Springer Proceedings in Mathematics & Statistics

Fatih Yilmaz · Araceli Queiruga-Dios ·
Jesús Martín Vaquero ·
Ion Mierluş-Mazilu · Deolinda Rasteiro ·
Víctor Gayoso Martínez *Editors*

Mathematical Methods for Engineering Applications

ICMASE 2022, Bucharest, Romania, July 4–7

Springer

Springer Proceedings in Mathematics & Statistics

Volume 414

This book series features volumes composed of selected contributions from workshops and conferences in all areas of current research in mathematics and statistics, including data science, operations research and optimization. In addition to an overall evaluation of the interest, scientific quality, and timeliness of each proposal at the hands of the publisher, individual contributions are all refereed to the high quality standards of leading journals in the field. Thus, this series provides the research community with well-edited, authoritative reports on developments in the most exciting areas of mathematical and statistical research today.

Fatih Yilmaz · Araceli Queiruga-Dios ·
Jesús Martín Vaquero · Ion Mierluş-Mazilu ·
Deolinda Rasteiro · Víctor Gayoso Martínez
Editors

Mathematical Methods for Engineering Applications

ICMASE 2022, Bucharest, Romania, July 4–7

Springer

Editors
Fatih Yilmaz
Faculty of Arts and Sciences
Ankara Hacı Bayram Veli University
Polatli, Ankara, Turkey

Araceli Queiruga-Dios
Department of Applied Mathematics
University of Salamanca
Salamanca, Spain

Jesús Martín Vaquero
Department of Applied Mathematics
University of Salamanca
Salamanca, Spain

Ion Mierluş-Mazilu
Department of Mathematics and Computer Science
Technical University of Civil Engineering
Bucharest, Romania

Deolinda Rasteiro
Department of Physics and Mathematics
Instituto Superior de Engenharia de Coimbra
Coimbra, Portugal

Víctor Gayoso Martínez
Centro Universitario de Tecnología y Arte Digital (U-tad)
Las Rozas de Madrid, Spain

ISSN 2194-1009 ISSN 2194-1017 (electronic)
Springer Proceedings in Mathematics & Statistics
ISBN 978-3-031-21699-2 ISBN 978-3-031-21700-5 (eBook)
https://doi.org/10.1007/978-3-031-21700-5

Mathematics Subject Classification: 08Axx, 11-xx, 65-xx, 97Uxx

This Springer imprint is published by the registered company Springer Nature Switzerland AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Contents

Mutual Generation of the Choice and Majority Functions

Elmira Yu Kalimulina

1 Introduction

The rapid growth of quantum computers and its application in the field of artificial intelligence has led k-valued (in particular ternary) computing to be relevant again [1]. Also, the research and development of algorithms based on k–valued logic are very relevant in many other fields such as, for example, telecommunications (developing of new protocols [2], choosing optimal network routing scheme [3], data aggregation schemes [4]), symbolic analysis of complex systems, software development and detecting design errors, machine learning, etc.

The detailed review of k-valued logic applications was given in [5, 6]. Thus, the problem of a full description of all closed classes of k-valued logic functions is very crucial for progress in many fields of science and engineering. A fundamentally essential problem—the problem of full description of closed classes of three-valued logic functions [7]—must be solved to make the implementation of circuits with the desired functional diagram possible [8]. The famous result by Emil Post relates to the full description of all closed classes of Boolean functions (with respect to the superposition operation) [9]. Later it had been described in detail in [10]. This result let many problems of two-valued logic to be solved. Then the special case of the finite generation of all closed two-valued logic classes with respect to a superposition operation had been proved. But with the transition to a k-valued logic ($k > 2$) a continuum of closed classes with respect to superposition operation appeared. And in that case a complete description is impossible. There are not finitely generated classes in k-valued logic case (see the example of Yanov and Muchnik [11]). Therefore, the description of all finitely generated classes for k-valued logic is an open problem [12].

There are many results related to the description of family of classes of functions closed with respect to a special operation. The operation of binary superposition determined for the k-valued logic functions on the basis of their representation in the binary number system has been considered in [13]. Criterion of implicit completeness

E. Y. Kalimulina (✉)
V. A. Trapeznikov Institute of Control Sciences of Russian Academy of Sciences, Moscow, Russia
e-mail: elmira.yu.k@ipu.ru
URL: https://www.ipu.ru/staff/elmira

F. Yilmaz et al. (eds.), *Mathematical Methods for Engineering Applications*,
Springer Proceedings in Mathematics & Statistics 414,
https://doi.org/10.1007/978-3-031-21700-5_6

in three-valued logic in terms of precomplete classes was considered in [14]. Several sufficient conditions of finite generation are known. The most famous of them are: the existence of a majority function in the class, of a choice function, and all unary functions (see [15]).

This paper considers the problem of verifying the finite generation of classes containing some subclass of one variable functions. We also give a description of overlattices of classes in P_k containing some precomplete class of unary functions, that has been given earlier by M.A. Posypkin in [16]. The finitely generation of overlattices has been proved. It is also shown that any class consisting of monotone functions and containing all monotone functions of one variable is finitely generated. The finite precompleteness of some closed classes can not be checked under the sufficient assumptions given above. These classes do not contain a choice function, any majority function, all functions of one variable, and the set of any one-variable function that is precompleted on all one-place functions. Some examples of such closed classes have been given in this paper. The proof of finite generation of such classes is based on the constant modelling method proposed in [17].

Let us introduce some standard notation and definitions [18]. Let E_k be the set $\{0, 1, \dots, k-1\}$. For every natural number n the set E_k^n is n-th Cartesian power of a set E_k, and the mapping $f : E_k^n \to E_k$ is an n-place k-valued logic function. The set of all functions of k-valued logic is denoted by P_k. Let R be an arbitrary set of k-valued logic functions. A superposition of functions over a set R is defined by induction: (1) every function f from R is a superposition over R; (2) if $g_0(x_1, \dots, x_n)$ is superposition over R and if $g_i(x_{i,1}, \dots, x_{i,m_i})$ is either a superposition over R for any $i = 1, \dots, n$, or $x_{i,l} (1 \leqslant l \leqslant m_i)$, then a function $g_0(g_1(x_{1,1}, \dots, x_{1,m_1}), \dots, g_n(x_{n,1}, \dots, x_{n,m_n}))$ is superposition over R.

The closure (with respect to superposition) of a set R is the set of all superpositions over R. The closure of a set R is denoted by $[R]$. Obviously, $R \subseteq [R]$. A set R of k-valued logic functions is called a (functionally) closed class if $[R] = R$.

We call a set of functions Q generates a closed class R (or the class R is generated by a set of functions Q) if $[Q] = R$. If a closed class R is generated by a finite set of functions, then R is called finitely generated. If the set Q generates a closed class R, then we say the set Q is complete in the class R.

A set Q is called a precomplete class in the closed class R if $Q \subseteq R$, $[Q] \neq R$ and $[Q \cup \{f\}] = R$ holds for every function f that does not belong to Q but belongs to R.

Let $e_i^n(x_1, \dots, x_i, \dots, x_n)$ denote a k-valued logic function for any natural n and any i, $1 \geqslant i \geqslant n$. The values of e_i^n coincide with the values of the variable x_i. The functions e_i^n are called selector functions. The function $e_1^1(x)$ is also denoted by x.

2 Majority Functions and the Choice Function

Definition 1 If $\mu(x, y \dots, y) = \mu(y, x \dots, y) = \dots = \mu(y, y \dots, x) = y$, then a function $\mu(x_1, \dots, x_n)$ is called a majority function for any $n \geqslant 3$.

For example, $d_3(x_1, x_2, x_3) = x_1 \cdot x_2 \vee x_2 \cdot x_3 \vee x_1 \cdot x_3$ is a majority function, where $x \cdot y = \min(x, y)$, $x \vee y = \max(x, y)$.

The set of all functions of no more than s variables obtained from the function f by identifying variables is denoted by $A_s(f)$ for any function $f(x_1, \ldots, x_n)$ and any s, $s \geqslant 1$.

If $s > n$, then we set $A_s(f) = \{f\}$.

The idea of proving the following theorem belongs to K. Baker and A. Pixley [19].

Theorem 1 (see also [20]) *If $\mu(x_1, \ldots, x_{m+1})$ is a majority function of $m + 1$ variables, where $m \geqslant 2$, then we have $f \in [\{\mu\} \cup A_{k^m}(f)]$ for any function $f(x_1, \ldots, x_n) \in P_k$, where $k \geqslant 2$.*

Corollary 1 *Let $F = [F] \subseteq P_k$, $k \geqslant 2$, $\mu(x_1, \ldots, x_{m+1}) \in F$, μ be a majority function. Then F is finite generated.*

Let $E \subset E_k$. Let us consider a special case of the majority function.

Definition 2 A function $g(x_1, \ldots, x_n) \in P_k$, $k \geqslant 3$ is called a majority function on the set E if

$$g : E_k^n \to E$$

and $g(x, y \ldots, y) = g(y, x \ldots, y) = \cdots = g(y, y \ldots, x) = y$ holds for any $n \geqslant 3$ and for all $x, y \in E$.

Let us show that the property similar to one considered in Theorem 1 holds for a majority function on the set E.

Let P_k^E for any $k \geqslant 2$ denote the set of all functions from P_k taking values from the set E and all selector functions from P_k.

Theorem 2 (see also [21]) *Let the closed class $F \subseteq P_k$, $k \geqslant 3$ contain a function $g(x_1, \ldots, x_m)$, that is a majority on the set E. Then the class $F \cap [P_k^E]$ is finitely generated.*

Consider the another function called a choice function.

Definition 3 (*Choice function*). If $y = i$, where $i = 0, 1, \ldots, k-1$, then the function $\varphi(y, x_0, \ldots, x_{k-1}) = x_i$ is called the choice function in P_k for any $k \geqslant 2$.

For example, $xy \vee \bar{x}z$ is the choice function in P_2. A sufficient condition for the finiteness of class can be obtained via the choice function. And the following theorem gives an answer.

Theorem 3 *Let $\varphi(y, x_0, \ldots, x_{k-1})$ be a choice function, and F is an arbitrary closed class in P_k, $k \geqslant 2$ such, that $\varphi, 0, \ldots, k-1 \in F$. Then F is finitely generated.*

Proof This statement follows from the decomposition, that holds for any arbitrary function $f(x_1, \ldots, x_n)$ from the class F. It may be checked by substituting the constants $0, \ldots, k-1$ instead of the first variable. Let $f(x_1, \ldots, x_n)$ be an arbitrary function from the class F, then $f(x_1, \ldots, x_n) = \varphi(x_1, f(0, x_2, \ldots, x_n), \ldots, f(k-1, x_2, \ldots, x_n))$. Applying a similar decomposition for $f(0, x_2, \ldots, x_n)$, …, $f(k-1, x_2, \ldots, x_n)$ and further for all other subfunctions, we can obtain that $F \subseteq [\{\varphi(y, x_0, \ldots, x_{k-1}), 0, \ldots, k-1\}]$. The reverse inclusion is obvious.

2.1 Mutual Generation of the Choice Function and Majority Functions

Theorem 4 *Let the choice function $\varphi(y, x_0, \dots, x_{k-1})$ in P_k, $k \geqslant 2$ belong to the closed class F. Then F contains some majority function.*

Proof Let $\varphi_l(z_2, z_3)$, where $l = 0, \dots, k-1$ denote the function obtained from $\varphi(y, x_0, \dots, x_{k-1})$ by the following identification of the variables: $y = x_l = z_2$, and for all $j \neq l$, $j \in \{0, 1, \dots, k-1\}$ $x_j = z_3$.

Let

$$\mu(z_1, z_2, z_3) = \varphi(z_1, \varphi_0(z_2, z_3), \dots, \varphi_{k-1}(z_2, z_3)).$$

Since $\varphi(x, x, \dots, x) = x$ and $\varphi(y, x, \dots, x) = x$, then

$$\mu(z_2, z_1, z_1) = \varphi(z_2, \varphi(z_1, z_1, \dots, z_1), \dots, \varphi(z_1, z_1, \dots, z_1)) = \varphi(z_2, z_1, \dots, z_1) = z_1.$$

We emphasize that if $z_1 = i$ for any $i \in \{0, 1, \dots, k-1\}$, then $\varphi_i(z_2, z_1) = \varphi_i(z_2, i) = i$, $\varphi_i(z_1, z_2) = \varphi_1(i, z_2) = i$.

Then

$$\mu(z_1, z_2, z_1) = \varphi(z_1, \varphi_0(z_2, z_1), \dots, \varphi_{k-1}(z_2, z_1)) = z_1,$$

and

$$\mu(z_1, z_1, z_2) = \varphi(z_1, \varphi_0(z_1, z_2), \dots, \varphi_{k-1}(z_1, z_2)) = z_1.$$

Hence, $\mu(z_1, z_2, z_3)$ is a majority function.

The converse statement is not true: the choice function cannot be generated by an arbitrary majority function. However, there are majority functions whose closure the choice function belongs to.

Theorem 5 *The majority function μ in P_k, $k \geqslant 2$, generation a choice function $\varphi(y, x_0, \dots, x_{k-1})$ exists.*

Proof Let us define the function $\mu(x_1, \dots, x_{2k+2})$ as a majority function on sets $(x, y \dots, y), (y, x \dots, y), \dots, (y, y \dots, x)$, $x, y \in E_k$. Let $i, a \in \{0, 1, \dots, k-1\}$. Then suppose that function μ takes the value a on all sets such that $x_1 = x_2 = i$, $x_{2i+3} = x_{2i+4} = a$.

It is obvious that a function

$$\varphi(x_1, x_3, \dots, x_{2k+1}) = \mu(x_1, x_1, x_3, x_3, \dots, x_{2k+1}, x_{2k+1})$$

is a desired choice function.

We denote by M the set of all functions from P_k that are monotone with respect to the linear order $(0 < \dots < k-1)$.

Let for $i = 0, 1, \dots, k-1$

$$J_i(x) = \begin{cases} k-1, & x \geqslant i \\ 0, & x < i. \end{cases}$$

Let for $i = 0, 1, \dots, k-1$

$$j_i(x) = \begin{cases} 1, & x \geqslant i \\ 0, & x < i. \end{cases}$$

Definition 4 For any $k \geqslant 3$ a monotone function $\varphi_M \in P_k$ defined as

$$\varphi_M(y, x_0, x_1, \dots, x_{k-1}) = J_{k-1}(y)x_{k-1} \vee \cdots \vee J_1(y)x_1 \vee x_0,$$

is called a monotone choice function in P_k.

Theorem 6 *(property of a monotone choice function)*

$$M = [\{\varphi_M, 0, 1, \dots, k-1\}].$$

Proof Let $f(x_1, \dots, x_n)$ be an arbitrary function from M. Then $f(x_1, \dots, x_n) = J_{k-1}(x_1)f(k-1, x_2, \dots, x_n) \vee \cdots \vee J_1(x_1)f(1, x_2, \dots, x_n) \vee (0, x_2, \dots, x_n)$.

This equality is verified directly by substituting the values of the variable x_1 and using the definition of monotonic function f. Then we apply this expansion to all subfunctions. Hence, it follows that $M \subseteq [\{\varphi_M, 0, 1, \dots, k-1\}]$. The reverse inclusion is obvious.

Consider a set consisting of functions f for which there exists a number i: $1 \leqslant i \leqslant n$ such that $f(x_1, \dots, x_{i-1}, k-1, x_{i+1}, \dots, x_n) = k-1$ independently of the values of the other variables. This set of functions will be denoted by F_{k-1}. It is easy to see that this is a closed class.

Note that $\varphi_M \in F_{k-1}$. It is enough to consider a set in which the value of the variable x_0 is equal to $k-1$, and the values of the other variables are arbitrary. On any set with this property, the monotone choice function takes a value equal to $k-1$.

Theorem 7 *Let μ to be arbitrary majority function, then $\mu \notin [\varphi_M]$.*

Proof Since $[\varphi_M] \subseteq F_{k-1}$, it is sufficient to show that a majority function doesn't lie in F_{k-1}.

Let some majority function $\mu(x_1, \dots, x_n)$, $n \geqslant 3$ lie in F_{k-1}. Then, due to the property that all functions from F_{k-1} have, there is a number $i : 1 \leqslant i \leqslant n$ such that if the variable $x_i = k-1$, then $\mu(x_1, \dots, x_{i-1}, k-1, x_{i+1}, \dots, x_n) = k-1$ regardless of the values of the other numbers.

Consider a set $\tilde{\alpha}$ such that $\alpha_i = k-1$, and $\alpha_j = 0$ for all $j \neq i$, $j = 0, 1, \dots, n$. Then, on the one hand $\mu(\tilde{\alpha}) = 0$, since μ is a majority function, and on the other hand $\mu(\tilde{\alpha}) = k-1$, since μ is contained in F_{k-1}. We've got a contradiction.

A monotone choice function cannot be generated by an arbitrary monotone majority function. However, there are monotonic majority functions, the closure of which belongs to a monotone choice function.

Theorem 8 *A monotone majority function μ in P_k generating a monotone choice function $\varphi_M(y, x_0, \dots, x_{k-1})$ exists.*

Proof Set $k = 3$. Define the function $\mu(x_1, \dots, x_8)$ as a majority on the following sets: $(x, y \dots, y)$, $(y, x \dots, y)$, $\dots$, $(y, y \dots, x)$, $x, y \in E_3$. Then set function $\mu = a$ for any $a, b, c \in \{0, 1, 2\}$ on all sets such that $x_1 = x_2 = 0$, $x_3 = x_4 = a$; set function $\mu = \max(a, b, c)$ on all sets such that $x_1 = x_2 = 1$, $x_3 = x_4 = a$, $x_5 = x_6 = b$; and set μ equal to $\max(a, b, c)$ on all sets where $x_1 = x_2 = 2$, $x_3 = x_4 = a$, $x_5 = x_6 = b$, $x_7 = x_8 = c$.

On the other sets, we can redefine the function by monotony. It's clear that $\varphi(x_1, x_3, x_5, x_7) = \mu(x_1, x_1, x_3, x_3, x_5, x_5, x_7, x_7)$ is the desired monotone choice function.

The majority function of $2n + 2$ variables is constructed similarly. By pairwise identification of variables (see $k = 3$) we obtain a function of $n + 1$ variable. That function is a desired monotonic choice function.

3 Description of Classes, that Include a Class of Unary Functions, or Pre-complete Class of Unary Functions

Firstly, we need the following definitions and notation.

We say that a function $f(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n)$ from P_k essentially depends on the variable x_i, if there are such values $a_1, a_2, \dots, a_{i-1}, a_{i+1}, \dots, a_n \in E_k$ of variables $x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n$ such that

$$h(x) = f(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_n)$$

doesn't equal to the constant identically.

In this case, the variable x_i is called essential. A variable is called dummy if the function $f(x_1, \dots, x_n)$ does not depend on it essentially.

Let $f, g \in P_k$. We say that $f = g$ if one of them can be obtained from the other by adding or removing dummy variables.

Let $\tilde{x}$ to denote the set of numbers $x_1, \dots, x_n$, $n \geqslant 1$.

Let F be a closed class in P_k, then $F(n)$ is the set of all functions from F that depend on the variables $x_1, \dots, x_n$. $F^{(n)}$ is the set of all functions F taking at most n values. $CR(F)$ is the set of all precomplete classes in the closed class $F \subseteq P_k$. PS_k is the set of all unary functions taking exactly k values.

Definition 5 A function $f(x_1, \dots, x_n)$ that takes no more than two values is called quasilinear if for any number of the variable i, where $(1 \leqslant i \leqslant n)$, and any two elements $\alpha, \beta \in E_k$ one of two following relations holds:
either for any $\gamma_1, \dots, \gamma_{i-1}, \gamma_{i+1}, \dots, \gamma_n \in E_k$

$$f(\gamma_1, \dots, \gamma_{i-1}, \alpha, \gamma_{i+1}, \dots, \gamma_n) = f(\gamma_1, \dots, \gamma_{i-1}, \beta, \gamma_{i+1}, \dots, \gamma_n),$$

or for any $\gamma_1, \ldots, \gamma_{i-1}, \gamma_{i+1}, \ldots, \gamma_n \in E_k$

$$f(\gamma_1, \ldots, \gamma_{i-1}, \alpha, \gamma_{i+1}, \ldots, \gamma_n) \neq f(\gamma_1, \ldots, \gamma_{i-1}, \beta, \gamma_{i+1}, \ldots, \gamma_n).$$

The set of all quasilinear functions in P_k will be denoted by LQ_k.

The problem of enumerating of all closed classes of k-valued logic containing all functions of one variable was solved by G.A. Burle [15].

Theorem 9 (see [15]) *Functionally closed classes of functions of k-valued logic that contain all functions of one variable are following classes: $P_k(1)$, $LQ_k \cup P_k(1)$, $P_k^{(2)} \cup P_k(1), \ldots, P_k^{(k-1)} \cup P_k(1)$, P_k and are only them.*

The k-valued classes $P_k(1)$, $LQ_k \cup P_k(1)$, $P_k^{(2)} \cup P_k(1), \ldots, P_k^{(k-1)} \cup P_k(1)$, P_k are called Burle's classes. The proof of the Theorem 9 is based on the following property of Burle's classes: $P_k(1)$ is precomplete class in $LQ_k \cup P_k(1)$, $LQ_k \cup P_k(1)$ is precomplete class in $P_k^{(2)} \cup P_k(1)$, $P_k^{(l)} \cup P_k(1)$ is precomplete class in $P_k^{(l+1)} \cup P_k(1)$ for $2 \leqslant l \leqslant k-2$, $P_k^{(k-1)} \cup P_k(1)$ is precomplete class in P_k. It's easy to see from this property that all Burle's classes are finitely generated. In other words, all closed classes of k -valued logic containing all functions of one variable are finitely generated.

The problem of finding functionally closed classes containing a given class of functions of one variable was formulated by S.G. Gindikin (as it was point out by G.A. Burle). For classes containing precomplete classes of the set of all one-place functions, this problem was solved by M.A. Posypkin [16].

Theorem 10 ([16]) *Let $k \geqslant 3$ and $CR(PS_k) = \{V_1, \ldots, V_{r_k}\}$. Then the set $CR(P_k(1))$ consists of classes $P_k^{(k-2)} \cup PS_k$, $V_1 \cup P_k^{(k-1)}, \ldots, V_{r_k} \cup P_k^{(k-1)}$.*

Corollary 2 *Let $G \in CR(P_k(1))$, $k \geqslant 3$. Then $1 \in G$.*

Proof The Theorem 10 describes the set $CR(P_k(1))$. Let $CR(PS_k) = \{V_1, \ldots, V_{r_k}\}$. Then the set $CR(P_k(1))$ consists of classes $P_k^{(k-2)} \cup PS_k$, $V_1 \cup P_k^{(k-1)}, \ldots, V_{r_k} \cup P_k^{(k-1)}$.

(1) The class $P_k^{(k-2)} \cup PS_k$ contains one, since $1 \in P_K^{(k-2)}$.

(2) Classes $V_1 \cup P_K^{(k-1)}, \ldots, V_{r_k} \cup P_k^{(k-1)}$ contain one, since $1 \in P_k^{(k-1)}$.

Remark In the case $k = 3$, the class $P_k^{(k-2)} \cup PS_K$ coincides with the class of all linear unary functions $L_3(1)$.

An overlattice of a class G is the set of all classes $F \subseteq P_k$ such that $G \subseteq F$.

A complete description of the overlattices of precomplete in $P_k(1)$ classes for $k = 3$ is described by the following theorem.

Theorem 11 ([16]) *1. Let $V \in CR(PS_3)$. Then the closed class $V \cup P_3^{(2)}(1)$ has a finite overlattice consisting of the following classes:*
$V \cup P_3^{(2)}(1)$, $P_3(1)$;

$V \cup P_3^{(2)}(1) \cup LQ_3$, $P_3(1) \cup LQ_3$;
$V \cup P_3^{(2)}(1) \cup P_3^{(2)}$, $P_3(1) \cup P_3^{(2)}$;
P_3.
2. *The class* $L_3(1)$ *has a finite overlattice consisting of the following closed classes:* $L_3(1)$, L_3, $P_3(1)$, $LQ_3 \cup P_3(1)$, $P_3^{(2)} \cup P_3(1)$, P_3.

4 Conclusion

In this paper, the problem of verifying the finite generation of classes containing some subclass of functions of one variable has been considered. We also give a description of the over lattices of classes in P_k containing some precomplete class of unary functions. The finite generation of overlattices has been proved. The completeness problem for this operator has a solution. It is possible to describe the sublattice of closed classes in the general case of closure of functions with respect to the classical superposition operator.

In further papers relying on the above results, we plan to show that any class containing any of the precomplete classes of the set of unary functions in P_3 is finitely generated.

Funding This publication was prepared with the support of the *Russian Foundation for Basic Research* according to the research project **No. 20-01-00575_A**.

References

1. Bykovsky, Al. Yu., Heterogeneous Network Architecture For Integration Of Ai And Quantum Optics By Means Of Multiple-valued Logic, Quantum Rep. vol.2., pp. 126–165. (2020) https://doi.org/10.3390/Quantum2010010.
2. Kalimulina, E.Y.: Application of Multi-Valued Logic Models in Traffic Aggregation Problems in Mobile Networks. IEEE 15th International Conference on Application of Information and Communication Technologies (AICT). pp. 1–6 (2021). https://doi.org/10.1109/AICT52784.2021.9620244.
3. Kalimulina, E.Y.: Analysis of Unreliable Open Queueing Network with Dynamic Routing. Distributed Computer and Communication Networks. Communications in Computer and Information Science, vol 700. Springer, Cham. (2017). https://doi.org/10.1007/978-3-319-66836-9_30.
4. Esin, A.; Yavorskiy, R.; Zemtsov, N. Brief Announcement Monitoring Of Linear Distributed Computations. In Distributed Computing; Dolev, S., Ed.; Lecture Notes in Computer Science; Springer: Berlin/Heidelberg, Germany, Volume 4167. (2006).
5. Kalimulina, E.Y.: Lattice Structure of Some Closed Classes for Three-Valued Logic and Its Applications. MDPI Mathematics. 10, 94. (2022). https://doi.org/10.3390/math10010094.
6. Kalimulina, E.Y.: Lattice Structure of Some Closed Classes for Non-binary Logic and Its Applications. Mathematical Methods for Engineering Applications. ICMASE-2021. Springer Proceedings in Mathematics & Statistics, vol 384. Springer, Cham. (2022). https://doi.org/10.1007/978-3-030-96401-6_2.

7. S.V. Yablonskiy, G.P. Gavrilov And V.B. Kudryavtsev "Logical Algebra Functions And Post Classes". Moscow.: "Nauka", 1966.
8. B. Parhami and M. McKeown, Arithmetic with binary-encoded balanced ternary numbers, 2013 Asilomar Conference on Signals, Systems and Computers, 2013, pp. 1130–1133, https://doi.org/10.1109/ACSSC.2013.6810470.
9. Post E.L., Introduction to a general theory of elementary propositions, American Journ. Mathem. V.43, 1921, p. 163–185.
10. Post E.L., Two-valued iterative systems of mathematical logic, Annals of Math. Studies. Princeton Univ. Press, 1942, V.5.
11. Yanov Yu. I., Yanov Yu. I.,On the existence of k-valued closed classes without a finite basis, Dokl. Akad. Nauk SSSR, 1959, 127, 1, p. 44–46.
12. Ugol'nkov, A. B., Some Problems in the Field of Mutivalued Logics, in Proc. X Int. Workshop "Discrete Mathematics and its Applications," Moscow, MSU, February 1–6, pp. 18–34 (2010).
13. Podol'ko, D.K., A family of classes of functions closed with respect to a strengthened superposition operation. Moscow Univ. Math. Bull. 70, 79–83 (2015). https://doi.org/10.3103/S0027132215020059.
14. Starostin, M.V., Implicit completeness criterion in three-valued logic in terms of maximal classes, (2021) arXiv:2103.16631pdf.
15. Burle G.A., Classes of k-valued logic containing all functions of one variable, Discrete analysis, iss. 10 (1967).
16. Posypkin M. A., On closed classes containing precomplete classes of the set of all one-place functions, Vestnik Moskov. Univ. Ser. 1. Mat. Mekh., 1997, no. 4, 58–59.
17. Ugolnikov, A.B. On closed Post classes, Proceedings of universities, Mathematics, iss. 7, pp. 79–88 (1988).
18. Enderton Herbert B., First-Order Logic (Ch.2), A Mathematical Introduction to Logic (Second Edition), Academic Press. (2001) https://doi.org/10.1016/B978-0-08-049646-7.50008-4.
19. Baker, K.A., Pixley, A.F., Polynomial interpolation and the Chinese remainder theorem for algebraic systems, Math. Zeitschrift, 143, 165–174 (1975).
20. Marchenkov S. S, Closed classes of Boolean functions. - Fizmatlit, 2000. - S. 130–130.
21. Demetrovics J., Hannak L., and L. Ronyai. On algebraic properties of monotone clones, Order 3, 219–225 (1986).