On semirecursive sets

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Semirecursive sets were defined by Carl Groos Jockusch in the 1960s. (Jockusch, 1965; Jockusch, 1968.)

A set $A \subseteq \omega$ is *semirecursive* iff there is a recursive *selector* function f of two variables, where for any $x, y \in \omega$ the following conditions are satisfied: $f(x, y) \in \{x, y\}$ and if $x \in A$ or $y \in A$, then $f(x, y) \in A$.

If a set A is semirecursive with selector function $f(x, y)$, then its complement $\omega \backslash A$ is also semirecursive with selector function

$$g(x, y) = f(y, x).$$

We assume an **additional condition:**

the selector function is polynomial-time computable.

Semirecursive sets satisfying this condition are also called p-selective. In general, this condition does not limit the computational complexity of the recognition problem. But if $P \neq NP$, then no semirecursive set with polynomial-time computable selector function is NP-complete. (Selman, 1979; Selman, 1982.)

Example. Let us fix some numbering $\nu : \omega \to \mathbb{Q}$ of rational numbers so that for any two numbers $x, y \in \omega$ the condition $\nu(x) \leqslant \nu(y)$ can be checked in polynomial time. For a real number α, let us denote by A_α the set of numbers of those rationals that are greater than α.
The set A_α is semirecursive. Its selector function is polynomial-time computable.

$$f(x,y) = \begin{cases} x, & \nu(x) > \nu(y) \\ y, & \nu(x) \leqslant \nu(y). \end{cases}$$

However, depending on the choice of α, the complexity of recognizing the set A_α can be arbitrarily high.

On the other hand, the set A_α belongs to the non-uniform class $P/poly$ because for any finite set $B \in \omega$, it suffices to specify one rational approximation $\beta \approx \alpha$, comparison with which determines the intersection $B \cap A_\alpha$. If the set B is not entirely contained in A_α, then $\nu(\beta)$ can be chosen from the relative complement $B \backslash A_\alpha$. Therefore, this number is bounded from above.

Theorem 1. *Let a non-empty set U of propositional formulae be closed under the substitution of a Boolean constant, i. e., False $\perp$ or True $\top$ instead of any variable. Moreover, let the numbering $\nu : \omega \to U$ be bijective and satisfy following conditions.*
(i) *For any formula $\varphi(p_1, \ldots, p_n) \in U$ the following inequality holds:*

$$\nu^{-1}(\varphi) \geqslant n.$$

(ii) *For any formula $\varphi(p_1, \ldots, p_n) \in U$, formulae obtained by substituting a constant instead of the variable p_n can be calculated in polynomial time.*
(iii) *Moreover, the numbers of the new formulae are less than the number of the original formula $\varphi(p_1, \ldots, p_n)$, i. e.,*

$$\nu^{-1}(\varphi(p_1, \ldots, p_{n-1}, \perp)) < \nu^{-1}(\varphi(p_1, \ldots, p_n)),$$

$$\nu^{-1}(\varphi(p_1, \ldots, p_{n-1}, \top)) < \nu^{-1}(\varphi(p_1, \ldots, p_n)).$$

If the set A of numbers of satisfiable formulae from U is semirecursive with a polynomial-time computable selector function, then the set A is polynomial-time decidable.

The class RP (Randomized Polynomial-time) consists of sets A that are recognized by probabilistic polynomial-time algorithms with the constraint: if $x \in A$, then x is accepted with probability at least $1/2$, otherwise x is accepted with probability 0.

Since there is a polynomial-time computable numbering of pairs of finite graphs given by adjacency matrices, the problem of recognizing isomorphism reduces to recognizing some set GraphIso $\subset \omega$ consisting of the numbers of pairs of isomorphic graphs.

Theorem 2. *If the set* GraphIso *is semirecursive with a polynomial-time computable selector function, then*

$$\text{GraphIso} \in RP.$$

For two graphs (G, H), we associate two pairs of graphs (G', G'') and (G', H'), where graphs G', G'', and H' are obtained by independent random permutations of the vertices of the original graphs. Graphs G' and G'' are always isomorphic to each other. The input is accepted if the value of the selector function encodes the pair (G', H').

It is also possible to consider the problem of isomorphism of any finite structures, where operations and relations are defined by tables.

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Thank you!