

$$R_{\mu,\infty}\mathbf{M}(\psi) = R_{\mu,0},\ R_{\mu,0}\mathbf{M}(\psi) = R_{\mu,\infty},\ R_{\mu,\gamma}\mathbf{M}(\psi) = R_{\mu,\gamma'},\ \gamma, \gamma' \in \mathbb{F}_q^*,\ \gamma' = \frac{\pm\sqrt{\mu^3}}{\gamma}.$$

If $\mu = -1/3$ and $q \equiv 1 \pmod{12}$ and $-1/3$ is a fourth power, then there are four distinct fourth roots of $-1/3$ and a matrix $\mathbf{M}'$ corresponding to ψ has either the form (7.1) *or the form* (2.7) *with*

$$a = -b/d,\ b = \pm\sqrt{1/3},\ c = 1,\ d = \textit{a fourth root of } -1/3. \tag{7.2}$$

If $\mathbf{M}'$ has the form (7.1) *it has order* 2*, otherwise it has order* 3*.*

Proof. We consider $\mu \in \mathbb{F}_q^* \setminus \{1, 1/9\}$ due to Lemma 4.4. Let $\mathbf{M}$ be given by (2.7). By hypothesis, there exist $\gamma, \zeta \in \mathbb{F}_q$ such that $[1, 0, 1, 0]\mathbf{M} = [\gamma, \mu, \gamma, 1]$ and $[\zeta, \mu, \zeta, 1]\mathbf{M} = [1, 0, 1, 0]$. This implies

$$a^3 + 3ab^2 - ac^2 - ad^2 - 2bcd = 0; \tag{7.3}$$

$$a^2c + 2abd + b^2c - \mu c^3 - 3\mu cd^2 = 0. \tag{7.4}$$
$$3\mu a^2b - 2\mu acd - \mu bc^2 + \zeta a^3 + 3\zeta ab^2 - \zeta ac^2 - \zeta ad^2 + b^3 - 2\zeta bcd - bd^2 = 0; \tag{7.5}$$
$$\mu a^2d + 2\mu abc + \zeta a^2c + 2\zeta abd + \zeta b^2c + b^2d = 0; \tag{7.6}$$
$$3\mu c^2d + \zeta c^3 + 3\zeta cd^2 + d^3 = 0. \tag{7.7}$$

If $c = 0$ then, by (7.7), $d^3 = 0$, so $ad - bc = 0$, contradiction. Therefore, , $c \neq 0$. As $\mathbf{M}$ is defined up to a factor of proportionality, we can put $c = 1$.

(a) Let $\zeta = 0$. By (7.7), $d(3\mu + d^2) = 0$.

(a1) Let $d = 0$. By (7.6), $2\mu ab = 0$.

(a11) Let $b = 0$. Then $ad - bc = 0$, contradiction.

(a12) Let $b \neq 0$. Then $a = 0$ and, by (7.4), $b^2 - \mu = 0$ that implies μ is a square and $b = \pm\sqrt{\mu}$. Then for $c = 1$, $a = d = \zeta = 0$, also (7.3) and (7.5) are satisfied and we obtain the matrix (7.1).

(a2) Let $d \neq 0$. Then, by (7.7), -3μ must be a square and $d^2 = -3\mu$. Equation (7.6) becomes $\mu a^2d + 2\mu ab + b^2d = 0$ that gives $a = -3b/d$ or $a = b/d$.

(a21) Let $a = b/d$. Then $ad - bc = 0$, contradiction.

(a22) Let $a = -3b/d$. From (7.3) we have $-9b(\mu - 1)(\mu + 3b^2)/d^3 = 0$.

(a221) Let $b = 0$. Then $a = 0$ and $ad - bc = 0$, contradiction.

(a222) Let $b \neq 0$. Then $-\mu/3$ must be a square and $b^2 = -\mu/3$. It follows that $a^2 = 1$ as $d^2 = -3\mu$. Also, $ad = -3b$. Then (7.5) becomes $32\mu b/3 = 0$. As q is odd, this implies $\mu = 0$, contradiction.

(b) Let $\zeta \neq 0$. By above, this means $[0, \mu, 0, 1]\mathbf{M} = [\gamma', \mu, \gamma', 1]$, $\gamma' \in \mathbb{F}_q$ where $\gamma' \neq \gamma$ with $[1, 0, 1, 0]\mathbf{M} = [\gamma, \mu, \gamma, 1]$. What has been said entails

$$3\mu a^2b + b^3 - 2\mu ad - \mu b - bd^2 = 0; \tag{7.8}$$
$$\mu a^2d + 2\mu ab + b^2d - 3\mu^2d - \mu d^3 = 0. \tag{7.9}$$

(b1) Let $d = 0$. By (7.7), $\zeta = 0$, contradiction.

(b2) Let $d \neq 0$. If $a = 0$ then, by (7.3), $-2bd = 0$ that implies $b = 0$, and $ad - bc = 0$, contradiction. So, we should put $a \neq 0$. Now, if $b = 0$ then, by (7.8), $-2\mu ad = 0$, contradiction. So, we should take also $b \neq 0$.
By (7.7), $\zeta(1 + 3d^2) = -3\mu d - d^3$.

(b21) Let $1 + 3d^2 = 0$. Then $-3\mu d - d^3 = 0$, $d^2 = -1/3$, $\mu = 1/9$, contradiction.

(b22) Let $1 + 3d^2 \neq 0$. Then

$$\zeta = (-3\mu d - d^3)/(1 + 3d^2). \tag{7.10}$$

Now, by (7.6), $(3\mu d^2 - 2\mu - d^2)a = bd(2d^2 + 1 - 3\mu)$. If $3\mu d^2 - 2\mu - d^2 = 0$, then $2d^2 + 1 - 3\mu = 0$, $\mu = (2d^2 + 1)/3$, and the coefficient of a becomes $2(d-1)(d+1)(d^2 + 1/3)$. If $d = \pm 1$, then $\mu = 1$, contradiction. As $d^2 + 1/3 \neq 0$ we have $3\mu d^2 - 2\mu - d^2 \neq 0$ and

$$a = \frac{2bd^3 + bd - 3\mu bd}{3\mu d^2 - 2\mu - d^2}. \tag{7.11}$$

Substituting this value of a in (7.4) (taking into account $d^2 + 1/3 \neq 0$) we obtain

$$b^2(3\mu^2 d^2 - 4\mu^2 - 4\mu d^4 + 6\mu d^2 - d^2) = -\mu(3\mu(3d^2 - 2) - d^2)^2; \tag{7.12}$$

If the coefficient of b^2 is 0, then also $\mu(3d^2 - 2) - d^2 = 0$. If $3d^2 - 2 = 0$, then $d^2 = 0$, contradiction; so $\mu = d^2/(3d^2 - 2)$. Substituting this value of μ in the coefficient of b^2 we obtain $d^2(d-1)(d+1)(d^2 + 1/3)/(d^2 - 2/3)^2 = 0$, contradiction, as $d \neq 0$, $d^2 + 1/3 \neq 0$ and $d = \pm 1$ implies $\mu = 1$. So, the coefficient of b^2 cannot be 0, and we obtain

$$b^2 = \frac{-\mu(3\mu d^2 - 2\mu - d^2)^2}{3\mu^2 d^2 - 4\mu^2 - 4\mu d^4 + 6\mu d^2 - d^2}. \tag{7.13}$$

Substituting the value of ζ of (7.10) in (7.5) we have

$$(ad - b)(3\mu a^2 + 9\mu abd - 3\mu d^2 + mu - a^2 d^2 - abd - 3b^2 d^2 - b^2 + d^4 + d^2) = 0. \tag{7.14}$$

Recalling that $ad - b \neq 0$ and substituting the value of a of (7.11) in the second factor of (7.14) we obtain

$$\begin{aligned}(\mu(3d^2 - 1) - d^2(d^2 + 1))(27\mu^2 b^2 d^2 + 9\mu^2 d^4 - 12\mu^2 d^2 \\ + 4\mu^2 - 18\mu b^2 d^2 - 4\mu b^2 - 6\mu d^4 + 4\mu d^2 - 4b^2 d^4 - b^2 d^2 + d^4) = 0.\end{aligned} \tag{7.15}$$

Suppose $\mu(3d^2 - 1) - d^2(d^2 + 1) = 0$. If $3d^2 - 1 = 0$ then also $d^2(d^2 + 1) = 0$. As $d \neq 0$ this implies $d^2 = -1$, so $-3 - 1 = -4 = 0$, contradiction, as q is odd. Therefore $\mu = (d^4 + d^2)/(3d^2 - 1)$. Substituting this value of μ in (7.13) we obtain $b^2 = (d^6 + d^4)/(3d^2 - 1) = \mu d^2$. As $b \neq 0$, from (7.11) we obtain $ab = (d^5 + d^3)/(3d^2 - 1)$ and finally $abd - b^2 = 0$, contradiction.

Substituting the value of b^2 (7.13) in the second factor of (7.15) we obtain

$$d^2(\mu-1)(\mu+1/3)(\mu-1/9)(3\mu d^2-2\mu-d^2)^2=0. \tag{7.16}$$

Suppose $3\mu d^2-2\mu-d^2=0$. Then, by (7.13), $b^2=0$, contradiction.
Suppose $\mu=-1/3$. Then by (7.11), (7.13),

$$\begin{aligned} a &= (3bd^3+3bd)/(1-3d^2); \\ b^2 &= (3d^2-1)^2/(9d^4-18d^2-3). \end{aligned} \tag{7.17}$$

Substituting this value of a in (7.3) we obtain

$$bd(d^2-1/3)^2(d^2+1/3)(d^4+1/3)=0. \tag{7.18}$$

By the constraints above, $d^4=-1/3$, so $-1/3$ must be a fourth power. Hence, by (7.17), $b^2=1/3$, so $1/3$ must be a square. Then $a=(d(3bd^3+3bd))/(d(1-3d^2))=(b(-1+3d^2))/(d(1-3d^2))=-b/d$ and $ad-b=-2b\neq 0$. As μ is a square and $\mu=-1/3$, by [20, Section 1.5(xi)] $q\equiv 1 \pmod 3$, so as $1/3$ is also a square, -1 must be a square that, by [20, Section 1.5(ix)], happens if and only if $q\equiv 1 \pmod 4$. The two conditions implies $q\equiv 1 \pmod{12}$. Then, by [20, Section 1.5(v)], if $-1/3$ is a fourth power the equation $d^4=-1/3$ has four distinct roots.
The other factors of (7.16) cannot be zero and this completes the proof about the structure of the matrix $\mathbf{M}(\psi)$.
The remaining assertions follow by direct computation using Magma. □

Remark 7.2. Note that in Lemma 7.1 the condition $q\equiv 1 \pmod{12}$ and $-1/3$ is a fourth power is not empty; the examples are $q=37,49,61,121,157,169,193,313,\ldots$

Theorem 7.3. *Let $q\not\equiv 0 \pmod 3$. Let the* EnΓ*-line ℓ_μ be as in (4.2) with $\mu\in\mathbb{F}_q^*\setminus\{1,1/9\}$. Let $G_q^{\ell_\mu}$ be the subgroup of G_q fixing ℓ_μ and let $\mathscr{O}_\mu$ be the orbit of ℓ_μ under G_q. Then $\#G_q^{\ell_\mu}$, $\#\mathscr{O}_\mu$ and the structure of $G_q^{\ell_\mu}$ are as follows:*

(i) *Let μ be a non-square in $\mathbb{F}_q$. Then $\#G_q^{\ell_\mu}=2$, $\#\mathscr{O}_\mu=(q^3-q)/2$. The elements of $G_q^{\ell_\mu}$ have the matrix form (4.6).*

(ii) *Let μ be a square in $\mathbb{F}_q$ and either $\mu\neq -1/3$, or $q\not\equiv 1 \pmod{12}$, or $-1/3$ is not a fourth power. Then $G_q^{\ell_\mu}$ is isomorphic to $C_2\times C_2$ and $\#\mathscr{O}_\mu=(q^3-q)/4$. Two elements of $G_q^{\ell_\mu}$ have the matrix form (4.6) and the other two ones are given by (7.1).*

(iii) *If $\mu = -1/3$ and $q \equiv 1 \pmod{12}$ and $-1/3$ is a fourth power, then μ is a square in $\mathbb{F}_q$, $G_q^{\ell_\mu}$ is isomorphic to A_4 and $\#\mathscr{O}_\mu = (q^3 - q)/12$. Two elements of $G_q^{\ell_\mu}$ have the matrix form (4.6), two other ones are given by (7.1), and the last eight elements have the matrix form (7.2).*

Proof. The assertions follow from Lemmas 4.5 , 7.1, and [20, Lemma 2.44(ii)]. Finally, the only group of order 12 having three elements of order two and eight elements of order three is A_4, see [22]. □

Lemma 7.4. *Let q be odd, $q \equiv -1 \pmod 3$, μ be not a square. The line $\mathcal{L}$ and a line ℓ_μ belong to the same orbit of G_q if and only if $\mu = -1/3$, $q \equiv -1 \pmod{12}$.*

Proof. Let $\psi \in G_q$, $\mathcal{L}\psi = \ell_\mu$ and let $\mathbf{M}(\psi)$ be a matrix corresponding to ψ. We find it as is a version of $\mathbf{M}$ of (2.7). By (2.7), we have

$$[0,0,1,0]\mathbf{M} = [3ab^2, b^2c + 2abd, ad^2 + 2bcd, 3cd^2]. \tag{7.19}$$

(a) Let $\mathbf{P}(0,0,1,0)\psi = \mathbf{P}(1,0,1,0)$. Then $ad^2 + 2bcd \neq 0$, $3cd^2 = 0$, $b^2c + 2abd = 0$ from which we sequently obtain $d \neq 0$, $c = 0$, $ab = 0$. But $3ab^2 \neq 0$, contradiction.

(b) Let $\mathbf{P}(0,0,1,0)\psi = \mathbf{P}(0,\mu,0,1)$. Then $b^2c + 2abd \neq 0$, $3ab^2 = 0$, $ad^2 + 2bcd = 0$ from which we sequently obtain $b \neq 0$, $a = 0$, $cd = 0$. But $3cd^2 \neq 0$, contradiction.

(c) Let $\mathbf{P}(0,0,1,0)\psi = \mathbf{P}(\zeta,\mu,\zeta,1)$, $\zeta \in \mathbb{F}_q^*$. This implies

$$3ab^2 - ad^2 - 2bcd = 0; \tag{7.20}$$
$$b^2c + 2abd - 3\mu cd^2 = 0; \tag{7.21}$$
$$3ab^2, 3cd^2 \neq 0. \tag{7.22}$$

By (7.22), $a, b, c, d \neq 0$. As $\mathbf{M}$ is defined up to a factor of proportionality, we can put $d = 1$. Then by (7.20)

$$c = (3ab^2 - a)/2b. \tag{7.23}$$

Substituting this value of c in (7.21) we obtain: $3a(-3\mu b^2 + \mu + b^4 + b^2)/2b = 0$ which implies $\mu(3b^2 - 1) = b^4 + b^2$. Suppose $3b^2 - 1 = 0$; then $b^4 + b^2 = 4/9 = 0$, contradiction as q is odd. Therefore $b^2 \neq 1/3$ and

$$\mu = (b^4 + b^2)/(3b^2 - 1). \tag{7.24}$$

If $b = \pm 1$, then $\mu = 1$, contradiction. Therefore in the following we assume $b \neq \pm 1$. By $d = 1$ and (7.23) we have:

$$[1,0,0,1]\mathbf{M} = [8a^3b^3 + 8b^6,\ 12a^3b^4 - 4a^3b^2 + 8b^5, \tag{7.25}$$

$$18a^3b^5 - 12a^3b^3 + 2a^3b + 8b^4,\ 27a^3b^6 - 27a^3b^4 + 9a^3b^2 - a^3 + 8b^3].$$

(c1) Let $\mathbf{P}(1,0,0,1)\psi = \mathbf{P}(1,0,1,0)$. Then, by (7.25),

$$2b(b-1)(b+1)(9a^3b^2 - a^3 - 4b^3) = 0; \tag{7.26}$$
$$4b^2(3a^3b^2 - a^3 + 2b^3) = 0; \tag{7.27}$$
$$27a^3b^6 - 27a^3b^4 + 9a^3b^2 - a^3 + 8b^3 = 0. \tag{7.28}$$

From (7.26), $a^3(9b^2-1) = 4b^3$. If $9b^2 - 1 = 0$ then $4b^3 = 0$, contradiction as q is odd and $b \neq 0$, so $a^3 = 4b^3/(9b^2-1)$. From (7.27), substituting the value of a^3, we obtain $6b^3(5b^2-1)/(9b^2-1) = 0$. If $q \equiv 0 \pmod 5$ then $-1 = 0$, contradiction. Suppose $q \not\equiv 0 \pmod 5$; then we obtain $b^2 = 1/5$. Then $a^3 = b$. Substituting the values of a^3 and b^2 in (7.28), we obtain: $192b/125 = 0$. As $192 = 2^6 3$, this implies $b = 0$, contradiction.

(c2) Let $\mathbf{P}(1,0,0,1)\psi = \mathbf{P}(\zeta', \mu, \zeta', 1)$, $\zeta' \in \mathbb{F}, \zeta' \neq \zeta$. This implies

$$2b(b-1)(b+1)(9a^3b^2 - a^3 - 4b^3) = 0; \tag{7.29}$$
$$-27a^3b^{10} + 54a^3b^6 - 32a^3b^4 + 5a^3b^2 + 16b^7 - 16b^5 = 0; \tag{7.30}$$

Equation (7.29) is the same as (7.26), so reasoning as above we obtain $b^2 \neq 1/9$ and $a^3 = 4b^3/(9b^2-1)$. Substituting the value of a^3 in (7.30), we obtain $b^5(b-1)^2(b+1)^2(3b^4+6b^2-1)$, that implies

$$3b^4 + 6b^2 - 1 = 0. \tag{7.31}$$

The excluded values $b = 0, \pm 1, 1/3$ and $b^2 = 1/3, 1/9$ are not roots of (7.31). Equation (7.31) is equianharmonic, as its I and Δ invariants are 0 and not 0, respectively; see [20, Section 1.11]. Therefore, by [20, Theorem 1.42], as $q \equiv -1 \pmod 3$, equation (7.31) has 0 or 2 roots.

Let $y = b^2$. The equation $3y^2 + 6y - 1 = 0$ has solution in $\mathbb{F}_q$ if and only if $6^2 + 12 = 16 \cdot 3$ is a square in $\mathbb{F}_q$. By [20, Section 1.5], if q is odd, $q \equiv -1 \pmod 3$, 3 is a square if and only if $q \equiv -1 \pmod{12}$. In this case, we obtain $b^2 = (-3 \pm 2\sqrt{3})/3$.

Exactly one of the two values of b^2 is a square in $\mathbb{F}_q$. In fact, if both the values are squares or not squares, $[(-3-2\sqrt{3})/3] \cdot [(-3+2\sqrt{3})/3] = -1/3$ would be a square, contradiction as 3 is a square and -1 is not a square. Therefore, equation (7.31) has exactly 2 solutions in $\mathbb{F}_q$. Anyway, substituting both the possible values of b^2 in (7.24), we obtain the same value $\mu = -1/3$.

If $ad - bc = 0$, then $a = (3ab^2 - a)/2$. As $a \neq 0$, this implies $b^2 = 1$, contradiction as $b \neq \pm 1$. □

Lemma 7.5. *Let $q \equiv 1 \pmod{12}$, $\mu = -1/3$, $-1/2$ be a cube and $-1/3$ be a fourth power. Then μ is a square and the lines $\mathcal{L}$ and $\ell_{-1/3}$ belong to the same orbit of G_q.*

Proof. Let $\psi \in G_q$, $\mathcal{L}\psi = \ell_{-1/3}$ and let $\mathbf{M}(\psi)$ be a matrix corresponding to ψ. We find it as is a version of $\mathbf{M}$ of (2.7). By (2.7) we have (7.19). Reasoning as in Lemma 7.4, we obtain $\mathbf{P}(0,0,1,0)\psi = \mathbf{P}(\zeta, -1/3, \zeta, 1), \zeta \in \mathbb{F}^*$, that implies

$$3ab^2 - ad^2 - 2bcd = 0; \tag{7.32}$$

$$b^2c + 2abd + cd^2 = 0; \tag{7.33}$$

$$3ab^2, 3cd^2 \neq 0. \tag{7.34}$$

By (7.34), $a, b, c, d \neq 0$. As $\mathbf{M}$ is defined up to a factor of proportionality, we can put $d = 1$. Then by (7.32), $c = (3ab^2 - a)/2b$. Substituting this value of c in (7.33) we obtain $a(3b^4 + 6b^2 - 1) = 0$ which implies

$$3b^4 + 6b^2 - 1 = 0. \tag{7.35}$$

As q is odd, the values $\pm 1, \pm 1/3$, are not roots of the equation (7.35), so in the following we assume $b \neq \pm 1, \pm 1/3$.

Reasoning as in Lemma 7.4, we obtain $\mathbf{P}(1,0,0,1)\psi = \mathbf{P}(\zeta', \mu, \zeta', 1)$, $\zeta' \in \mathbb{F}, \zeta' \neq \zeta$, and

$$a^3 = 4b^3/(9b^2 - 1). \tag{7.36}$$

As $q \equiv 1 \pmod{12}$, -3 and -1 are squares in $\mathbb{F}_q$ by [20, Section 1.5] . Therefore also 3 is a square. Let α be a fourth root of $-1/3$ and β be a square root of 3 and γ be a cubic root of $-1/2$. Then Equation (7.35) can be factorized in the following way:

$$(2b + (-\alpha^3 - \alpha)\beta - 3\alpha^3 - \alpha)(2b + (\alpha^3 - \alpha)\beta + 3\alpha^3 + \alpha)$$
$$(2b + (\alpha^3 + \alpha)\beta - 3\alpha^3 - \alpha)(2b + (\alpha^3 + \alpha)\beta + 3\alpha^3 + \alpha) = 0.$$

Choosing, for example,

$$b = ((\alpha^3 + \alpha)\beta + 3\alpha^3 + \alpha)/2,$$

Equation (7.36) can be factorized in the following way:

$$(2a + (\gamma\alpha^3 - \gamma\alpha)\beta + 3\gamma\alpha^3 + \gamma\alpha)(2a + (\gamma\alpha^3 + \gamma\alpha)\beta - 3\gamma\alpha^3 + \gamma\alpha)$$
$$(a - \gamma\alpha^3\beta - \gamma\alpha) = 0$$

and we can choose, for example, $a = \gamma\alpha^3\beta + \gamma\alpha$. Note that $a \neq 0$. In fact $a = \gamma\alpha(\alpha^2\beta + 1) = 0$ implies $\alpha^2\beta = -1$ whence $(\alpha^2\beta)^2 = (-1)^2$ which simplifies to $-1 = 1$, contradiction, as q is odd.
Finally, if $ad - bc = 0$, then $a = (3ab^2 - a)/2$. As $a \neq 0$, this implies $b^2 = 1$, contradiction as $b \neq \pm 1$.

$\square$

Remark 7.6. Note that in Lemma 7.5 the condition $q \equiv 1 \pmod{12}$, $-1/2$ is a cube, and $-1/3$ is a fourth power is not empty; the examples are $q = 121, 157, 397, 433, 529, 601, 625, 961, 997, \ldots$ For $q = 121$ we actually tested by Magma that $\mathcal{L}$ and $\ell_{-1/3}$ give the same orbit.

Theorem 7.7. *The line $\mathcal{L}$ and a line ℓ_μ belong to the same orbit of G_q if and only if q is odd and $\mu = -1/3$, $q \equiv -1 \pmod{12}$ and μ is not a square, or $q \equiv 1 \pmod{12}$ and μ is a square and $1/2$ is a cube and $-1/3$ is a fourth power.*

Proof. If q is even, or q is odd and $q \equiv 1 \pmod 3$ and $-1/2$ is not a cube, or μ is a square and $\mu \neq -1/3$, or μ is a square and $\mu = -1/3$ and $q \not\equiv 1 \pmod{12}$, or μ is a square and $\mu = -1/3$ and $q \equiv 1 \pmod{12}$ and $-1/3$ is not a fourth power, by Theorems (3.5), (5.2) and (7.3), the line $\mathcal{L}$ and a line ℓ_μ have different stabilizer groups, so they cannot belong to the same orbit.

Then apply Lemmas (7.4) and (7.5). □

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