Generalized power series solutions of q-difference equations and the small divisors phenomenon[1]

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Abstract

The problem of the convergence of generalized formal power series (with complex power exponents) solutions of q-difference equations is studied in the situation where the small divisors phenomenon arises; a sufficient condition of convergence generalizing corresponding conditions for classical power series solutions is obtained; an illustrating example is given.

Keywords: q-difference equation, generalized power series solution, convergence, small divisors

Our talk is based on the joint work with A. Lastra, see [4]. We consider a q-difference equation

$$F(z, y, \sigma y, \sigma^2 y, \dots, \sigma^n y) = 0, \qquad z \in \mathbb{C}, \tag{1}$$

where $F = F(z, y_0, y_1, \dots, y_n)$ is a polynomial and σ stands for the dilatation operator

$$\sigma : \; y(z) \mapsto y(qz),$$

$q \neq 0, 1$ being a fixed complex number. We study the question of the convergence of its *generalized* formal power series solutions $y = \varphi$ of the form

$$\varphi = \sum_{j=0}^{\infty} c_j z^{\lambda_j}, \qquad c_j, \lambda_j \in \mathbb{C}, \tag{2}$$

where $c_0 \neq 0$ and the sequence of the exponents λ_j possesses the following two properties:

(i) $\operatorname{Re}\lambda_j \leqslant \operatorname{Re}\lambda_{j+1}$ for all $j \geqslant 0$,

(ii) $\lim_{j\to\infty} \operatorname{Re}\lambda_j = +\infty$.

We note that the conditions (i), (ii) make the set of all generalized formal power series an algebra over $\mathbb{C}$. The definition of the dilatation operator extends naturally to this algebra after fixing the value of $\ln q$ by the condition $0 \leqslant \arg q < 2\pi$:

$$\sigma\Big(\sum_{j=0}^{\infty} c_j z^{\lambda_j}\Big) = \sum_{j=0}^{\infty} c_j q^{\lambda_j} z^{\lambda_j}.$$

Thus the notion of a generalized formal power series solution of (1) is correctly defined in view of the above remarks: such a series φ is said to be a *formal solution* of (1) if the substitution of $y_i = \sigma^i \varphi$, $i = 0, 1, \dots, n$, into the polynomial F leads to a generalized power series with zero coefficients.

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Formal solutions (2) generalize classical power series solutions of the form $\sum_{j=0}^{\infty} c_j z^j$. The convergence of the latter was widely studied within the last decades: there are two principally different cases, that of $|q| \neq 1$ (see [8], [5]) and that of $|q| = 1$, q not being a root of unity, where the small divisors phenomenon may arise (see [1], [2]). Namely, the coefficients c_j of a formal power series solution $\sum_{j=0}^{\infty} c_j z^j$ of (1) are determined recurrently via relations $Q(q^j)\, c_j = P_j(c_0, c_1, \ldots, c_{j-1})$, with some polynomials $Q, \{P_j\}$. Therefore the sequence q^j tends neither to infinity nor to zero if $|q| = 1$ and may come arbitrarily close to a root of Q, which may cause a high growth of the coefficients c_j obstructing the convergence of the series.

For the generalized formal power series solution (2) of (1), assume that each $F'_{y_i}(z, \varphi, \sigma\varphi, \ldots, \sigma^n\varphi)$ is of the form

$$\frac{\partial F}{\partial y_i}(z, \varphi, \sigma\varphi, \ldots, \sigma^n\varphi) = A_i z^{\gamma} + B_i z^{\gamma_i} + \ldots, \qquad \operatorname{Re}\gamma_i > \operatorname{Re}\gamma \geqslant 0,$$

$\gamma \in \mathbb{C}$ being the same for all $i = 0, 1, \ldots, n$, and at least one of the A_i's being non-zero. Then under a generic assumption on the power exponents λ_j of (2) that, starting with some $j_0 \in \mathbb{Z}_+$, the q^{λ_j}'s are not the roots of a non-zero polynomial

$$L(\xi) = A_n \xi^n + \ldots + A_1 \xi + A_0$$

of degree $\leqslant n$, one can assert that all $\lambda_j - \lambda_{j_0}$, $j > j_0$, belong to a finitely generated additive semi-group $\Gamma \subset \mathbb{C}$ whose generators $\alpha_1, \ldots, \alpha_s$ all have a positive real part (see Lemmas 1, 2 in [3]). Thus we may initially consider the formal solution (2) in the form

$$\varphi = \sum_{j=0}^{\infty} c_j z^{\lambda_j} = \sum_{j=0}^{j_0} c_j z^{\lambda_j} + \sum_{(m_1,\ldots,m_s)\in\mathbb{Z}_+^s\setminus\{0\}} c_{m_1,\ldots,m_s} z^{\lambda_{j_0}+m_1\alpha_1+\ldots+m_s\alpha_s}. \tag{3}$$

For such a formal series solution the small divisors phenomenon does not arise if all the α_k's lie *strictly above* or *strictly under* the line $\mathcal{L}$ passing through $0 \in \mathbb{C}$ and having the slope $\ln|q|/\arg q$ (or, equivalently, all the q^{α_k}'s lie *strictly inside* or *strictly outside* the unit circle). This condition defines an analogue (and generalization) of the case of $|q| \neq 1$ in the classical situation. The convergence of (3) under such a condition was studied in our previous work [3]. Contrariwise, the placement of α_k's on both sides of (or on) $\mathcal{L}$ may cause the small divisors phenomenon. The study of this situation is the main subject of our present talk and we propose the following theorem on the convergence of φ.

Theorem 1. *Let the generalized formal power series* (3) *satisfy* (1). *If* $\deg L = n$, $L(0) \neq 0$, *and for each root* $\xi = a$ *of the polynomial* $(\xi - q^{\lambda_{j_0}})L(\xi)$ *the following diophantine condition is fulfilled:*

$$|(\lambda_{j_0} + m_1\alpha_1 + \ldots + m_s\alpha_s)\ln q - \ln a - 2\pi m i| > c\,(m_1 + \ldots + m_s)^{-\nu} \quad \text{for all } m_i \in \mathbb{Z}_+,\ m \in \mathbb{Z} \tag{4}$$

(*with the exception of* $m_1 = \ldots = m_s = 0$), *where* c *and* ν *are some positive constants, then* (3) *has a non-zero radius of convergence* (*that is, it converges uniformly in any sector* $S \subset \mathbb{C}$ *of sufficiently small radius with the vertex at the origin and of the opening less than* 2π *defining there a germ of a holomorphic function*).

Remarks 1. The diophantine condition of Theorem 1 is generically fulfilled. As for concrete examples, one can apply in particular Schmidt's result [6] from which it follows that (4) holds for $a = q^{\lambda_{j_0}}$, if

1. the real parts of all $\frac{1}{2\pi i}\alpha_1 \ln q, \dots, \frac{1}{2\pi i}\alpha_s \ln q$ are algebraic and together with 1 linearly independent over $\mathbb{Z}$ or

2. the imaginary parts of all $\frac{1}{2\pi i}\alpha_1 \ln q, \dots, \frac{1}{2\pi i}\alpha_s \ln q$ are algebraic and linearly independent over $\mathbb{Z}$.

(If L has roots $\xi = a$ other than $q^{\lambda_{j_0}}$ then the number $\frac{1}{2\pi i}\ln(q^{\lambda_{j_0}}/a)$ should be added to the set of numbers in the above conditions 1, 2 for each such $a \neq q^{\lambda_{j_0}}$.)

The proof of the theorem is based on Siegel's ideas [7] of studying a first order equation $\sigma y = f(y)$ describing the linearization of a diffeomorphism f of $(\mathbb{C}, 0)$. This uses the majorant method adapted to our "multi-index case" for the construction of a convergent series majorizing (3).

Some particular placements of the α_k's with respect to the line $\mathcal{L}$ allow one to weak assumptions of Theorem 1. Therefore we formulate a separate statement which follows from Theorem 1 and distinguishes all these particular cases of the placement of the α_k's on the plane.

Theorem 2. *The statement of Theorem 1 holds in the following particular cases:*

a) $L(0) \neq 0$ *and all the* α_k*'s lie* strictly above *the line* $\mathcal{L}$*;*

b) $\deg L = n$ *and all the* α_k*'s lie* strictly under *the line* $\mathcal{L}$*;*

c) *all the* α_k*'s lie* on *the line* $\mathcal{L}$ *and the condition* (4) *is fulfilled for those roots* $\xi = a$ *of the polynomial* $(\xi - q^{\lambda_{j_0}})L(\xi)$ *that lie on the circle* $\{|\xi| = |q^{\lambda_{j_0}}|\}$*;*

d) $L(0) \neq 0$*, all the* α_k*'s lie* above *or* on *the line* $\mathcal{L}$*, and the condition* (4) *is fulfilled for those roots* $\xi = a$ *of the polynomial* $(\xi - q^{\lambda_{j_0}})L(\xi)$ *that lie inside the closed disk* $\{|\xi| \leqslant |q^{\lambda_{j_0}}|\}$*;*

e) $\deg L = n$*, all the* α_k*'s lie* under *or* on *the line* $\mathcal{L}$*, and the condition* (4) *is fulfilled for those roots* $\xi = a$ *of the polynomial* $(\xi - q^{\lambda_{j_0}})L(\xi)$ *that lie outside the open disk* $\{|\xi| < |q^{\lambda_{j_0}}|\}$.

Note that the small divisors phenomenon for classical power series solutions of (1) arising in the case of $q = e^{2\pi i\omega}$, $\omega \in \mathbb{R} \setminus \mathbb{Q}$, and studied in [1], [2], is contained in the case c) of Theorem 2: the line $\mathcal{L}$ coincides with the Ox axis, $\lambda_{j_0} = 0$, the set of power exponents is generated by the unique $\alpha_1 = 1 \in \mathcal{L}$ and the condition (4) is reduced to

$$|j\omega - (1/2\pi i)\ln a - m| > cj^{-\nu} \qquad \text{for all } j \in \mathbb{N},\ m \in \mathbb{Z},$$

in this case (see Th. 6.1 in [1] and Th. 8 in [2]).

Example 1. Consider a kind of a q-difference analogue of the Painlevé III equation with $a = b = 0$, $c = d = 1$:

$$y\,\sigma^2 y - (\sigma y)^2 - z^2 y^4 - z^2 = 0,$$

where $q = e^{2i\pi\omega}$, $\omega \in \mathbb{R} \setminus \mathbb{Q}$. This equation possesses a two-parameter family of formal solutions:

$$\varphi = \sum_{m_1, m_2 \in \mathbb{Z}_+} c_{m_1,m_2}\, z^{r + m_1(2-2r) + m_2(2+2r)},$$

where the complex coefficient $c_{0,0} \neq 0$ is arbitrary, $-1 < \operatorname{Re} r < 1$, the other complex coefficients c_{m_1,m_2} are uniquely determined by $c_{0,0}$ and r. The numbers $q^{2\pm 2r}$ lie on the opposite sides of the unit circle (if $\operatorname{Im} r \neq 0$) or on the unit circle (if $\operatorname{Im} r = 0$), whereas the second degree polynomial $L(\xi) = c_{0,0}(\xi - q^r)^2$ does not vanish at 0. Therefore taking r, ω in such a way that the condition of Theorem 1 holds,

$$|(m_1(2-2r)\omega + m_2(2+2r)\omega - m| > c\,(m_1 + m_2)^{-\nu}$$

for some positive c and ν, we obtain the convergent φ. For example, it is sufficient for ω to be algebraic and for r simply to have a non-zero imaginary part. Indeed, then for any $m_1 \neq m_2$ one has

$$|(m_1(2-2r)\omega + m_2(2+2r)\omega - m| \geqslant 2|\omega \operatorname{Im} r| \cdot |m_2 - m_1| > c\,(m_1+m_2)^{-\nu},$$

whereas for $m_1 = m_2$ it follows that

$$|(m_1(2-2r)\omega + m_2(2+2r)\omega - m| = |4\omega\, m_1 - m| > c\, m_1^{-\nu}.$$

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