DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **10**, Numbers **1&2**, January & March **2004**

Website: http://AIMsciences.org

pp. **423–433**

UNIFORM NONAUTONOMOUS ATTRACTORS UNDER DISCRETIZATION

Peter E. Kloeden
FB Mathematik
Johann Wolfgang Goethe Universität
D-60054 Frankfurt am Main, Germany

and

Victor S. Kozyakin
Institute for Information Transmission Problems
Russian Academy of Sciences
Bolshoj Karetny lane, 19, 101447 Moscow, Russia

Abstract. A nonautonomous or cocycle dynamical system that is driven by an autonomous dynamical system acting on a compact metric space is assumed to have a uniform pullback attractor. It is shown that discretization by a one-step numerical scheme gives rise to a discrete time cocycle dynamical system with a uniform pullback attractor, the component subsets of which converge upper semi continuously to their continuous time counterparts as the maximum time step decreases to zero. The proof involves a Lyapunov function characterizing the uniform pullback attractor of the original system.

1. **Introduction.** The effects of discretization on autonomous dynamical systems has been extensively investigated over the past fifteen years and are now well understood. In particular, an autonomously discretized system has a maximal attractor that converges upper semi continuously to the maximal attractor of the original system [11, 19]. Matters are not so clear or straightforward for nonautonomous systems, even for a variable time step (hence nonautonomous) discretization of an autonomous system. A basic difficulty here is to find an appropriate definition of a nonautonomous attractor. A far deeper difficulty is simply that essentially anything can happen in the general nonautonomous setting, so attention needs to be focused of special classes of nonautonomous systems that are nevertheless general enough to encapture nontrivial dynamics. But how should one select or characterize such classes of systems?

Recent developments in random dynamical systems provide two important insights that can be exploited in the deterministic setting [1, 5, 7]. Very closely related ideas can be found in the work on nonautonomous systems of Mark Vishik and his coworkers, see for example [4, 6, 20]. The first is the cocycle formulation of a nonautonomous dynamical system, in which the state space dynamics is driven by an underlying autonomous dynamical system on some "parameter" space, with

1991 *Mathematics Subject Classification.* 34D10, 34D45, 37C60.
Key words and phrases. Cocycle dynamical systems, attractors, perturbations, discretization.
Dedicated to Marko Josephovich Vishik on the occasion of his 80th birthday.
The work was supported by the DFG Forschungsschwerpunkt "Ergodentheorie, Analysis und effiziente Simulation dynamischer Systeme" and the Russian Foundation for Basic Research grants No. 00-15-96116 and 0001-00571.

the resulting product system forming a skew-product flow. The second is the definition of a nonautonomous attractor, now called a pullback attractor, consisting of parametrized component subsets of the state space that are carried into each other under the cocycle dynamics and are attracting in a "pullback" sense that allows convergence to a fixed component subset by starting progressively earlier. These ideas have already been used to investigate such nonautonomous dynamical systems under discretization, but the results obtained are only for very specific examples or under very restricted structural assumptions. See [2, 12, 13, 14] and the references therein.

In this paper we establish an analog of the autonomous result in [11] for general uniform pullback attractors under discretization. As in [11], we make extensive use of a Lyapunov function that characterizes the attractor of the continuous time system.

The paper is organized as follows. The cocycle formulation of a nonautonomous dynamics is recalled in the next section, and pullback attractors in Section 3, while one-step numerical schemes with variable time steps are formulated as discrete time cocycle systems in Section 4. The main result is stated in Section 5 and then proved in section 6. The appendix contains the necessary modifications to the uniform case of the construction of a Lyapunov function characterizing a pullback attractor from [10]

The following notation and definitions will be used. $H^*(A,B)$ denotes the Hausdorff separation or semi-metric between nonempty compact subsets A and B of $\mathbb{R}^d$, and is defined by

$$H^*(A,B) := \max_{a\in A} \operatorname{dist}(a,B)$$

where $\operatorname{dist}(a,B) := \min_{b\in B} \|a-b\|$. For a nonempty compact subset A of $\mathbb{R}^d$ and $r>0$, the open and closed balls about A of radius r are defined, respectively, by

$$B(A;r) := \{x\in\mathbb{R}^d :\ \operatorname{dist}(x,A) < r\}, \qquad B[A;r] := \{x\in\mathbb{R}^d :\ \operatorname{dist}(x,A) \le r\}.$$

2. **The cocycle formalism.**

We consider a parametrized differential equation

$$\dot{x} = f(p,x)$$

on $\mathbb{R}^d$, where p is a parameter that is allowed to vary with time in a certain way.

In particular, let P be a compact metric space and consider a group $\{\theta_t\}_{t\in\mathbb{R}}$ of mappings $\theta_t : P \to P$ for each $t\in\mathbb{R}$ such that $(t,p)\mapsto\theta_t p$ is continuous. The autonomous dynamical system $\{\theta_t\}_{t\in\mathbb{R}}$ on P acts as a driving mechanism that generates the time variation in the parameter p in the parametrized differential equation above to form a nonautonomous differential equation

$$\dot{x} = f(\theta_t p, x) \tag{2.1}$$

on $\mathbb{R}^d$ for each $p\in P$.

A simple example is a triangular system of autonomous differential equations

$$\dot{x} = f(p,x), \qquad \dot{p} = g(p), \tag{2.2}$$

for $x\in\mathbb{R}^d$ and $p\in P$, where P is a compact manifold in $\mathbb{R}^n$ for some $n\ge 2$. The mappings $\theta_t p_0$ here are defined as the translation along the solution of the second equation, namely $\theta_t p_0 := p(t,p_0)$. A less trivial example of the above skew product formalism is given by Sell's investigations of almost periodic differential equations [17], in which P is a compact metric space of admissible vector field functions and θ_t is a temporal shift operator acting on these vector field functions. See also [18].

Random dynamical systems [1, 5, 7] also provide examples, but with a measure space rather than a topological space as the parameter space.

We make the following assumption.

Assumption 2.1. *The mapping $f : P \times \mathbb{R}^d \to \mathbb{R}^d$, where P is a compact metric space, and the mappings $\theta_t : P \to P$, $t \in \mathbb{R}$, satisfy:*

- $(p, x) \mapsto f(p, x)$ *is continuous in* $(p, x) \in P \times \mathbb{R}^d$*;*
- $x \mapsto f(p, x)$ *is globally Lipschitz continuous on* $\mathbb{R}^d$ *with Lipschitz constant* $L(p)$ *for each* $p \in P$*;*
- $p \mapsto L(p)$ *is continuous;*
- $(t, p) \mapsto \theta_t p$ *is continuous.*

For the present discussion we will also assume the global forwards existence and uniqueness of solutions of (2.1), e.g., due to an additional dissipativity structural assumption (later we will assume the existence of a global uniform pullback attractor).

The solution mapping $\Phi : \mathbb{R}^+ \times P \times \mathbb{R}^d \to \mathbb{R}^d$ of (2.1), for which

$$\frac{d}{dt}\Phi(t, p, x_0) = f\left(\theta_t p, \Phi(t, p, x_0)\right), \qquad x_0 \in \mathbb{R}^d, p \in P, t \in \mathbb{R}^+, \tag{2.3}$$

with the *initial condition property*

$$\Phi(0, p, x_0) = x_0, \qquad x_0 \in \mathbb{R}^d, p \in P, \tag{2.4}$$

is continuous in all of its variables and satisfies the *cocycle property*

$$\Phi(s + t, p, x_0) = \Phi(s, \theta_t p, \Phi(t, p, x_0)), \qquad x_0 \in \mathbb{R}^d,\ p \in P,\ s, t \in \mathbb{R}^+. \tag{2.5}$$

That is, Φ is a *cocycle mapping* on $\mathbb{R}^d$ with respect to the autonomous dynamical system $\{\theta_t\}_{t\in\mathbb{R}}$ on P. In fact, the product mapping $\Pi = (\Phi, \theta)$ then forms an autonomous semi–dynamical system, called a skew product flow, on the product space $\mathbb{R}^d \times P$.

Note that the t variable in Φ is now the time that has elapsed since starting rather than absolute time. Although solutions of initial value problems may also be (at least partially) extendable backwards in time, interest in this paper is on what happens forwards in time since starting, as is typical in investigations of systems with some kind of dissipative behaviour.

3. **Pullback attractors.** Attractors concern the asymptotic behaviour of a dynamical system in the neighbourhood of an invariant set. However, the well known definitions of attractors and invariant sets used for autonomous systems are too restrictive in the nonautonomous context. Instead, it is more useful to say that a family $\widehat{A} = \{A_p;\, p \in P\}$ of nonempty compact subsets of $\mathbb{R}^d$ is *invariant under* Φ, or Φ*–invariant,* if

$$\Phi(t, p, A_p) = A_{\theta_t p}, \qquad p \in P, t \in \mathbb{R}^+.$$

The natural generalization of convergence then seems to be the *forwards convergence* defined by

$$H^*(\Phi(t, p, x_0), A_{\theta_t p}) \to 0 \quad \text{as} \quad t \to \infty,$$

but this does not ensure convergence to a specific component set A_p for a fixed p. For this one needs to start "progressively earlier" at $\theta_{-t}p$ in order to "finish" at p, which leads to the concept of *pullback convergence* defined by

$$H^*(\Phi(t, \theta_{-t}p, x_0), A_p) \to 0 \quad \text{as} \quad t \to \infty.$$

The Φ– invariant family $\widehat{A}$ is then called a *pullback attractor* in the case of pullback convergence and a *forwards attractor* in the case of forwards convergence. The concepts of forwards and pullback convergence are usually independent of each other.

A more general definition of a pullback attractor [3, 10, 13] that encompasses local attraction as well as parametric dependent regions of pullback attraction and the attraction of compact sets will be used in this paper. A Φ–invariant family of nonempty compact subsets $\widehat{A} = \{A_p;\ p \in P\}$ will be called a *pullback attractor with respect to a basin of attraction system* $\mathcal{D}_{att}$ if it satisfies the pullback attraction property

$$\lim_{t\to\infty} H^*\left(\Phi(t, \theta_{-t}p, D_{\theta_{-t}p}), A_p\right) = 0 \tag{3.6}$$

for all $p \in P$ and all $\widehat{D} = \{D_p;\ p \in P\}$ belonging to a *basin of attraction system* $\mathcal{D}_{att}$, i.e., a collection of families of nonempty sets $\widehat{D} = \{D_p;\ p \in P\}$ where D_p is compact in $\mathbb{R}^d$ for each $p \in P$ with the property that $\widehat{D}^{(1)} = \left\{D_p^{(1)}\ ;\ p \in P\right\} \in \mathcal{D}_{att}$ if $\widehat{D}^{(2)} = \left\{D_p^{(2)}\ ;\ p \in P\right\} \in \mathcal{D}_{att}$ and $D_p^{(1)} \subseteq D_p^{(2)}$ for all $p \in P$. Obviously, $\widehat{A} \in \mathcal{D}_{att}$. In fact, $A_p \subset \mathrm{int}\mathcal{D}_{att}(p)$, where $\mathcal{D}_{att}(p) := \bigcup_{\widehat{D}=\{D_p;\ p\in P\}\in\mathcal{D}_{att}} D_p$, for each $p \in P$.

Similarly, a Φ–invariant family of nonempty compact subsets $\widehat{A} = \{A_p;\ p \in P\}$ will be called a *forwards attractor with respect to a basin of attraction system* $\mathcal{D}_{att}$ if it satisfies the pullback attraction property

$$\lim_{t\to\infty} H^*\left(\Phi(t, \theta_t p, D_p), A_{\theta_t p}\right) = 0 \tag{3.7}$$

for all $p \in P$ and all $\widehat{D} = \{D_p;\ p \in P\} \in \mathcal{D}_{att}$.

The forwards and pullback attractors will be a called *uniform* if the limits (3.6) and (3.7) are uniform in $p \in P$. If one of these uniform convergences holds, then so does the other and the family $\widehat{A}$ is both forwards and pullback attracting [3].

A family $\widehat{B} = \{B_p\ ;\ p \in P\} \in \mathcal{D}_{att}$ of nonempty compact subsets B_p of $\mathbb{R}^d$ with nonempty interiors is called a *pullback absorbing neighbourhood system* in $\mathcal{D}_{att}$ if it is Φ–positively invariant, i.e. if

$$\Phi(t, p, B_p) \subset B_{\theta_t p} \qquad \text{for all} \quad t \geq 0, \quad p \in P,$$

and if it *pullback absorbs* all families $\widehat{D} = \{D_p;\ p \in P\} \in \mathcal{D}_{att}$ of compact subsets, i.e., for each such family $\widehat{D}$ and $p \in P$ there exists a $T(\widehat{D}, p) \in \mathbb{R}^+$ such that

$$\Phi\left(t, \theta_{-t}p, D_{\theta_{-t}p}\right) \subset \mathrm{int}B_p \qquad \text{for all} \quad t \geq T(\widehat{D}, p).$$

The existence of *pullback absorbing neighbourhood system* $\widehat{B} = \{B_p\ ;\ p \in P\} \in \mathcal{D}_{att}$ ensures the existence of a unique pullback attractor $\widehat{A} = \{A_p\ ;\ p \in P\}$ with respect to the basin of attraction system $\mathcal{D}_{att}$ and the component subset A_p is determined by

$$A_p = \bigcap_{t\geq 0} \Phi(t, \theta_{-t}p, B_{\theta_{-t}p}) \qquad \text{for each} \quad p \in P, \tag{3.8}$$

see [5, 7, 12, 13].

The following lemma from [9] shows that there always exists such a pullback absorbing neighbourhood system for any given cocycle attractor.

Lemma 3.1. *If $\widehat{A}$ is a cocycle attractor with a basin of attraction system $\mathcal{D}_{att}$ for a cocycle dynamical system (Φ, θ) for which $(t, p, x) \mapsto \Phi(t, \theta_{-t}p, x)$ is continuous,* *then there exists a pullback absorbing neighbourhood system $\widehat{B} \subset \mathcal{D}_{att}$ of $\widehat{A}$ with respect to Φ.*

Note that for a pullback attractor $\widehat{A}$ the mapping $t \mapsto A_{\theta_t p}$ is continuous for each fixed $p \in \Phi$ due to the continuity of Φ in t and the Φ-invariance of $\widehat{A}$. However, the mapping $p \mapsto A_p$ is usually only upper semi continuous, see [2, 14].

4. **One–step numerical schemes.** We also consider a variable timestep one-step explicit numerical scheme corresponding to the differential equation (2.1), such as the Euler scheme, which we write as

$$x_{n+1} = x_n + h_n F\left(h_n, \theta_{t_n} p, x_n\right) \tag{4.9}$$

with stepsizes $h_n = t_{n+1} - t_n \in (0, 1]$, where $F : [0,1] \times P \times \mathbb{R}^d \to \mathbb{R}^d$ is the increment function,

Assumption 4.1. *The numerical scheme (4.9) satisfies:*

- $F : [0,1] \times P \times \mathbb{R}^d \to \mathbb{R}^d$ *is continuous in all of its variables;*
- *a local discretization error estimate of the form*

$$|\Phi(h, p, x_0) - x_1| \le h\ \mu_R(h), \qquad |x_0| \le R, \tag{4.10}$$

  for each $R > 0$, where $\mu_R : [0,1] \to \mathbb{R}^+$ is a strictly increasing function with $\mu_R(h) > 0$ for $h > 0$ with $\mu_R(0) = 0$.

Note that for one-step order schemes such as the Euler and Runge–Kutta schemes, $\mu(h)$ is typically of the form $K_R h^\nu$ for some integer $\nu \ge 1$. In our case here this would require the differentiability of F in p as well as x and of θ_t in t, which is too restrictive for certain applications, e.g. random differential equations [2, 8].

To show that the numerical scheme (4.9) with variable time steps generates a discrete time cocycle we first restrict the choice of admissible stepsize sequences. As in [2, 12, 13], for each $\delta > 0$ we define $\mathcal{H}^\delta$ to be the set of all two sided sequences $\mathbf{h} = \{h_n\}_{n \in \mathbb{Z}}$ satisfying

$$\frac{1}{2}\delta \le h_n \le \delta \tag{4.11}$$

for each $n \in \mathbb{Z}$ (the particular factor 1/2 here is chosen just for convenience [13]). The set $\mathcal{H}^\delta$ is compact metric space with the metric

$$\rho_{\mathcal{H}^\delta}\left(\mathbf{h}^{(1)}, \mathbf{h}^{(2)}\right) = \sum_{n=-\infty}^{\infty} 2^{-|n|} \left| h_n^{(1)} - h_n^{(2)} \right|.$$

We then consider the shift operator $\tilde{\theta} : \mathcal{H}^\delta \to \mathcal{H}^\delta$ defined by $\tilde{\theta}\mathbf{h} = \tilde{\theta}\{h_n\}_{n\in\mathbb{Z}} := \{h_{n+1}\}_{n\in\mathbb{Z}}$, which is a homeomorphism on the compact metric space $\left(\mathcal{H}^\delta, \rho_{\mathcal{H}^\delta}\right)$ and its iterates $\{\tilde{\theta}_n\}_{n\in\mathbb{Z}}$ forms a discrete time group on $\mathcal{H}^\delta$. For a given sequence $\mathbf{h} = \{h_n\}_{n\in\mathbb{Z}}$ we set $t_0 = 0$ and define $t_n = t_n(\mathbf{h}) := \sum_{j=0}^{n-1} h_j$ and $t_{-n} = t_{-n}(\mathbf{h}) := -\sum_{j=1}^{n} h_{-j}$ for $n \ge 1$.

Finally, we introduce the parameter space $\mathcal{Q}^\delta = \mathcal{H}^\delta \times P$ for a fixed $\delta > 0$ and define a mapping $\psi : \mathbb{Z}^+ \times \mathcal{Q}^\delta \times \mathbb{R}^d \to \mathbb{R}^d$ by

$$\psi(0, q, x_0) := x_0, \qquad \psi(n, q, x_0) = \psi(n, (\mathbf{h}, p), x_0) := x_n \qquad n \ge 1,$$

where x_n is the nth iterate of the numerical scheme (4.9) with initial value $x_0 \in \mathbb{R}^d$, initial parameter $p \in P$ and stepsize sequence $\mathbf{h} \in \mathcal{H}^\delta$. These mappings are continuous on $\mathcal{Q}^\delta \times \mathbb{R}^d$ and satisfy a cocycle property with respect to a group of continuous mappings $\Theta = \{\Theta_n\}_{n\in\mathbb{Z}}$ on $\mathcal{H}^\delta \times P$ with $\Theta_n : \mathcal{Q}^\delta \to \mathcal{Q}^\delta$ for $n \in \mathbb{Z}$ defined by iteration of the basic mappings

$$\Theta_0 := \mathrm{id}_{\mathcal{Q}^\delta}, \qquad \Theta_1(\mathbf{h}, p) := \left(\tilde{\theta}_1 \mathbf{h}, \theta_{h_0} p\right), \qquad \Theta_{-1}(\mathbf{h}, p) := \left(\tilde{\theta}_{-1} \mathbf{h}, \theta_{-h_{-1}} p\right).$$

For details see [2].

How one actually chooses or should choose the step sizes is a very important issue with actual computations. Difficulties arise when one tries to incorporate the mechanism for choosing the step sizes into the dynamics of the numerical system, see Lamba [16]. Consequently, we believe that it is better to keep this mechanism separate from the formulation of the numerical dynamical system. Whatever mechanism one uses to select an admissible step size sequence, once one has chosen such a sequence the subsequent dynamics are included in our cocycle formalism proposed here.

5. **Discretization of uniform pullback attractors.** Our main result is a nonautonomous generalization of the result in [11] for the autonomous case.

Theorem 5.1. *Let Assumptions 2.1 and 4.1 hold and suppose that the continuous time cocycle system* (Φ, θ) *generated by the differential equation (2.1) has a uniform pullback attractor* $\widehat{A} = \{A_p\}_{p\in P}$ *with respect to the basin of attraction system* $\mathcal{D}_{att}$ *containing all nonempty compact subsets of* $\mathbb{R}^d$.

Then the discrete time cocycle system (Ψ, Θ) *generated by the numerical scheme (4.9) has a uniform pullback attractor* $\widehat{A}^\delta = \{A^\delta_q\}_{q\in\mathcal{Q}^\delta}$*, provided the maximal step-size* δ *is sufficiently small, such that*

$$\lim_{\delta\to 0+} \sup_{p\in P} \sup_{\mathbf{h}\in\mathcal{H}^\delta} H^*\left(A^\delta_{(p,\mathbf{h})}, A_p\right) = 0.$$

The global Lipschitz property in Assumption 2.1 here is not a major limitation, since the essential dynamics that is being approximated occurs in a (possibly very large) compact subset of $\mathbb{R}^d$ on which a local Lipschitz property could be used.

6. **Proof of Theorem 5.1.** The proof of Theorem 5.1 uses a Lyapunov function that characterizes the uniform pullback attractor, in much the same way as a Lyapunov function characterizing the uniformly asymptotically stable set was used in [11] for the autonomous case.

The existence of a Lyapunov function characterizing a uniform pullback attractor is provided by the following theorem. Its proof requires a modification of the construction of the Lyapunov function in [9, 10] from general to uniform pullback attractors and a compact parameter space. The essential difference is that the lower bound in Property 2 is now independent of the parameter p. The proof is given in the Appendix.

Theorem 6.1. *Let* $\widehat{A}$ *be a uniform pullback attractor of a cocycle dynamical system* (Φ, Θ) *on* $\mathbb{R}^d \times P$ *generated by a differential equation (2.1) for which Assumption 2.1 holds.*

Then there exists a function $V : P \times \mathbb{R}^d \to \mathbb{R}^+$ *such that*

Property 1 (upper bound): *For all* $p \in P$ *and* $x_0 \in \mathbb{R}^d$

$$V(p, x_0) \le \mathrm{dist}(x_0, A_p); \tag{6.12}$$

Property 2 (lower bound): *There exists an increasing function* $a : \mathbb{R}^+ \to \mathbb{R}^+$ *with* $a(0) = 0$ *and* $a(r) > 0$ *for* $r > 0$ *such that*

$$a(\mathrm{dist}(x_0, A_p)) \leq V(p, x_0) \tag{6.13}$$

for all $x_0 \in \mathbb{R}^d$ *and all* $p \in P$*;*

Property 3 (Lipschitz condition): *For all* $p \in P$ *and* x_0*,* $y_0 \in \mathbb{R}^d$

$$|V(p, x_0) - V(p, y_0)| \leq \|x_0 - y_0\|; \tag{6.14}$$

Property 4 (pullback convergence): *For all* $p \in P$ *and any bounded subset* D *of* $\mathbb{R}^d$

$$\limsup_{t\to\infty} \sup_{z\in D} V(p, \Phi(t, \theta_{-t}p, z)) = 0. \tag{6.15}$$

In addition,

Property 5 (forwards convergence): *There exists a family* $\widehat{N}$ *of nonempty compact sets* N_p*,* $p \in P$*, which are* Φ*–positively invariant in the sense that* $\Phi(t, p, N_p) \subseteq N_{\theta_t p}$ *for all* $t \geq 0$*,* $p \in P$*, and satisfy* $A_p \subset \mathrm{int} N_p$ *for each* $p \in P$ *such that*

$$V(\theta_t p, \Phi(t, p, x_0)) \leq e^{-t} V(p, x_0) \tag{6.16}$$

for all $x_0 \in N_p$ *and* $t \geq 0$*.*

Note that for a global pullback attractor the Φ–positively invariant family $\widehat{N}$ in the forwards convergence Property 5 and the pullback absorbing family $\widehat{B}$ in Lemma 3.1 can be constructed to be arbitrarily large, i.e. with $B(A_p; R) \subset N_p \subset B_p$ for each $p \in P$ given an arbitrary $R > 0$.

6.1. **A Lyapunov inequality for the discretized dynamics.** Similarly as in [11], the key tool in our proof is provided by the following Lyapunov function inequality in which the function $\mu = \mu_R$ is from the local discretization error estimate (4.10) of the numerical scheme (4.9) with R chosen so large such that $\bigcup_{p\in P} N_p \subset B[0; R]$.

Lemma 6.2.

$$V(\theta_h p, x_1(h, p)) \leq e^{-ch}\, V(p, x_0) + h\, \mu(h) \tag{6.17}$$

where $x_1(h, p)$ *is the first iteration of the numerical scheme (4.9) with any stepsize* $h > 0$ *and initial state* $x_0 \in N_p$ *for parameter* p*.*

Proof: From the Lipschitz property of V with Lipschitz constant $L = 1$ we have

$$V(\theta_h p, x_1(h, p)) \leq V(\theta_h p, \Phi(h, p, x_0)) + \|x_1(h, p) - \Phi(h, p, x_0)\|.$$

Applying the forwards exponential decay in equality (Property 5) of the Lyapunov function V in Theorem 6.1 to the first term on the right side and the local discretization error estimate (4.10) to the second then gives (6.17) for all $h \geq 0$.

6.2. **Construction of an absorbing family for the discretized dynamics.** Fix $\delta > 0$ and define

$$\eta = \eta(\delta) := \frac{2\delta\, \mu(\delta)}{1 - e^{-c\delta/2}}.$$

Lemma 6.3. $\Lambda_p^\delta := \left\{x \in \mathbb{R}^d : V(p, x) \leq \eta(\delta)\right\}$ *is a nonempty compact subset of* $\mathbb{R}^d$ *with*

$$H^*\left(\Lambda_p^\delta, A_p\right) \leq \alpha^{-1}(\eta). \tag{6.18}$$

Proof: Λ_p^δ is nonempty because it contains $A_p = \left\{x \in \mathbb{R}^d : V(p,x) = 0\right\}$ and is closed in view of the continuity of V in x. Its boundedness and the inequality (6.18) follow from the inequality

$$\alpha(\mathrm{dist}(x, A_p)) \leq V(p,x)$$

in the second property of V in Theorem 6.1 and the definition of Λ_p^δ, giving

$$\mathrm{dist}(x, A_p) \leq \alpha^{-1}\left(V(p,x)\right) \leq \alpha^{-1}\left(\eta\right)$$

for all $x \in \Lambda_p(\eta)$.

The family $\widehat{\Lambda}^\delta = \left\{\Lambda_p^\delta;\, p \in P\right\}$ is positively invariant with respect to the discrete time cocycle mapping ψ formed by iterating the numerical scheme, i.e.

Lemma 6.4. $\psi\left(1, (p, \mathbf{h}), \Lambda_p^\delta\right) \subseteq \Lambda_{\theta_h p}^\delta$.

Proof: It suffices to consider any single iterate $x_1 = x_1(h,p)$ for an arbitrary $x_0 \in \Lambda_p^\delta$, so $V(p, x_0) \leq \eta = \eta(\delta)$. Then by the key inequality (6.17) with $h \in [\delta/2, \delta]$ and the definition of $\eta(\delta)$ we have

$$\begin{aligned} V\left(\theta_h p, x_1(h,p)\right) &\leq e^{-ch}\, V\left(p, x_0\right) + h\, \mu(h) \\ &\leq e^{-c\delta/2}\, V\left(p, x_0\right) + \delta\, \mu(\delta) \\ &\leq e^{-c\delta/2}\, \eta(\delta) + \frac{1}{2}\left(1 - e^{-c\delta/2}\right)\eta(\delta) \\ &= \frac{1}{2}\left(1 + e^{-c\delta/2}\right)\eta(\delta) \leq \eta(\delta), \end{aligned}$$

so $x_1(h,p) \in \Lambda_{\theta_h p}^\delta$.

The family $\widehat{\Lambda}^\delta = \left\{\Lambda_p^\delta;\, p \in P\right\}$ is in fact forwards absorbing under ψ uniformly in p provided δ is chosen small enough.

Lemma 6.5. *There is a $\delta^* > 0$ such that for each family $\widehat{D} = \{D_p;\, p \in P\}$ of compact subsets with $D_p \subset N_p$ and $\delta \in (0, \delta^*)$ there exists an integer $N_{\widehat{D},\delta} \geq 1$ for which $\psi\left(n, (p, \mathbf{h}), D_p\right) \subseteq \Lambda_{\theta_{t_n} p}^\delta$ for all $n \geq N_{\widehat{D},\delta}$ uniformly in $p \in P$.*

Proof: As in Lemma 3.4 of [11] there exists a $\gamma > 0$ such that

$$1 + e^{-c\gamma/2} = 2e^{-c\gamma/4} \qquad \text{and} \qquad 1 + e^{-c\delta/2} < 2e^{-c\delta/4}$$

for all $0 < \delta < \delta^* := \gamma$.

Consider $x_1(h_0, p) = \psi\left(1, (p, \mathbf{h}), x_0\right)$ with $x_0 \notin \Lambda_p^\delta$. Then by the key inequality (6.17) and the definition of $\eta = \eta(\delta)$ we have

$$\begin{aligned} V\left(\theta_h p, x_1(h_0,p)\right) &\leq e^{-c\delta/2}\, V\left(p, x_0\right) + \delta\, \mu(\delta) \\ &= e^{-c\delta/2}\, V\left(p, x_0\right) + \frac{1}{2}\left(1 - e^{-c\delta/2}\right)\eta(\delta) \\ &< \frac{1}{2}\left(1 + e^{-c\delta/2}\right) V\left(p, x_0\right) \\ &< e^{-c\delta/4}\, V\left(p, x_0\right) \end{aligned}$$

since $V(p, x_0) > \eta(\delta)$ and $0 < \delta < \delta^*$. Repeating this argument,

$$V\left(\theta_{t_n} p, x_n\right) < e^{-cn\delta/4}\, V\left(p, x_0\right),$$

where $x_n = \psi\left(n, (p, \mathbf{h}), x_0\right)$, as long as $x_0 \notin \Lambda_p^\delta$, $x_1(h_0, p) \notin \Lambda_{\theta_{t_1} p}^\delta$, $\ldots$, $x_{n-1} = \psi\left(n-1, (p, \mathbf{h}), x_0\right) \notin \Lambda_{\theta_{t_{n-1}} p}^\delta$. Now

$$V\left(p, x_0\right) \leq \mathrm{dist}(x_0, A_p) \leq H^*(D_p, A_p) < \infty$$

for all $x_0 \in D$, so

$$V\left(\theta_{t_n} p, x_n\right) < e^{-cn\delta/4}\, H^*(D_p, A_p)$$

as long as $x_0 \notin \Lambda_p^\delta$, $x_1(h_0, p) \notin \Lambda_{\theta_{t_1} p}^\delta$, ..., $x_{n-1} = \psi\left(n-1, (p, \mathbf{h}), x_0\right) \notin \Lambda_{\theta_{t_{n-1}} p}^\delta$. Define $N_{\widehat{D},\delta}$ to be the smallest integer n for which

$$e^{-cn\delta/4}\, H^*(D_p, A_p) \leq \eta(\delta) < e^{-c(n-1)\delta/4}\, H^*(D_p, A_p).$$

Thus for each $x_0 \in D_p$ there exists an integer $n \geq N_{\widehat{D},\delta}$, possibly 0, such that $\psi(n, (p, \mathbf{h}), x_0) \in \Lambda_{\theta_{t_n} p}^\delta$. By the positive invariance of the Λ_p^δ proved in Lemma 6.4 it follows that the jth iterate of ψ then remains in $\Lambda_{\theta_{t_j} p}^\delta$ for $j \geq n$, so the proof of Lemma 6.5 is complete.

6.3. **Existence and convergence of the discretized pullback attractor.** We apply known existence results to the numerical cocycle mapping ψ and the absorbing family $\widehat{\Lambda}^\delta = \left\{\Lambda_p^\delta;\, p \in P\right\}$ defined in the previous subsection to obtain the existence of a uniform pullback attractor $\widehat{A}^\delta = \left\{A_q^\delta;\, q = (p, \mathbf{h}) \in \mathcal{Q}^\delta\right\}$ for ψ, namely with

$$A_q^\delta = \bigcap_{n \geq 0} \psi\left(\left|t_{-n}\right|, \Theta_{-n} q, \Lambda_{\theta_{t_{-n}} p}^\delta\right).$$

(Note $t_{-n} < 0$ here).

Let $\widehat{A} = \{A_p;\, p \in P\}$ be the pullback attractor for Φ. Then $A_p \subseteq \Lambda_{(p,\mathbf{h})}(\eta)$ for all $p \in P$ and $\mathbf{h} \in \mathcal{H}^\delta$, so

$$H^*\left(A_{(p,\mathbf{h})}^\delta, A_p\right) \leq H^*\left(\Lambda_p^\delta, A_p\right) \leq \alpha^{-1}(\eta(\delta)) \to 0 \quad \text{as} \quad \delta \to 0$$

uniformly in $p \in P$ and $\mathbf{h} \in \mathcal{H}^\delta$.

This completes the proof of Theorem 5.1.

7. Appendix: The Lyapunov function for a uniform pullback attractor.

The construction of the Lyapunov function for a pullback attractor given in [10] will be modified here for unform pullback attractors and a compact parameter space P. The essential change is to show that there exists a lower bound function $a(r)$ in Property 2 that is independent of the parameter p.

As in [10] define $V(p, x_0)$ for all $p \in P$ and $x_0 \in \mathbb{R}^d$ by

$$V(p, x_0) := \sup_{t \geq 0} e^{-T_{p,t}} \operatorname{dist}\left(x_0, \Phi(t, \theta_{-t} p, B_{\theta_{-t} p})\right),$$

where $\widehat{B} = \{B_p\,;\, p \in P\} \in \mathbb{R}^d$ is a pullback absorbing neighbourhood system for the Φ–pullback attractor $\widehat{A}$ and

$$T_{p,t} = t + \int_0^t L(\theta_{-s} p)\, ds \qquad \text{with} \quad T_{p,0} = 0.$$

Note that $T_{p,t}$ satisfies $t \leq T_{p,t} \leq (1 + L^*)t$ for all $t \geq 0$ and $p \in P$, where $L^* := \max_{p \in P} L(p)$ exists and is finite by the continuity assumptions and the compactness of P.

7.1. **Proof of property 2.** By Property 1, $V(p,x_0) = 0$ for $x_0 \in A_p$. Assume instead that $x_0 \in \mathbb{R}^d \setminus A_p$. Now the supremum in

$$V(p,x_0) = \sup_{t\geq 0} e^{-T_{p,t}} \mathrm{dist}\left(x_0, \Phi(t,\theta_{-t}p, B_{\theta_{-t}p})\right)$$

involves the product of an exponentially decreasing quantity bounded below by zero and a bounded increasing function, since the $\Phi(t,\theta_{-t}p, B_{\theta_{-t}p})$ are a nested family of compact sets decreasing to A_p with increasing t. Hence there exists a $T^* = T^*(p,x_0) \in \mathbb{R}^+$ such that

$$\frac{1}{2}\mathrm{dist}(x_0, A_p) \leq \mathrm{dist}\left(x_0, \Phi(t,\theta_{-t}p, B_{\theta_{-t}p})\right)$$

for all $t \geq T^*$, but not for $t < T^*$. Thus, from above,

$$V(p,x_0) \geq e^{-T_{p,T^*}} \mathrm{dist}\left(x_0, \Phi(T^*,\theta_{-T^*}p, B_{\theta_{-T^*}p})\right) \geq \frac{1}{2} e^{-T_{p,T^*}} \mathrm{dist}\left(x_0, A_p\right).$$

The lower bound $a(p,r)$ in [10] was defined by

$$a(p,r) := \frac{1}{2} r \; e^{-T_{p,\widehat{T}(p,r)}}$$

where

$$\widehat{T}(p,r) := \sup\{T^*(p,x_0) : x_0 \in \mathbb{R}^d, \quad \mathrm{dist}\left(x_0, A_p\right) = r\},$$

which is finite.

Now by uniform pullback convergence there exists a finite $T(r/2)$, which is independent of $p \in P$ and can be chosen to be nonincreasing in r, such that

$$H^*(\Phi(t,\theta_{-t}p, B_{\theta_{-t}p}), A_p) < \frac{1}{2} r$$

for all $t \geq T(r/2)$ and all $p \in P$. Hence $r \leq \mathrm{dist}(x_0, \Phi(t,\theta_{-t}p, B_{\theta_{-t}p})) + \frac{1}{2} r$ for $\mathrm{dist}(x_0, A_p) = r$ and $t \geq T(r/2)$, i.e. $\frac{1}{2} r \leq \mathrm{dist}(x_0, \Phi(t,\theta_{-t}p, B_{\theta_{-t}p}))$. Thus, $\widehat{T}(p,r) \leq T(r/2) < \infty$ and so $T_{p,\widehat{T}(p,r)} \leq T_{p,T(r/2)} \leq (1+L^*)T(r/2) < \infty$ for all $p \in P$. Finally, define

$$a(r) := \frac{1}{2} r \; e^{-(1+L^*)T(r/2)} \leq a(p,r),$$

which satisfies the stated properties.

Remark 7.1. *As already noted, a uniform pullback attractor is also a uniform forwards attractor, and vice versa. When the pullback attraction is not uniform, the lower bound function a in Property 2 of Theorem 6.1 depends explicitly on p as well as x and there is no guarantee that* $\inf_{p\in P} a(p,x) > 0$*, which is needed to deduce forwards attraction from Property 5 of the Lyapunov function. Essentially, the lower bound* $a(\theta_t p, \mathrm{dist}(\Phi(t,p,x_0), A_{\theta_t p}))$ *then may converge to zero as* $t \to \infty$ *without* $\mathrm{dist}(\Phi(t,p,x_0), A_{\theta_t p})$ *converging to zero as* $t \to \infty$*. See the example in* [10]*. This situation cannot occur in the uniform case.*

REFERENCES

[1] L. Arnold, *Random Dynamical Systems.* Springer–Verlag, Heidelberg, 1998.

[2] D.N. Cheban, P.E. Kloeden and B. Schmalfuß, Pullback attractors in dissipative nonautonomous differential equations under discretization. *J. Dyn. Systems & Diff. Eqns.* **13** (2001),185-213.

[3] D.N. Cheban, P.E. Kloeden and B. Schmalfuß, The Relationship between Pullback, Forwards and Global Attractors of Nonautonomous Dynamical Systems. *Nonlinear Dynamics & Systems Theory.* **2** (2) (2002), 9–28.

[4] V. Chepyzhov and M.I. Vishik, Attractors of non-autonomous dynamical systems and their dimension, *J. Math. Pures Appl.*, **73** (1994), 279–333.

[5] H. Crauel and F. Flandoli, Attractors for random dynamical systems, *Probab. Theory Relat. Fields,* **100** (1994), 365–393.

[6] M.A. Efendiev and S.V. Zelik, Attractors of the Reaction-Diffusion Systems with Rapidly Oscillating Coefficients and Their Homogenization, *Mathematische Nachrichten* (2003) (to appear)

[7] F. Flandoli and B. Schmalfuß, Random attractors for the 3d stochastic Navier Stokes equation with multiplicative white noise, *Stochastics and Stochastics Reports,* **59** (1996), 21–45.

[8] L. Grüne und P.E. Kloeden, Pathwise approximation of random ordinary differential equations. *BIT* **41** (2001), 710–721.

[9] P.E. Kloeden, Lyapunov functions for cocycle attractors in nonautonomous difference equation, *Izvestiya Akad Nauk Republ. Moldavia Matematika* **26** (1998), 32–42.

[10] P.E. Kloeden, Lyapunov functions for cocycle attractors in nonautonomous differential equations, *Electronic J. Differential Eqns.*Conference **05** (2000), 91–102.

[11] P.E. Kloeden and J. Lorenz, Stable attracting sets in dynamical systems and in their one-step discretizations, *SIAM J. Numer. Anal.* **23** (1986), 986-995.

[12] P.E. Kloeden and B. Schmalfuß, Lyapunov functions and attractors under variable time–step discretization, *Discrete & Conts. Dynamical Systems* **2** (1996), 163–172.

[13] P.E. Kloeden and B. Schmalfuß, Nonautonomous systems, cocycle attractors and variable time–step discretization, *Numer. Algorithms* **14** (1997), 141–152.

[14] P.E. Kloeden and D.J. Stonier, Cocycle attractors in nonautonomously perturbed differential equations. *Dynamics of Discrete, Continuous and Impulsive Systems.* **4** (1998), 211–226.

[15] M.A. Krasnosel'skii, *The Operator of Translation along Trajectories of Differential Equations*, Translations of Mathematical Monographs, Volume 19. American Math. Soc., Providence, R.I., 1968.

[16] H. Lamba, Dynamical systems and adaptive timestepping in ODE solvers. *BIT* **40** (2000), 314–335

[17] G.R. Sell, *Lectures on Topological Dynamics and Differential Equations.* Van Nostrand–Reinbold, London, 1971.

[18] Wenxian Shen and Yingfei Yi, *Almost Automorphic and Almost Periodic Dynamics in Skew-Product Semiflows,* AMS Memoirs, vol. 647, 1998.

[19] A.M. Stuart and A.R. Humphries, *Numerical Analysis and Dynamical Systems,* Cambridge University Press, Cambridge 1996.

[20] M.I. Vishik, *Asymptotic Behaviour of Solutions of Evolutionary Equations,* Cambridge Univ. Press, Cambridge, 1992.

[21] T. Yoshizawa, *Stability Theory by Lyapunov's Second Method.* Mathematical Soc. Japan, Tokyo, 1966

Received October 2001; revised February 2003.

E-mail address: kloeden@math.uni-frankfurt.de
E-mail address: kozyakin@iitp.ru