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Invariant Finitely Additive Measures for General Markov Chains and the Doeblin Condition

Alexander Zhdanok

Institute for Information Transmission Problems (A.A. Kharkevich Institute) of Russian Academy of Sciences, 19/1 Bolshoy Karetny Per., 127051 Moscow, Russia; zhdanok@iitp.ru

Abstract: In this paper, we consider general Markov chains with discrete time in an arbitrary measurable (phase) space. Markov chains are given by a classical transition function that generates a pair of conjugate linear Markov operators in a Banach space of measurable bounded functions and in a Banach space of bounded finitely additive measures. We study sequences of Cesaro means of powers of Markov operators on the set of finitely additive probability measures. It is proved that the set of all limit measures (points) of such sequences in the weak topology generated by the preconjugate space is non-empty, weakly compact, and all of them are invariant for this operator. We also show that the well-known Doeblin condition (D) for the ergodicity of a Markov chain is equivalent to condition $(*)$: all invariant finitely additive measures of the Markov chain are countably additive, i.e., there are no invariant purely finitely additive measures. We give all the proofs for the most general case.

Keywords: general Markov chains; Markov operators; finitely additive measures; invariant measures; Doeblin condition; ergodicity; probability theory

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1. Introduction

We study general Markov chains understood as random Markov processes with discrete time in an arbitrary phase space and homogeneous in time. Markov chains are given by a transition probability (function) that is countably additive by its second argument, i.e., we consider only "classical" Markov chains. Transition probabilities generate integral Markov operators acting in spaces of bounded measurable functions, bounded countably additive, and bounded finitely additive measures.

In our papers [1,2], we proved several theorems about the ergodicity of Markov chains in one sense or another depending on the properties of subspaces of invariant finitely additive measures. In the present paper, we continue to investigate these problems within the framework of the operator approach proposed by Kryloff and Bogoliouboff [3,4] in 1937, developed by Yosida and Kakutani [5] in 1941, that is used in many works on this subject (see, for example, Revuez's book [6]).

In Sections 2 and 3, we introduce the basic information and constructions used below, which are not fully represented in the known and available literature. The content and form of presentation of this preliminary information are approximately the same as in our previous articles on this topic, i.e., we repeat them for the convenience of readers.

The main results of this paper are contained in Sections 4–6. We present these results here too.

In Theorem 5, we consider sequences of Cesàro means of powers of a Markov operator in the space of finitely additive measures. It is proved that the set of all limit measures of such sequences in the $*$-weak topology is non-empty, $*$-weakly compact, and all of them are invariant for this Markov operator. Here, the $*$-weak topology τ_B is the topology in the space of finitely additive measures generated by the preconjugate space of measurable functions.

The main statement of the whole article is proved in Theorem 7: the Doeblin condition (D) of ergodicity for general Markov chains is equivalent to the condition $(*)$ which states that all invariant finitely additive measures of a Markov chain are countably additive. Condition $(*)$ (Corollary 1), in turn, is equivalent to the condition $(\tilde{*})$ which states that the Markov chain does not have invariant purely finitely additive measures.

In Section 6, Theorem 9, on the finite dimension of the set of invariant finitely additive measures for Markov chains satisfying the condition $(*)$ is given. Thereafter, in Theorems 10 and 11, we also give an inversion of Theorem 9 for the case when the invariant finitely additive measure is unique.

We present some theorems and definitions from other publications used in Sections 4–6 in exact and complete formulations (and with their new numbering) in order to make the text of this article more autonomous.

In Section 7, we discuss the results of Lin [7] and Horowitz [8], which are close to our theorems in the present paper. We also comment on recently published papers [9,10], in which the authors use finitely additive measures for classical Markov chains.

2. Finitely Additive Measures

2.1. Definitions, Designations and Some Information

Here are some of the basic definitions and concepts used by us, as well as their symbolism, focusing on [11,12].

Let us begin with notation and definitions.

Let X be an arbitrary infinite set and Σ be some sigma-algebra of its subsets. The pair (X, Σ) is called measurable space and each set $A \in \Sigma$ is called measurable. We shall always (by default) assume that the sigma-algebra Σ contains all one-point sets $\{x\} \in \Sigma, x \in X$. Everywhere below, $R = R^1$ is the set of real numbers (number line).

We denote by $B(X, \Sigma)$ the Banach space of bounded Σ-measurable functions $f\colon X \to R$ with sup-norm $\|f\| = sup|f(x)|$.

Any function $\varphi\colon \Sigma \to R$ will be called a set function on Σ. The set function $\varphi(E), E \in \Sigma$, can take both positive and negative values. In this paper, we use only bounded set functions, i.e., such that $sup|\varphi(E)| < \infty$, where the supremum is taken over all sets $E \in \Sigma$.

Definition 1. *Let (X, Σ) be an arbitrary measurable space. A bounded set function $\mu\colon \Sigma \to R$ is called finitely additive measure if $\mu(\varnothing) = 0$ and $\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2)$ for any sets $E_1, E_2 \in \Sigma$ such that $E_1 \cap E_2 = \varnothing$.*

Definition 2. *Let (X, Σ) be an arbitrary measurable space. A bounded set function $\mu\colon \Sigma \to R$ is called countably additive measure if it is a finitely additive measure and the following condition is satisfied: if $E_1, E_2, \ldots \in \Sigma$, $E_i \cap E_j = \varnothing$ for $i \neq j$ then*

$$\mu(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mu(E_n).$$

Definition 3 (see [12]). *A finitely additive non-negative measure $\mu\colon \Sigma \to R$ is called purely finitely additive (pure charge, pure mean) if any countably additive measure λ satisfying the condition $0 \le \lambda \le \mu$ is identically zero. The alternating measure μ is called is purely finitely additive if both non-negative measures μ^+ and μ^- of its Jordan decomposition $\mu = \mu^+ - \mu^-$ are purely finitely additive.*

If the measure μ is purely finitely additive, then it is equal to zero on every one-point set: $\mu(\{x\}) = 0, \forall x \in X$ (see [13], Lemma 1). The converse, generally speaking, is not true (for example, for the Lebesgue measure on the segment $[0, 1]$).

Theorem 1 (Yosida-Hewitt decomposition, see [12])**.** *Any finitely additive measure μ uniquely decomposes into the sum $\mu = \mu_1 + \mu_2$, where μ_1 is a countably additive measure and μ_2 is a purely finitely additive measure.*

The first versions of such a decomposition of finitely additive measures on some topological spaces were constructed in the articles of Alexandroff ([14], (Chapter III, Section 13, Theorems 1–5)).

In this article, we also consider Banach spaces of bounded measures $\mu\colon \Sigma \to R$ with a norm equal to the total variation in the measure μ:

$ba(X, \Sigma)$ is the space of finitely additive measures, and

$ca(X, \Sigma)$ is the space of countably additive measures.

If $\mu \geq 0$, then the norm is $\|\mu\| = \mu(X)$. These measure spaces are studied in detail in [11].

We can formally consider a measure that is identically equal to zero to be both countably additive and purely finitely additive.

The purely finitely additive measures also form a Banach space $pfa(X, \Sigma)$ with the same norm, and $ba(X, \Sigma) = ca(X, \Sigma) \oplus pfa(X, \Sigma)$.

We denote the sets of measures:

$S_{ba} = \{\mu \in ba(X, \Sigma)\colon \mu \geq 0, ||\mu|| = 1\}$,
$S_{ca} = \{\mu \in ca(X, \Sigma)\colon \mu \geq 0, ||\mu|| = 1\}$,
$S_{pfa} = \{\mu \in pfa(X, \Sigma)\colon \mu \geq 0, ||\mu|| = 1\}$.

All measures from these sets will be called probabilistic.

2.2. Order Properties of Measure Spaces

We present here some order properties of measure spaces that we will use in the next sections.

The measure space $ba(X, \Sigma)$ is semi-ordered with respect to the natural order relation: for $\mu_1, \mu_2 \in ba(X, \Sigma)$ we write $\mu_1 \leq \mu_2$ if $\mu_1(E) \leq \mu_2(E)$ for all $E \in \Sigma$.

In one paper ([12], Theorem 1.11), it was proved that the space of finitely additive measures $ba(X, \Sigma)$ is a vector lattice and formulas were given for finding $(\mu_1 \wedge \mu_2) = inf\{\mu_1, \mu_2\}$ and $(\mu_1 \vee \mu_2) = sup\{\mu_1, \mu_2\}$, where $\mu_1, \mu_2 \in ba(X, \Sigma)$.

In one paper ([12], Theorem 1.14 and Theorem 1.17), it was proved that the subspaces $ca(X, \Sigma)$ and $pfa(X, \Sigma)$ are also vector lattices.

Definition 4. *Two non-negative elements x_1 and x_2 from the vector lattice X are called disjoint if $x_1 \wedge x_2 = 0$, and denoted by $x_1 d x_2$. We also use this term for the corresponding pairs of measures $\mu_1, \mu_2 \in ba(X, \Sigma)$.*

Definition 5 (see, for example, ([11], Chapter III, Item 4.12))**.** *Nonnegative measures μ_1 and μ_2 are called singular if there exist sets $D_1, D_2 \in \Sigma$ such that $\mu_1(D_1) = \mu_1(X), \mu_2(D_2) = \mu_2(X)$ and $D_1 \cap D_2 = \emptyset$, and denoted by $\mu_1 \perp \mu_2$.*

It is easy to check that if the measures μ_1 and μ_2 are singular then they are also disjoint, i.e., $\mu_1 \perp \mu_2 \Rightarrow \mu_1 d \mu_2$. However, the converse, generally speaking, is not true (in [15] there are corresponding examples).

It follows from ([12], Theorem 1.21) that the countably additive measures μ_1 and μ_2 are disjunctive if and only if they are singular. In one paper ([12], Theorem 1.16), it is also stated that every purely finite additive measure is disjoint with any countably additive measure.

Let us give examples of families of purely finitely additive measures.

Example 1. *Let $X = [0, 1] \subset R$ and $\Sigma = \mathfrak{B}$, where $\mathfrak{B}$ is the Borel sigma-algebra. There exists a finitely additive measure $\mu\colon \mathfrak{B} \to R$, $\mu \in S_{ba}$ such that, for any $\varepsilon > 0$, the following holds: $\mu((0, \varepsilon)) = 1, \mu([\varepsilon, 1]) = 0, \mu(\{0\}) = 0$.*

It is easy to check that such a measure is purely finitely additive but it is not unique.

It is known (see [12]) that the cardinality of a family of such measures located "near zero (on the right)" is not less than $2^{2^{\aleph_0}} = 2^c$ *(hypercontinuum).*

Example 2. *Let* $X = R$ *and* $\Sigma = \mathfrak{B}$*. There exists a finitely additive measure* $\mu\colon \mathfrak{B} \to R$, $\mu \in S_{ba}$ *such that for any* $x \in R$ *the following holds:* $\mu((x, \infty)) = 1, \mu((-\infty, x)) = 0, \mu(\{x\}) = 0$.

This measure is also purely finitely additive. And there are also a lot of such measures.

A detailed exposition of the foundations of the general theory of finitely additive measures is contained in the monograph K.P.S.B. Rao and M.B. Rao [15], in which such measures are called charges.

3. Markov Operators and Invariant Measures

Markov chains on a (phase) measurable space (X, Σ) are given by their transition function (probability) $p(x, E)$, for $x \in X$ and $E \in \Sigma$ under ordinary conditions:

1. $0 \leq p(x, E) \leq 1, p(x, X) = 1, \forall x \in X, \forall E \in \Sigma$;
2. $p(\cdot, E) \in B(X, \Sigma), \forall E \in \Sigma$;
3. $p(x, \cdot) \in ca(X, \Sigma), \forall x \in X$.

The numerical value of the function $p(x, E)$ is the probability that the system will move from the point $x \in X$ to the set $E \in \Sigma$ in one step (per unit of time).

We emphasize that transition function is a countably additive measure with respect to the second argument, i.e., we consider classical Markov chains.

The transition function generates two Markov linear bounded positive integral operators:
$T : B(X, \Sigma) \to B(X, \Sigma), (Tf)(x) = Tf(x) = \int_X f(y)p(x, dy)$,
$\forall f \in B(X, \Sigma), \forall x \in X$;
$A : ca(X, \Sigma) \to ca(X, \Sigma), (A\mu)(E) = A\mu(E) = \int_X p(x, E)\mu(dx)$,
$\forall \mu \in ca(X, \Sigma), \forall E \in \Sigma$.

Let $\mu_0 \in S_{ca}$ be the initial measure. Then the iterative sequence of countably additive probability measures $\mu_n = A\mu_{n-1} \in S_{ca}, n \in N$, is usually identified with the Markov chain. We will call $\{\mu_n\}$ a Markov sequence of measures.

Topologically conjugate to the space $B(X, \Sigma)$ is the (isomorphic) space of finitely additive measures: $B^*(X, \Sigma) = ba(X, \Sigma)$ (see, for example, [11]). In this case, the operator $T^*\colon ba(X, \Sigma) \to ba(X, \Sigma)$ serves as topologically conjugate to the operator T.

The operator T^* is the only bounded continuation of the operator A to the entire space $ba(X, \Sigma)$ while preserving its analytic form, i.e.,

$$T^*\mu(E) = \int_X p(x, E)\mu(dx), \forall \mu \in ba(X, \Sigma), \forall E \in \Sigma.$$

The operator T^* has its own invariant subspace $ca(X, \Sigma)$, i.e., $T^*[ca(X, \Sigma)] \subset ca(X, \Sigma)$, on which it matches the original A operator. The construction of the Markov operators T and T^* is now functionally closed. Where it does not cause misunderstandings, we will continue to denote the operator T^* as A.

In such a setting, it is natural to admit to consideration the Markov sequences of probabilistic finitely additive measures: $\mu_0 \in S_{ba}, \mu_n = T^*\mu_{n-1} \in S_{ba}, n \in N$, keeping the countable additivity of the transition function $p(x, \cdot)$ with respect to the second argument.

Such a Markov chain can have cycles consisting of finitely additive measures. The properties of such cycles Markov chain are considered in detail in our paper [16].

It is permissible and cardinally to change the formulation of the problem: to allow the transition function $p(x, \cdot)$ itself to be only a finitely additive measure with respect to the second argument. Such Markov chains are also studied (see ([1], (Chapter II, Section 5)) and [13,17]), and they are called "finitely additive Markov chains". In this paper, we do not consider such Markov chains.

Thus, in our case, the following terminology is appropriate: we study countably additive Markov chains with operators defined on the space of finitely additive measures.

Definition 6. *If $A\mu = \mu$ for some measure $\mu \in S_{ba}$, then we will call such a measure invariant for the operator A (and for the Markov chain). An invariant countably additive measure is often called the stationary distribution of the Markov chain.*

We denote the sets of invariant probability measures of the Markov chain for the operator A as follows:
$\Delta_{ba} = \{\mu \in S_{ba} : \mu = A\mu\}$,
$\Delta_{ca} = \{\mu \in S_{ca} : \mu = A\mu\}$, and
$\Delta_{pfa} = \{\mu \in S_{pfa} : \mu = A\mu\}$.

Let M_{ba} be the linear subspace of invariant measures of the Markov chain in the space $ba(X, \Sigma)$. Obviously, M_{ba} is generated by the set Δ_{ba}: $M_{ba} = Sp\Delta_{ba}$. We will also use the notation M_{ca} and M_{pfa} with a similar meaning.

The linear dimension of a set Δ_{ba} is the algebraic dimension of the linear space M_{ba} generated by it, and we will denote it $dim\Delta_{ba}$, $dim\Delta_{ba} = dimM_{ba}$. Similarly, we will talk about the dimension of the sets Δ_{ca} and Δ_{pfa}.

The classical countably additive Markov chain may or may not have invariant probability countably additive measures, i.e., possibly $\Delta_{ca} = \varnothing$ (for example, for a symmetric walk on Z).

Šidak was one of the first to extend the Markov operator A to the space of finitely additive measures in the framework of the operator approach and proved the following two important theorems in [18] (1962).

Theorem 2 (Šidak ([18], Theorem 2.2))**.** *Any countably additive Markov chain on an arbitrary measurable space (X, Σ) has at least one invariant finitely additive probability measure, i.e., always $\Delta_{ba} \neq \varnothing$.*

This result was then briefly proved in our paper [1] as a simple corollary of the Krein-Rutman theorem ([19], Theorem 3.1).

Theorem 3 (Šidak ([18], (Theorem 2.5)))**.** *If a finitely additive measure μ for an arbitrary Markov chain is invariant $A\mu = \mu$, and $\mu = \mu_1 + \mu_2$ is its decomposition into countably additive and purely finitely additive components, then each of them is also invariant: $A\mu_1 = \mu_1$, $A\mu_2 = \mu_2$.*

Therefore, in many cases, it is sufficient to study invariant measures from Δ_{ca} and from Δ_{pfa} separately.

Remark 1. *In [20], (1966), Foguel considered the same operator construction of a general Markov chain as Šidak in [18] (1962). However, in the main part of [20] it is assumed that X is a locally compact space. In one study[20], Foguel also studied the properties of invariant finitely additive measures of Markov chains. In this paper, we use the same construction of general Markov chains and their operators as discussed in the referenced papers by Šidak and Foguel. However, here we are solving our own problems not covered in the works of Šidak [18] and Foguel [20].*

We note that in several other papers Foguel (see, for example, [21]) develops an operator approach for studying Feller Markov chains on locally compact topological spaces X with a Baire σ-algebra Σ.

Remark 2. *Hernández-Lerma and Lasserre proved in ([22], Theorem 6.3.1) (2003) that for a Markov chain defined on a separable metrizable phase space (X, Σ), under certain assumptions, there exists an invariant finite additive measure. It is also shown that if a finitely additive measure is invariant, then both its countably additive and purely finitely additive components are invariant. In this article, we consider general Markov chains and do not separately single out the particular case of the topological phase space.*

4. Doeblin Condition and Invariant Measures

4.1. Doeblin Condition, Its Modification and Condition $(*)$

In 1937, Doeblin published a large work in two parts [23,24]. In one paper [24], [Chapter 2], for general Markov chains, condition (D) is formulated under which the Markov chain has the maximum set of ergodic properties.

In subsequent works, different authors constructed and used different versions of the *Doeblin condition* (D). We use the version suggested by Doob ([25], Chapter 5):

$$(D)\begin{cases} \textit{There is a bounded measure } \varphi \in ca(X,\Sigma),\ \varphi \geq 0, \varepsilon > 0 \textit{ and } k \in N, k \geq 1, \\ \textit{such that, if } \ \varphi(E) \leq \varepsilon \textit{ for } E \in \Sigma, \textit{ then } p^k(x,E) \leq 1-\varepsilon \textit{ for all } x \in X. \end{cases}$$

The superscript k in p^k denotes the order of the integral convolution (iteration) of the transition function, not its degree. Let us call the number k the parameter of the condition (D).

Another ergodicity condition for Markov operators was introduced by Kryloff and Bogoliouboff in their two studies, also in 1937, already mentioned above, refs. [3,4]. In these studies, quasi-compactness condition (K) for the Markov operator $T\colon B(X,\Sigma) \to B(X,\Sigma)$ are formulated and it is shown very briefly that this condition is sufficient for a certain uniform (mean) ergodic theorem to hold for Cesaro means of powers T^n of the operator T.

Let us give this condition (K) for the operator T.

A Markov operator $T\colon B(X,\Sigma) \to B(X,\Sigma)$ is called quasi-compact (quasi-completely continuous) if the following condition is satisfied

$$(K)\begin{cases} \text{There is a compact (completely continuous) operator } T_1 \\ \text{and an integer } k \geq 1 \text{ such that } ||T^k - T_1|| < 1. \end{cases}$$

In these cases, the Markov chain itself will be called quasi-compact.

In one paper [5], (1941) it is noted that the given condition of quasi-compactness, denoted by (K), can be used not only for the Markov operator T on the space of functions, as is performed in [3,4], but for the "dual" Markov operator A on the space of countably additive measures, and also for arbitrary linear operators in Banach spaces.

After the publication of [5], studies on the comparison of the conditions (K) and (D) under certain assumptions or in other formulations of the problem were continued.

Today, the equivalence of the conditions (K) and (D) in the general case can be considered established.

However, we will not actively use the condition (K) in this paper.

Let an arbitrary Markov chain with a transition function $p(x,E)$ and Markov operators T and A be given on (X,Σ). For any fixed $m \in N$, we define a new Markov chain with a transition function $q_m(x,E)$ and Markov operators T_m and A_m according to the rules for constructing Cesaro means:

$$q_m(x,E) = \frac{1}{m}\sum_{k=1}^{m} p^k(x,E), \qquad T_m = \frac{1}{m}\sum_{k=1}^{m} T^k, \qquad A_m = \frac{1}{m}\sum_{k=1}^{m} A^k.$$

We will call such a Markov chain a finitely averaged Markov chain (by the original Markov chain).

In [2], we formulate one more ergodicity condition $(\tilde{D})$ for a finitely averaged Markov chain.

$$(\tilde{D})\begin{cases} \textit{There is bounded measure } \varphi \in ca(X,\Sigma),\ \varphi \geq 0,\ \varepsilon > 0 \textit{ and } m \in N, \\ \textit{such that, if } \varphi(E) \leq \varepsilon,\ E \in \Sigma, \textit{ then } q_m(x,E) \leq 1-\varepsilon \textit{ for all } x \in X. \end{cases}$$

Obviously, the condition$(\tilde{D})$ is Doeblin condition (D) for a finitely averaged Markov chain (for fixed $m \geq 1$) with parameter $k = 1$. In one paper ([2], Theorem 12.1) it is shown that if condition (D) is satisfied, then condition $(\tilde{D})$ is also satisfied. However, the converse statement was not proved in [2], so this condition $(\tilde{D})$ had to be introduced. We have obtained such a converse statement in this paper and it will be shown below.

In one author's paper [2], the following statement was proved:

Theorem 4 ([2], Theorem 12.2)**.** *For an arbitrary Markov chain, the condition $(\tilde{D})$ is equivalent to the condition $(*)$:*

$$(*) \qquad \Delta_{ba} \subset ca(X, \Sigma),$$

it means that all invariant finitely additive measures of the original Markov chain are countably additive.

In Section 5, we will prove a new statement that the classical Doeblin condition (D) is also equivalent to the condition $(*)$. But for this, we need some information from functional analysis (see, for example, [11]) and the two preceding theorems.

4.2. Theorem on Invariance of All Weakly Limiting Measures for Cesàro Means

The space of measures $ba(X, \Sigma)$ is topologically conjugate to the space of functions $B(X, \Sigma)$, i.e., $B^*(X, \Sigma) = ba(X, \Sigma)$ (up to isomorphism), as we have already noted. Therefore, in Banach space $ba(X, \Sigma)$, we can consider not only the strong (metric) topology, but also the $*$-weak topology τ_B generated by the preconjugate space $B(X, \Sigma)$.

This topology is given by the Tikhonov base of neighborhoods of the point (measure) $\eta \in ba(X, \Sigma)$ of the form

$$V(\eta, f_1, f_2, \ldots, f_k, \varepsilon)$$
$$= \{\mu \in ba(X, \Sigma) \colon |\langle f_i, \mu\rangle - \langle f_i, \eta\rangle| < \varepsilon, \;\; i = 1, 2, \ldots, k; \;\; k \in N, \;\; \varepsilon > 0\},$$

where ε and k are arbitrary, and $f_1, f_2, \ldots, f_k \in B(X, \Sigma)$. The notation $\langle f_i, \mu\rangle$ denotes the value of the function f_i, as a linear functional on the measure μ, calculated by the formula

$$\langle f_i, \mu\rangle = \int_X f_i(x)\mu(dx), i = 1, 2, \ldots, k.$$

We will also use the notation $f(\mu) = \langle f, \mu\rangle$.

If a certain sequence of measures $\{\lambda_n\} = \{\lambda_1, \lambda_2, \ldots\} \subset S_{ba}$ is given, then the symbol $\mathfrak{M}\{\lambda_n\}$ denote the set of all limit points (measures) in the τ_B-topology of the sequence of measures $\{\lambda_n\}$. Note that the fact that the measure η is the limit point of the sequence of measures $\{\lambda_n\}$, generally speaking, does not imply that in $\{\lambda_n\}$ there is a subsequence converging to η in the τ_B-topology.

Let a Markov chain with operator A be given on (X, Σ). We introduce the notation for Cesaro means for some initial measure $\mu \in S_{ba}$:

$$\lambda_n = \lambda_n^\mu = \frac{1}{n}\sum_{k=1}^{n} A^k\mu, n \in N.$$

The proofs of further theorems are based on Theorem 7.2 from [1], its proof in [1] is not given. It only says that it can be carried out by analogy with the proof of another theorem for a Feller–Markov chain on a topological space. However, the difference between general Markov chains and topological Markov chains is very large, and we restore below the unpublished proof of Theorem 7.2 from [1].

Theorem 5. *Let an arbitrary Markov chain and an initial finitely additive measure $\mu \in S_{ba}$ be given on an arbitrary (X, Σ). Then each τ_B-limit point (in the topology τ_B) of the sequence $\{\lambda_n^\mu\}$ in $ba(X, \Sigma)$ will be the fixed point of the operator A, i.e., $\mathfrak{M}\{\lambda_n\} \subset \Delta_{ba}$. The set of such measures is nonempty, i.e., $\mathfrak{M}\{\lambda_n\} \neq \varnothing$ and $\mathfrak{M}\{\lambda_n\}$ is τ_B-compact.*

Proof. Let us choose some measure $\mu \in S_{ba}$. It is obvious that $\|\lambda_n\| = \lambda_n(X) = 1$, $n = 1, 2, \ldots$, i.e., the set $\{\lambda_n\}$ is metrically bounded in $ba(X, \Sigma)$. Hence, the τ_B-closure of the set $\{\lambda_n\}$ is compact in the τ_B-topology (see [11], Chapter V, Item 4, Corollary 3).

By Kelley ([26], Chapter V, Theorem 5), any subsequence of λ_{n_i}, including λ_n, in a compact set has an ω-limit point $\eta = \eta\{\lambda_{n_i}\}$ such that any of its neighborhood contains infinitely many elements of the sequence. It means that for each τ_B-neighborhood $V(\eta, f_1, f_2, \ldots, f_k, \varepsilon)$, where $\varepsilon > 0$, and $f_i \in B(X, \Sigma), i = 1, 2, \ldots, k$, the set $\{i : \lambda_{n_i} \in V(\eta, f_1, f_2, \ldots, f_k, \varepsilon)\}$ is infinite, i.e., there is a subsequence $\{\lambda_{n_{i_j}}\}, \lambda_{n_{i_j}} \in V(\eta, f_1, f_2, \ldots, f_k, \varepsilon)$, $j = 1, 2, \ldots$

Let the measure η be τ_B-limit for the sequence $\{\lambda_n\}$.

Now, let us do the following transformation.

$$A\lambda_n = \frac{1}{n} \sum_{k=1}^{n} A^{k+1}\mu + \frac{1}{n}[A^{n+1}\mu - A\mu] = \lambda_n + \frac{1}{n}[A^{n+1}\mu - A\mu], n \in N.$$

Let $f \in B(X, \Sigma)$. Then for any $\varepsilon > 0$ there is a strictly increasing sequence of natural numbers $\{n_i\} = \{n_i\}(f, \varepsilon)$ such that $\lambda_{n_i} \in V(\eta, f, Tf, \varepsilon)$, $i = 1, 2, \ldots$

Now

$$\begin{gathered} |f(\eta) - f(A\eta)| = |f(\eta) - Tf(\eta)| \\ \leq |f(\eta) - f(\lambda_{n_i})| + |f(\lambda_{n_i}) - Tf(\lambda_{n_i})| + |Tf(\lambda_{n_i}) - Tf(\eta)| \\ \leq \varepsilon + |f(\lambda_{n_i}) - f(A\lambda_{n_i})| + \varepsilon = 2\varepsilon + |f(\lambda_{n_i}) - f(\lambda_{n_i}) - f(\frac{1}{n_i}[A^{n_i+1}\mu - A\mu])| \\ \leq 2\varepsilon + \frac{1}{n_i}|f(A^{n_i+1}\mu - A\mu)| \leq 2\varepsilon + \frac{2\|f\|}{n_i}. \end{gathered}$$

Since $n_i \to \infty$ for $i \to \infty$, then $|f(\eta) - f(A\eta)| \leq 2\varepsilon$. As ε is arbitrary, then $|f(\eta) - f(A\eta)| = 0$.

So, for each $f \in B(X, \Sigma)$, the equality $f(\eta) = f(A\eta)$ holds. The set $B(X, \Sigma))$ is total on $ba(X, \Sigma)$. Therefore, $\eta = A\eta$. Moreover, $\eta \in ba(X, \Sigma)$.

Let us show that $\eta \in S_{ba}$, i.e., the measure η is normalized and positive.

Consider τ_B-neighborhood of the point η of the form $V(\eta, f, \varepsilon)$, where $\varepsilon > 0$ is arbitrary, $f \in B(X, \Sigma)$ and $f(x) \equiv 1$. Then there is n_i such that $\lambda_{n_i} \in V(\eta, f, \varepsilon)$, i.e.,

$$|\langle f, \eta \rangle - \langle f, \lambda_{n_i} \rangle| = |\eta(X) - \lambda_{n_i}(X)| < \varepsilon.$$

Since $\lambda_n(X) = 1$ for all $n \in N$, then $|\eta(X) - 1| < \varepsilon$ for any $\varepsilon > 0$, whence $\eta(X) = 1$.

Suppose there is $E \in \Sigma$ and $r > 0$ such that $\eta(E) = -r < 0$.

Let us take as $f \in B(X, \Sigma)$ the characteristic function $f = \chi_E$ of the set E and the number $\varepsilon = \frac{r}{2}$. Then $\langle f, \eta \rangle = \eta(E) = -r$, $\langle f, \lambda_n \rangle = \lambda_n(E)$ and

$$|\langle f, \lambda_n \rangle - \langle f, \eta \rangle| = \lambda_n(E) + r \geq r > \varepsilon \text{ for all } n \in N.$$

Therefore, $\lambda_n \notin V(\eta, f, \varepsilon)$ for $n \in N$, i.e., the measure η is not τ_B-limit for the sequence $\{\lambda_n\}$. The resulting contradiction proves that $\eta(E) \geq 0$ for all $E \in \Sigma$.

Summing up all the previous conclusions, we obtain as a result that $\mathfrak{M}\{\lambda_n\} \subset \Delta_{ba}$, $\mathfrak{M}\{\lambda_n\} \neq \emptyset$ and $\mathfrak{M}\{\lambda_n\}$ is τ_B-compact. The theorem is proved. □

We also need one more theorem ([1], Theorem 7.3). Note that in this theorem the means are taken for Markov chains with different initial measures μ_n for each $n \in N$.

This is a generalization of Theorem 5. Its proof follows the same scheme as the proof of Theorem 5.

Theorem 6 ([1], Theorem 7.3). *Suppose that we have an arbitrary Markov chain on an arbitrary (X, Σ), a sequence of measures $\mu_n \in ba(X, \Sigma)$, $\mu_n \in S_{ba}$, and*

$$\lambda_n = \lambda_n^{\mu_n} = \frac{1}{n}\sum_{k=1}^{n} A^k \mu_n, \quad n = 1, 2, \ldots.$$

Then each τ_B-limit measure of the sequence $\{\lambda_n\}$ is invariant for the operator A, i.e., $\mathfrak{M}\{\lambda_n\} \subset \Delta_{ba}$, the set of such measures is nonempty, $\mathfrak{M}\{\lambda_n\} \neq \emptyset$ and $\mathfrak{M}\{\lambda_n\}$ is τ_B-compact.

5. Equivalence of Doeblin Condition to Condition $(*)$ and to Condition $(\tilde{*})$

The aim of this section is to obtain the main result of the paper.

At the beginning, let us prove one technical lemma, that we need in the proof of the second part (necessity) of Theorem 7.

Lemma 1. *For any Markov chain, for any set $G \in \Sigma$, and for all $m \in N$, the following equality holds*

$$\sup_{x \in X} p^m(x, G) = \sup_{\eta \in S_{ba}} A^m \eta(G).$$

Proof. It is easy to see that the transition function for any $m \in N$, $x \in X$, and $G \in \Sigma$ can be represented in the following form

$$p^m(x, G) = A^m \delta_x(G),$$

where δ_x is the Dirac measure at the point $x \in X$.

Hence, it follows that

$$\sup_{x \in X} p^m(x, G) = \sup_{x \in X} A^m \delta_x(G).$$

Then, it is obvious, that with an increase in the class of measures over which the supremum is sought, this supremum can only increase:

$$\sup_{x \in X} A^m \delta_x(G) \leq \sup_{\eta \in S_{ba}} A^m \eta(G).$$

Let us show that, in fact, equality holds here. Suppose the above inequality is strict. Let us introduce numbers $\alpha, \beta > 0$ such that $\alpha < \beta$ and

$$\sup_{x \in X} p^m(x, G) = \sup_{x \in X} A^m \delta_x(G) = \alpha < \beta = \sup_{\eta \in S_{ba}} A^m \eta(G).$$

Then there is a measure $\eta_0 \in S_{ba}$ such that

$$\alpha < \alpha + \frac{\beta - \alpha}{2} \leq A^m \eta_0(G) \leq \beta.$$

But, at the same time,

$$A^m \eta_0(G) = \int_X p^m(x, G) \eta_0(dx) \leq \int_X \alpha \cdot \eta_0(dx) = \alpha.$$

The resulting inequality contradicts the previous inequality: $A^m \eta_0(G) \geq \alpha + \frac{\beta - \alpha}{2}$. Therefore, the equality in the formulation of the Lemma 1 is true. The lemma is proven. □

The main result of our paper reads as follows:

Theorem 7. *For an arbitrary general Markov chain, the Doeblin condition (D) is equivalent to the condition $(*)$: $\Delta_{ba} \subset ca(X, \Sigma)$, which means that all invariant finitely additive measures of the Markov chain are countably additive.*

Proof. First, let us prove that the fulfillment of condition $(*)$ implies the fulfillment of condition (D) (sufficiency).

Let the condition $(*)$ be satisfied. Then, by Theorem 8.2 from [1], we have the dimension $dim\Delta_{ba} = dim\Delta_{ca} = n < \infty$. Let $H_{ba} = H_{ca} = \{\mu_1, \dots, \mu_n\}$ a singular basis of the space of invariant measures $M_{ba} = M_{ca}$, existing by Theorem 6.3 from [1].

Let $\varphi = \mu_1 + \dots + \mu_n$ be a countably additive measure (this measure may be not normalized). We want to prove that for this measure φ and for some numbers $m \geq 1 (m \in N)$ and $\varepsilon > 0$, the Doeblin condition (D) is satisfied.

Suppose that the given measure φ does not satisfy condition (D). Then for each $m \geq 1 (m \in N)$ and for each $0 < \varepsilon < 1$, there is a set $E_{m,\varepsilon} \in \Sigma$ and there is a point $x_{m,\varepsilon} \in X$, such that

$$\varphi(E_{m,\varepsilon}) \leq \varepsilon \quad \text{and} \quad p^m(x_{m,\varepsilon}; E_{m,\varepsilon}) > 1 - \varepsilon.$$

For each $m = 1, 2, \dots$ and for some fixed $\delta \in (0, 1)$ we choose $\varepsilon = \varepsilon(m) = \frac{\delta}{2^m}$, and $E_\delta = \cup_{m=1}^{\infty} E_{m,\epsilon(m)}$.

Since the measure φ is countably additive, and therefore it is also countably semi-additive, then

$$\varphi(E_\delta) = \varphi(\bigcup_{m=1}^{\infty} E_{m,\varepsilon(m)}) \leq \sum_{m=1}^{\infty} \varphi(E_{m,\varepsilon(m)}) \leq \sum_{m=1}^{\infty} \varepsilon(m) = \delta \sum_{m=1}^{\infty} \frac{1}{2^m} = \delta,$$

i.e., $\varphi(E_\delta) \leq \delta$.

At the same time, for all $m = 1, 2, \dots$ we have:

$$p^m(x_{m,\varepsilon(m)}; E_\delta) \geq p^m(x_{m,\varepsilon(m)}; E_{m,\varepsilon(m)}) > 1 - \varepsilon(m) = 1 - \frac{\delta}{2^m}.$$

Now we construct a sequence of Dirac measures $\eta_m \in S_{ca}$ concentrated at the points $x_{m,\varepsilon(m)}$, i.e., $\eta_m = \delta_{x_{m,\varepsilon(m)}}$. Then for powers of the Markov operator A^s for $s = 1, 2, \dots, m$ we have

$$A^s \eta_m(E) = p^s(x_{m,\varepsilon(m)}; E)$$

for arbitrary sets $E \in \Sigma$.

Accordingly, for the Cesaro means for $E = E_\delta$ and $m = 1, 2, \dots$, the following will be true:

$$\lambda_m^{\eta_m}(E_\delta) = \frac{1}{m} \sum_{s=1}^{m} A^s \eta_m(E_\delta) = \frac{1}{m} \sum_{s=1}^{m} p^s(x_{m,\varepsilon(m)}; E_\delta)$$

$$> \frac{1}{m} \sum_{s=1}^{m} (1 - \frac{\delta}{2^s}) = \frac{1}{m}(m - \sum_{s=1}^{m} \frac{\delta}{2^s}) = 1 - \frac{\delta}{m}(1 - \frac{\delta}{2^m}) > 1 - \frac{\delta}{m},$$

i.e., $\lambda_m^{\eta_m}(E_\delta) > 1 - \frac{\delta}{m}, m = 1, 2, \dots$

Now we turn to the τ_B-weak topology in the space $ba(X, \Sigma)$ generated by the preconjugate space $B(X, \Sigma)$. By Theorem 6, the set $\mathfrak{M}\{\lambda_m^{\eta_m}\}$ of all τ_B-limit points (measures) of the sequence $\lambda_m^{\eta_m}$ is nonempty and is contained in the set of all finitely additive measures invariant for the Markov chain, $\mathfrak{M}\{\lambda_m^{\eta_m}\} \subset \Delta_{ba} = \Delta_{ca}$, i.e., all τ_B-limit points (measures) of the sequence $\{\lambda_m^{\eta_m}\}$ are invariant and, by the condition $(*)$, countably additive.

Let the measure $\mu \in \mathfrak{M}\{\lambda_m^{\eta_m}\}$ be arbitrary. Then, if for some $E \in \Sigma$ there is a limit of the numerical sequence $\lambda_m^{\eta_m}(E)$ for $m \to \infty$, then, obviously, $\mu(E) = \lim_{m\to\infty} \lambda_m^{\eta_m}(E)$.

Take the set $E = E_\delta$. Then from the above estimate $\lambda_m^{\eta_m}(E_\delta) > 1 - \frac{\delta}{m}, m = 1, 2, \dots$ it follows that $\lambda_m^{\eta_m}(E_\delta) \to 1$ for $m \to \infty$. Consequently, for the τ_B-limit (invariant) measure μ we have $\mu(E_\delta) = 1$. Moreover, as we obtained in the above, $\varphi(E_\delta) \leq \delta < 1$.

Since the measure μ is invariant, it is decomposable on a singular basis $H_{ba} = \{\mu_1, \ldots, \mu_n\}, \mu = \sum_{i=1}^n \alpha_i \mu_i$, where $0 \leq \alpha_i \leq 1, i = 1, 2, \ldots, n$. Hence, $\mu = \sum_{i=1}^n \alpha_i \mu_i \leq \sum_{i=1}^n \mu_i = \varphi$. Therefore, $\mu(E_\delta) \leq \varphi(E_\delta) \leq \delta < 1$. But in the above, we obtained that $\mu(E_\delta) = 1$. From this contradiction, we can see that the condition $(*)$ implies the fulfillment of condition (D) for the measure $\varphi = \mu_1 + \mu_2 + \ldots + \mu_n$. The first part of the theorem is proved.

Now let us prove that the fulfillment of condition (D) implies the fulfillment of condition $(*)$ (necessity).

Let condition (D) be satisfied for some $\varphi \in ca(X, \Sigma), \varphi \geq 0, \varepsilon > 0$ and $m \geq 1 (m \in N)$.

Assume that condition $(*)$ is not satisfied. Then the Markov chain has invariant purely finitely additive measure $\lambda \in S_{pfa}, \lambda = A\lambda$.

Any purely finitely additive measure is disjoint with any countably additive measure ([12], Theorem 1.16), whence $\lambda \wedge \varphi = 0$. Then, by another theorem ([12], Theorem 1.19), for any number $\varepsilon > 0$ (and hence for our ε from the "triple" $(\varphi, \varepsilon, m)$) there is a set $G \in \Sigma$ such that $\varphi(G) < \varepsilon$ and $\lambda(X \setminus G) = 0$.

Hence, $\lambda(G) = 1$.

Now we use Lemma 1. Let us make the following transformations and estimates:

$$1 \geq \sup_{x \in X} p^m(x, G) = \sup_{\eta \in S_{ba}} A^m \eta(G) \geq A^m \lambda(G) = \lambda(G) = 1.$$

Hence,

$$\sup_{x \in X} p^m(x, G) = 1.$$

Therefore, there is at least one point $x_0 \in X$ such that

$$1 = \sup_{x \in X} p^m(x, G) \geq p^m(x_0, G) > 1 - \frac{\varepsilon}{2} > 1 - \varepsilon.$$

But by the condition (D), which we assume to be true, there must be $p^m(x, G) \leq 1 - \varepsilon$ for all $x \in X$, including for $x = x_0$. The resulting contradiction proves that condition $(*)$ is also satisfied. The theorem is proved. $\square$

We introduce one more condition:

$$(\tilde{*}) \quad \Delta_{pfa} = \varnothing.$$

The fulfillment of the condition $(\tilde{*})$ means that the Markov chain has no invariant purely finitely additive probability measures. Then all its invariant finitely additive measures (and there are always such) are countably additive, i.e., condition $(*)$ is satisfied.

Obviously, the converse is also true. Thus, we obtain the next corollary.

Corollary 1. *For an arbitrary Markov chain, the conditions $(*)$ and $(\tilde{*})$ are equivalent.*

From the proved Theorem 7 and from Theorem 4, as a consequence, we obtain the following promised and psychologically important statement for us.

Theorem 8. *For an arbitrary general Markov chain, the Doeblin condition (D) is equivalent to the condition $(\tilde{D})$.*

Remark 3. *Obviously, the condition $(*)$ is satisfied if and only if $\Delta_{ba} = \Delta_{ca}$.*

Now, let us collect all the results obtained in this section into one statement.

Proposition 1. *For any general Markov chain on an arbitrary measurable space* (X,Σ)*, the following conditions are equivalent:*

(i) *Doeblin condition (D) for the original Markov chain.*
(ii) *Doeblin condition* $(\tilde{D})$ *for a finitely averaged Markov chain.*
(iii) *Condition* $(*)$*: all invariant finitely additive probability measures of a Markov chain (for a Markov operator* T^**) are countably additive.*
(iv) *Condition* $(\tilde{*})$*: The Markov chain (the Markov operator* T^**) has no invariant purely finitely additive probability measures.*
(v) *Condition:* $\Delta_{ba} = \Delta_{ca}$.

6. Dimension of the Set of Invariant Measures

Theorem 7 and other results obtained above show that invariant finitely additive measures play a big role in the ergodic theory of Markov chains. However, this picture will be incomplete if we do not point to our already published results on the dimension of spaces of invariant measures. We present here two of our theorems on the dimension of spaces of invariant measures with new additions.

In one paper [1], the following statement was proved.

Theorem 9 (see [1], Theorem 8.2)**.** *For an arbitrary Markov chain, if condition* $(*)$ *is satisfied, i.e., if* $\Delta_{ba} \subset ca(X,\Sigma)$*, then the following condition is also satisfied*

$$(**) \quad dim\Delta_{ba} = n, \text{ where } 1 \le n < \infty.$$

Corollary 2. *For an arbitrary Markov chain, if the condition* (D) *or the condition* $(\tilde{D})$ *is satisfied, then the condition* $(**)$ *is also satisfied.*

Corollary 3. *For an arbitrary Markov chain, if the condition* $(*)$ *is satisfied, then*

$$dim\Delta_{ba} = dim\Delta_{ca} = n, \text{ where } 1 \le n < \infty.$$

An intuitive assumption arose that the statement of Theorem 9 can be reversed. In the same paper, such a reversal was proved, but only for the case of dimension $n = 1$. We present this theorem below.

Theorem 10 (see ([1], (Theorem 8.3)) and ([2], (Theorem 12.3)))**.** *Let an arbitrary Markov chain be given on some* (X,Σ)*. If* $dim\Delta_{ba} = 1$*, i.e., if the Markov chain has a unique invariant measure* μ *in* S_{ba}*,* $\Delta_{ba} = \{\mu\}$*, then the condition* $(*)$ *is satisfied:* $\Delta_{ba} \subset ca(X,\Sigma)$*, i.e., the invariant measure* μ *is countably additive.*

We recall that, by Theorem 7, the condition $(*)$ is equivalent to Doeblin condition (D). This implies the following theorem.

Theorem 11. *Let for an arbitrary Markov chain* $dim\Delta_{ba} = 1$*. Then the conditions* (D) *and* $(\tilde{D})$ *are satisfied.*

Please note that if $dim\Delta_{ba} = 1$ then the Markov chain can have only one cycle of any dimension consisting of countably additive measures.

7. Discussion

1. In the second part of [7] (1975), M. Lin considered an arbitrary measurable space (X,Σ) and investigated the properties of the Markov operator P defined on the space of functions $B(X,\Sigma)$ and the properties of the conjugate operator P^* defined on the conjugate space. It should be noted that the symbols $B^*(X,\Sigma)$, $ba(X,\Sigma)$, and also the isomorphism $B^*(X,\Sigma) = ba(X,\Sigma)$ were not used explicitly in the text of article [7].

We are also using this formulation of the problem with the corresponding addition.

In Theorem 5 of [7], a number of assertions under some hard a priori condition of "ergodicity" on the Markov chain are considered. In particular, it is proved that the Doeblin condition (viii) is equivalent to the condition (vi): "The space of P^* invariant functionals (i.e., finitely additive measures) is one-dimensional". But this is true only due to the a priori "ergodicity" condition in ([7], Theorem 5). In the general case, only the finite-dimensionality of the space of invariant finitely additive measures, and hence also of invariant countably additive measures, follows from the Doeblin condition (see, for example, ref. [25] and also our theorems in Sections 4 and 5).

In the same Theorem 5 from [7] it was proved (under the same a priori ergodicity condition) that the Doeblin condition (viii) and the condition (vi) are equivalent to the condition (v): "Every P^*-invariant functionals is a measure" (here it is countably additive measures) (viii).

If we translate these statements into the language we use, then we obtain a special case of our Theorem 7, but for a one-dimensional space of invariant finitely additive measures.

Our Theorems 10 and 11 also generalize the corresponding assertions from ([7], Theorem 5), since they do not assume that the Markov chain is ergodic.

Additionally, we would like to note that during the preparation of our articles [1,2] the author was not familiar with this article by Michael Lin [7]. Therefore, we did not cite this article in our works [1,2]. We apologize to Michael Lin and fill this gap.

2. In the work of S. Horowitz [8] (1972), some problems are solved that are close to our subject, but in a different formulation. Horowitz considered Markov chains defined on a measurable space (X, Σ, m) with a given fixed bounded non-negative countably additive measure m. The transition probability of the Markov chain $p(x, \cdot)$ is assumed to be absolutely continuous with respect to the measure m for each $x \in X$.

The Markov operator P generated by the function $p(x, E)$ acts (left and right) in the spaces $L_1(X, \Sigma, m)$ and in $L_\infty(X, \Sigma, m)$, respectively.

In one paper ([8], Theorem 4.1), it is proved that if the Markov chain is ergodic and conservative (the definitions were given at the beginning of the article [8], we will not repeat them here), then the quasi-compactness of the Markov operator P on space $L_\infty(m)$ (condition $\{h\}$) is equivalent to the absence of an invariant purely finitely additive measure (condition $\{a\}$) (pure charge) and is equivalent to the condition: the operator P has a unique invariant countably additive measure $\mu = \mu P$ and μ is a measure equivalent to m (condition $\{d\}$).

Thus, in Theorem 4.1 [8] a particular case of the one-dimensional space of invariant measures was considered. Our Theorem 7 and Corollary 1, as well as Theorems 10 and 11 contain similar statements, but for a different formulation of the problem and for other types of Markov operators.

In particular, in this paper and in our papers [1,2], it is not supposed to specify a priori some fixed countably additive measure to which all other objects are attached.

We also do not assume a priori ergodicity or conservativeness for the considered Markov chain.

It can be said that we consider a more general case of defining a Markov chain than Horowitz [8]. But it is better to talk about the parallel formulation of the problem.

The above Horowitz theorem from [8] is given in transformed form in Revuze's book (see [6], (Chapter 6, Section 3)) with new additions, conditions, and corollaries.

3. Let us point out the possibilities of applying our main theorems in some physical problems using the article [9] as an example. Its authors develop a special mathematical apparatus, which they call "Thermodynamic Formalism", to study the corresponding physical processes. In particular, a special classical Markov chain is constructed on some infinite-dimensional topological space, but finitely additive measures (purely finitely additive) are used as invariant measures.

In one paper [9], it is proved that an invariant finitely additive probability measure for a given special Markov chain exists, it is unique, and it does not have a purely finitely additive component. Based on these facts, the authors prove the "asymptotic stability" of the corresponding Markov operators, i.e., the convergence of a Markov sequence of countably additive measures to an invariant countably additive measure in some special metric.

We see here the points of contact of the construction developed in [9] with our results from the articles [1,2] (and from this article), since the main theorems from these articles are proved for any Markov chain on an arbitrary phase space, including on an arbitrary topological space, i.e., they are also applicable to the Markov chain studied in [9].

4. Questions of quasi-compactness of general linear operators are studied in [10]. In one paper ([10], Section 5), the results obtained are applied to Markov chains. In this case, finitely additive measures and the Doeblin condition are used. However, there is no analog of our Theorem 7.

8. Conclusions

Thus, if condition $(*)$ or condition $(\tilde{*})$ is satisfied for the Markov chain, then Doeblin condition (D) is satisfied, and the condition of quasi-compactness (K) is also satisfied. Consequently, for such a Markov chain, the corresponding ergodic theorems are valid, including uniform ones. Such theorems are studied well and in detail in the literature in different versions.

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