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POLYNOMIAL REFORMULATION OF THE KUO CRITERIA FOR V-SUFFICIENCY OF MAP-GERMS

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Dedicated to Peter E. Kloeden on his 60th birthday

Abstract. In the paper a set of necessary and sufficient conditions for v-sufficiency (equiv. sv-sufficiency) of jets of map-germs $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^m, 0)$ is proved which generalize both the Kuiper-Kuo and the Thom conditions in the function case ($m = 1$) so as the Kuo conditions in the general map case ($m > 1$). Contrary to the Kuo conditions the conditions proved in the paper do not require to verify any inequalities in a so-called horn-neighborhood of the (a'priori unknown) set $f^{-1}(0)$. Instead, the proposed conditions reduce the problem on v-sufficiency of jets to evaluating the local Łojasiewicz exponents for some constructively built polynomial functions.

1. **Introduction.** In theory of dynamical systems and nonlinear analysis quite a number of problems depending on parameters require analyzing the structure of the set of solutions of nonlinear equations, the number of variables in which exceeds the number of equations. As a rule, arising equations are rather complicated for investigation and need to be simplified in one or another way. Clearly, such "simplification" may lead as to correct conclusions about the structure of solutions as to wrong ones. Often, small solutions of equations are of interest. In this case one of the most popular methods of simplification of equations is their truncation, when one casts out high order terms in power-series expansions of the corresponding equations. In the paper polynomial necessary and sufficient conditions are proved allowing to judge in which cases, in analysis of systems of real nonlinear equations of finite smoothness, truncation is permissible.

Given a map $f : \mathbb{R}^n \to \mathbb{R}^m$ with $f(0) = 0$, let us consider the set of solutions of the equation

$$f(x) = 0. \tag{1}$$

Even locally, this set is very complicated in general. As usual, the map f is called C^k-smooth if all its components have continuous partial derivatives up to the order k inclusive. If $f \in C^k$ in a neighborhood of the origin then for each $r \leq k$ it is defined the r-th Taylor polynomial $f^{(r)}(x)$ of $f(x)$ about the point $x = 0$ which will

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be called the r-*truncation* of $f(x)$. Transition from equation (1) to the truncated equation

$$f^{(r)}(x) = 0 \tag{2}$$

is similar to the first-approximation method in the theory of stability and to the method of studying bifurcations by the passage to linearized equations in nonlinear analysis, and so on. As demonstrates the next example the sets of solutions of equations (1) and (2) may be topologically different.

Example. Let us discard in the next equations

$$x_1^2 - 2x_1x_2^2 + x_1^4 + x_2^4 + x_2^8 = 0, \qquad x_1^2 - 2x_1x_2^2 + x_1^4 + x_2^4 - x_2^8 = 0$$

the terms of order higher than 4, that is perform the 4-truncation of the left-hand parts. Then the truncated equation

$$x_1^2 - 2x_1x_2^2 + x_1^4 + x_2^4 = (x_1 - x_2^2)^2 + x_1^4 = 0$$

has a single solution, $x_1 = x_2 = 0$. The first of the full equations also has the same single solution, $x_1 = x_2 = 0$, while the second of the full equations has a continuum of solutions, $x_1 = x_2^2$. Thus, truncation of equations is not always permissible.

Therefore it is natural to ask when the structure of the zero-set of the truncated map $f^{(r)}$ is similar to that of the full map f. This problem concerns the property of sufficiency of jets. Roughly speaking, sufficiency of jets is the property that all maps with the same truncation have the same structure.

Following to [3] we recall briefly some definitions and results on sufficiency of jets. Let $\mathscr{E}_{[k]}(n,m)$ denote the set of C^k map-germs $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^m, 0)$. Given $r \le k$, let $j^r f(0)$ denote the r-jet of $f \in \mathscr{E}_{[k]}(n,m)$ at $0 \in \mathbb{R}^n$ which can be identified with the polynomial $f^{(r)}$, and let $J^r(n,m)$ denote the set of r-jets in $\mathscr{E}_{[k]}(n,m)$. We say $f, g \in \mathscr{E}_{[k]}(n,m)$ are C^0-equivalent, if there is a local homeomorphism $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that $f = g \circ h$. We further say $f, g \in \mathscr{E}_{[k]}(n,m)$ are v-equivalent (resp. sv-equivalent), if $f^{-1}(0)$ is homeomorphic to $g^{-1}(0)$ as germs at $0 \in \mathbb{R}^n$ (resp. there is a local homeomorphism $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that $h(f^{-1}(0)) = g^{-1}(0)$). Given $r \le k$, we call an r-jet $w \in J^r(n,m)$ C^0-sufficient (resp. v-sufficient, sv-sufficient) in $\mathscr{E}_{[k]}(n,m)$, if any two maps $f, g \in \mathscr{E}_{[k]}(n,m)$ with $j^r f(0) = j^r g(0) = w$ are C^0-equivalent (resp. v-equivalent, sv-equivalent).

Clearly, C^0-sufficiency of jets implies sv-sufficiency, while the latter implies v-sufficiency. In fact, according to D.J.A. Trotman and L.C. Wilson [25], v-sufficiency is equivalent to sv-sufficiency.

Concerning C^0-sufficiency of jets in the function case (i.e. $m = 1$), we have

Theorem 1.1 (N. Kuiper [17], T.-C. Kuo [18], J. Bochnak & S. Łojasiewicz [5])**.** *For $f \in \mathscr{E}_{[r]}(n,1)$, the jet $j^r f(0)$ is C^0-sufficient in $\mathscr{E}_{[r]}(n,1)$ if and only if there are positive numbers C, ε such that*

$$|\operatorname{grad} f(x)| \ge C|x|^{r-1} \quad \textit{for} \quad |x| < \varepsilon. \tag{3}$$

For $f \in \mathscr{E}_{[r+1]}(n,1)$, the jet $j^r f(0)$ is C^0-sufficient in $\mathscr{E}_{[r+1]}(n,1)$ if and only if there are numbers $C, \delta, \varepsilon > 0$ such that

$$|\operatorname{grad} f(x)| \ge C|x|^{r-\delta} \quad \textit{for} \quad |x| < \varepsilon. \tag{4}$$

K. Bekka and S. Koike [3] proved that the Kuiper-Kuo condition (3) is equivalent to the following Thom condition: there are numbers $K, \varepsilon > 0$ such that

$$\sum_{i<j} \left| x_i \frac{\partial f}{\partial x_j} - x_j \frac{\partial f}{\partial x_i} \right|^2 + |f(x)|^2 \geq K|x|^{2r} \quad \text{for} \quad |x| < \varepsilon. \tag{5}$$

Verification of the Kuiper-Kuo conditions (3) and (4), so as of the Thom condition (5), may be reduced to the problem on evaluation of the rate of growth of a polynomial about one of its roots, which is equivalent to calculation of the so-called local Łojasiewicz exponents of a polynomial. Recall, that according to the Łojasiewicz theorem [21, 22, 23] for any polynomial $p : \mathbb{R}^n \to \mathbb{R}$ with $p(0) = 0$ there are constants $C, \kappa > 0$ such that

$$|p(x)| \geq C|x|^\kappa$$

in a neighborhood of the zero root. The least κ for which the above inequality holds is called the *local Łojasiewicz exponent* for p and is denoted by $\mathscr{L}_0(p)$. If the zero root of p is isolated then such a least value of κ exists and is rational [11, 21, 22, 23]. Moreover, in this case $\mathscr{L}_0(p) \leq (d-1)^n + 1$ [12] where d is the degree of p. There is quite a number of publications devoted to evaluation of the Łojasiewicz exponent, see, e.g., [1, 6, 7, 9, 10, 12, 13, 20] and the bibliography therein.

Concerning v-sufficiency (equiv. sv-sufficiency) of jets in the general map case (i.e. $n \geq m$ but otherwise arbitrary), we have

Theorem 1.2 (T.-C. Kuo [19])**.** *For $f = (f_1, f_2, \ldots, f_m) \in \mathscr{E}_{[r]}(n, m)$ with $n \geq m$, the jet $j^r f(0)$ is v-sufficient (equiv. sv-sufficient) in $\mathscr{E}_{[r]}(n, m)$ if and only if there are numbers $C, \varepsilon, \sigma > 0$ such that*

$$\mathscr{D}(\operatorname{grad} f_1^{(r)}(x), \operatorname{grad} f_2^{(r)}(x), \ldots, \operatorname{grad} f_m^{(r)}(x)) \geq C|x|^{r-1} \tag{6}$$

in $\mathscr{H}_r(f^{(r)}; \sigma) \cap \{|x| < \varepsilon\}$.

For $f = (f_1, f_2, \ldots, f_m) \in \mathscr{E}_{[r+1]}(n, m)$ with $n \geq m$, the jet $j^r f(0)$ is v-sufficient (equiv. sv-sufficient) in $\mathscr{E}_{[r+1]}(n, m)$ if and only if for any polynomial map $g = (g_1, g_2, \ldots, g_m)$ of degree $r+1$ satisfying $j^r g(0) = j^r f(0)$ there are numbers C, δ, ε, $\sigma > 0$, all depending on g, such that

$$\mathscr{D}(\operatorname{grad} f_1^{(r)}(x), \operatorname{grad} f_2^{(r)}(x), \ldots, \operatorname{grad} f_m^{(r)}(x)) \geq C|x|^{r-\delta} \tag{7}$$

in $\mathscr{H}_{r+1}(g; \sigma) \cap \{|x| < \varepsilon\}$.

In the above theorem, $\mathscr{H}_s(f; \sigma)$ denotes the horn-neighbourhood of $f^{-1}(0)$,

$$\mathscr{H}_s(f; \sigma) = \{x \in \mathbb{R}^n : |f(x)| < \sigma|x|^s\},$$

and

$$\mathscr{D}(v_1, \ldots, v_m) = \min_i \{\text{distance of } v_i \text{ to } V_i\} \tag{8}$$

where V_i is the span of the v_j's, $j \neq i$.

Unfortunately, verification of the Kuo conditions (6) and (7) is not as "simple" as verification of the Kuiper-Kuo conditions (3), (4) or the Thom condition (5). The first problem here, not the major one, is that the function $\mathscr{D}(v_1, \ldots, v_m)$ is not defined explicitely, by a "simple" formula. This causes problems in practical evaluation of $\mathscr{D}(v_1, \ldots, v_m)$. The second problem, which is more serious, is that one need evaluate the values of $\mathscr{D}(\operatorname{grad} f_1^{(r)}(x), \operatorname{grad} f_2^{(r)}(x), \ldots, \operatorname{grad} f_m^{(r)}(x))$ not in a neighborhood of the origin but in horn-neighbourhoods of the sets $(f^{(r)})^{-1}(0)$ or $g^{-1}(0)$ which are a'priory unknown in general. At last, in the case of v-sufficiency in $\mathscr{E}_{[r+1]}(n,m)$ one need to verify condition (7) not for a single horn-neighbourhood but for a variety of horn-neighbourhoods defined for infinite number of polynomial maps g of degree $r+1$ satisfying $j^r g(0) = j^r f(0)$.

Not knowing about the works of N. Kuiper, T.-C. Kuo, J. Bochnak and S. Łojasiewicz, the author had sketched in [15], and later proved in [4, Ch. 8], a bit different criteria (in a bit different terms) for *sv*-sufficiency of map-germs.

Theorem 1.3 (V.S. Kozyakin [15], [4, Ch. 8])**.** *For $f \in \mathscr{E}_{[r]}(n,m)$ with $n \geq m$, the jet $j^r f(0)$ is sv-sufficient in $\mathscr{E}_{[r]}(n,m)$, $r \geq 2$, if and only if there is a number $q > 0$ such that*

$$|f^{(r)}(x)|^2|y|^2 + |(df^{(r)})^*(x)y|^2|x|^2 \geq q|x|^{2r}|y|^2 \tag{9}$$

for small x and all y.

For $f \in \mathscr{E}_{[r+1]}(n,m)$ with $n \geq m$, the jet $j^r f(0)$ is sv-sufficient in $\mathscr{E}_{[r+1]}(n,m)$, $r \geq 1$, if and only if

$$\frac{|f^{(r)}(x)|^2|y|^2 + |(df^{(r)})^*(x)y|^2|x|^2}{|x|^{2r+2}|y|^2} \to \infty \tag{10}$$

as $x \to 0$, $x \neq 0$, uniformly with respect to $y \neq 0$.

In the above theorem $(df)^*(x)$ denotes the matrix conjugate to $df(x)$. Clearly, the matrix $(df)^*(x)$ consists of m column vectors $\operatorname{grad} f_j(x)$, $j = 1, 2, \ldots, m$. If the norm $|\cdot|$ in the above theorem is Euclidean then all the functions in (9), (10) are polynomial. Hence, to verify conditions (9), (10) one can apply the technique of estimating the Łojasiewicz exponent mentioned above. As can be proved by standard reasoning [11, 22] condition (10) is equivalent, in fact, to the following condition: there are numbers $q, \delta > 0$ such that

$$|f^{(r)}(x)|^2|y|^2 + |(df^{(r)})^*(x)y|^2|x|^2 \geq q|x|^{2r+2-2\delta}|y|^2 \tag{11}$$

for small x and all y, which is similar to (7).

Remark that the technique used in proving Theorem 1.3 is much the same that used in proving Theorem 1.2. Moreover, since both theorems, Theorem 1.2 and Theorem 1.3, provide necessary and sufficient conditions for *sv*-sufficiency of map-germs under the same assumptions then condition (6) must be equivalent to (9) while condition (7) must be equivalent to (11). Nevertheless, no direct proofs of such an equivalence, to the best of the author's knowledge, are known.

The aim of the present paper is quite modest. First, we would like to reformulate the Kuo conditions (6), (7) in such a way to avoid verification of any inequalities in a horn-neighborhood of the a'priori unknown set $f^{-1}(0)$. Second, we would like to replace the function $\mathscr{D}(\cdot)$ in (6), (7) by something easier computable in applications.

To implement this program we firstly formulate in Section 2 "qualified" versions for the notions of regularity of the set of small non-zero solutions of equation (1) and transversality of this set to small spheres. The corresponding notions will play a key role in the further considerations. In Lemma 2.1 we show also that for polynomial maps these regularity and transversality conditions are equivalent to each other. Then, in Theorem 3.1 we formulate a set of equivalent to each other conditions (18), (19) for *v*-sufficiency (equiv. *sv*-sufficiency) of map-germs which are direct (and trivial) generalization of conditions (9), (11) from Theorem 1.3. Here we demonstrate also that these conditions may be treated as a natural generalization of both the Kuo conditions (6), (7) and the Thom conditions (5). At last, in Section 4 to prove Theorem 3.1 we establish equivalence between conditions (18), (19) and the Kuo conditions (6), (7).

2. **Qualified regularity and transversality.** Before to start formulating main results of the paper, let us introduce some notions.

From now on $\langle \cdot, \cdot \rangle$ stands for the Euclidean scalar product in $\mathbb{R}^n$, and $|\cdot|$ denotes the corresponding norm. If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a smooth map then $(df)^*(x)$ is the matrix conjugate to $df(x)$. Clearly, the matrix $(df)^*(x)$ consists of m column vectors $\operatorname{grad} f_j(x)$, $j = 1, 2, \ldots, m$.

Given a map-germ $f : \mathbb{R}^n \to \mathbb{R}^m$ with $f(0) = 0$ and an integer $p \geq 1$ let us consider two auxiliary functions of variables $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$:

$$\mathscr{R}_p(f; x, y) = |f(x)|^p |y|^p + |(df)^*(x)y|^p |x|^p \tag{12}$$

and

$$\mathscr{T}_p(f; x, y) = |f(x)|^p |y|^p + |(df)^*(x)y|^p |x|^p - |\langle (df)^*(x)y, x\rangle|^p. \tag{13}$$

Note, that both the functions $\mathscr{R}_p(f; x, y)$ and $\mathscr{T}_p(f; x, y)$ are homogeneous in y. These functions are polynomials in x and y if f is a polynomial and p is even.

Positivity of the function $\mathscr{R}_p(f; x, y)$ for $y \neq 0$ and small $x \neq 0$ means that $|(df)^*(x)y| > 0$ for each $y \neq 0$ and all small non-vanishing solutions x of equation (1), that is the derivative of the map $f(x)$ is regular on small solutions x of equation (1). So, the inequality $\mathscr{R}(f; x, y) > 0$ for $x, y \neq 0$ may be treated as a condition of regularity [8] of small non-zero solutions of equation (1). Therefore the relation

$$\mathscr{R}_p(f; x, y) \geq C|x|^{pq}|y|^p, \tag{14}$$

valid for small x and all y with some $C, q > 0$, can be called the *condition of qualified regularity* of small non-zero solutions of equation (1).

Similarly, positivity of the function $\mathscr{T}_p(f; x, y)$ for $y \neq 0$ and small $x \neq 0$ means that $|(df)^*(x)y| \cdot |x| > |\langle (df)^*(x)y, x\rangle|$ for each $y \neq 0$ and all small non-vanishing solutions x of equation (1). The latter inequality is an algebraic representation of the fact that the set of small solutions of equation (1) is transversal to any small sphere $|x| = \varepsilon$ [8]. Therefore the relation

$$\mathscr{T}_p(f; x, y) \geq C|x|^{pq}|y|^p, \tag{15}$$

valid for small x and all y with some $C, q > 0$, can be called the *condition of qualified transversality* of small non-zero solutions of equation (1) to the small spheres $|x| = \varepsilon$.

If the map f is polynomial then the functions $\mathscr{R}_p(f; x, y)$ and $\mathscr{T}_p(f; x, y)$ are comparable, in a natural sense, for small x.

Lemma 2.1. *If a map $f : \mathbb{R}^n \to \mathbb{R}^m$ with $f(0) = 0$ is polynomial then for any $p \in \mathbb{N}$ (i.e. p is a natural number) there is a constant $\mu_p > 0$ such that*

$$2^{1-p}\mathscr{R}_1^p(f; x, y) \leq \mathscr{R}_p(f; x, y) \leq 2\mathscr{R}_1^p(f; x, y), \tag{16}$$

$$\mu_p \mathscr{R}_p(f; x, y) \leq \mathscr{T}_p(f; x, y) \leq \mathscr{R}_p(f; x, y) \tag{17}$$

for small x and all y.

If a map f is polynomial then by Lemma 2.1 the set of small non-zero solutions of equation (1) is regular if and only if it is transversal to the small spheres $|x| = \varepsilon$, which is a well known fact [24]. In this case the set of small non-zero solutions of equation (1) is also qualifiedly regular (with some parameter $q > 0$) if and only if it is qualifiedly transversal (with the same parameter q) to the small spheres $|x| = \varepsilon$. Moreover, all conditions (14) and (15) with a given $q > 0$ but different $p \in \mathbb{N}$ are equivalent to each other.

3. **Main results.**

Theorem 3.1. *For $f \in \mathscr{E}_{[r]}(n,m)$ with $n \geq m$, the jet $j^r f(0)$ is v-sufficient (equiv. sv-sufficient) in $\mathscr{E}_{[r]}(n,m)$ if and only if for any $p \in \mathbb{N}$ there is a number $q > 0$ such that*

$$\mathscr{K}(f^{(r)}; x, y) \geq q|x|^{pr}|y|^p \tag{18}$$

for small x and all y where $\mathscr{K}$ is any one of the functions $\mathscr{R}_p$ or $\mathscr{T}_p$.

For $f \in \mathscr{E}_{[r+1]}(n,m)$ with $n \geq m$, the jet $j^r f(0)$ is v-sufficient (equiv. sv-sufficient) in $\mathscr{E}_{[r+1]}(n,m)$ if and only if for any $p \in \mathbb{N}$

$$\frac{\mathscr{K}(f^{(r)}; x, y)}{|x|^{pr+p}|y|^p} \to \infty \tag{19}$$

as $x \to 0$, $x \neq 0$, uniformly with respect to $y \neq 0$, where $\mathscr{K}$ is any one of the functions $\mathscr{R}_p$ or $\mathscr{T}_p$.

Clearly, each function $\mathscr{K}(f^{(r)}; x, y)$ in Theorem 3.1 is a polynomial in x and y, homogeneous in y. This allows to simplify the formulation of Theorem 3.1 in the function case ($m = 1$). Set

$$\mathscr{R}_p^*(f^{(r)}; x) = (f^{(r)}(x))^p + |\operatorname{grad} f^{(r)}(x)|^p |x|^p$$

and

$$\mathscr{T}_p^*(f^{(r)}; x) = (f^{(r)}(x))^p + |\operatorname{grad} f^{(r)}(x)|^p |x|^p - |\langle \operatorname{grad} f^{(r)}(x), x\rangle|^p.$$

Theorem 3.2. *For $f \in \mathscr{E}_{[r]}(n,1)$, the jet $j^r f(0)$ is v-sufficient (equiv. sv-sufficient) in $\mathscr{E}_{[r]}(n,1)$ if and only if for any $p \in \mathbb{N}$ there is a number $q > 0$ such that*

$$\mathscr{K}^*(f^{(r)}; x) \geq q|x|^{pr} \tag{20}$$

for small x where $\mathscr{K}^*$ is any one of the functions $\mathscr{R}_p^*$ or $\mathscr{T}_p^*$.

For $f \in \mathscr{E}_{[r+1]}(n,1)$, the jet $j^r f(0)$ is v-sufficient (equiv. sv-sufficient) in $\mathscr{E}_{[r+1]}(n,1)$ if and only if for any $p \in \mathbb{N}$

$$\frac{\mathscr{K}^*(f^{(r)}; x)}{|x|^{pr+p}} \to \infty \tag{21}$$

as $x \to 0$, $x \neq 0$, where $\mathscr{K}^*$ is any one of the functions $\mathscr{R}_p^*$ or $\mathscr{T}_p^*$.

Remark 1. Given an analytic map-germ $h : \mathbb{R}^n \to \mathbb{R}^1$ with $h(0) = 0$ and $0 < \theta < 1$, then the following *Bochnak-Łojasiewicz inequality*

$$|\operatorname{grad} h(x)| \cdot |x| \geq \theta |h(x)|$$

holds for small x [5, Lem. 2]. Hence for the polynomial $f^{(r)}(x)$ in Theorem 3.2 there is a number $\gamma > 0$ such that

$$\gamma \mathscr{R}_1^*(f^{(r)}; x) \leq |\operatorname{grad} f^{(r)}(x)| \cdot |x| \leq \mathscr{R}_1^*(f^{(r)}; x)$$

for small x. The latter inequalities mean that conditions (20) and (21) with $\mathscr{K}^* = \mathscr{R}_1^*$ are equivalent to the Kuiper-Kuo conditions (3) and (4), respectively.

So, conditions (18) and (19) in Theorem 3.1 may be treated as a natural generalization of the Kuiper-Kuo conditions (3) and (4), respectively.

Remark 2. Direct verification shows that

$$\mathscr{T}_2^*(f^{(r)}; x) = \sum_{i<j} \left| x_i \frac{\partial f^{(r)}}{\partial x_j} - x_j \frac{\partial f^{(r)}}{\partial x_i} \right|^2 + |f^{(r)}(x)|^2,$$

and condition (20) with $\mathscr{K}^* = \mathscr{T}_2^*$ is nothing else than the Thom condition (5) for the map $f^{(r)}$.

So, conditions (18) and (19) in Theorem 3.1 may be treated also as a natural generalization for the map case ($m > 1$) of the Thom condition (5).

As an example of application of the formulated above theorems, consider the well known problem on bifurcation of small auto-oscillations from an equilibrium in a system described by a differential equation with a parameter.

Example. Consider the differential equation

$$u'' + \varepsilon u' + \omega^2 u + U(\varepsilon, u, u') = 0.$$

Let ε be a small real parameter, the function $U(\varepsilon, u, v)$ be smooth and $U(\varepsilon, 0, 0) \equiv U_u'(\varepsilon, 0, 0) \equiv U_v'(\varepsilon, 0, 0) \equiv 0$. By rescaling of time and ε the equation under consideration can take the following form

$$u'' + \frac{\lambda}{\pi} u' + u + \frac{1}{\omega^2} U\left(\frac{\lambda\omega}{\pi}, u, \omega u'\right) = 0. \tag{22}$$

Let $u = u(t, \lambda, \xi, \eta)$ be the solution of equation (22) satisfying the initial conditions $u(0, \lambda, \xi, \eta) = \xi$, $u_t'(0, \lambda, \xi, \eta) = \eta$. Then the problem on existence of T-periodic solutions of equation (22) is equivalent [2, 16] to the problem on solvability of the following underspecified system of nonlinear equations

$$u(T, \lambda, \xi, \eta) = \xi, \quad u_t'(T, \lambda, \xi, \eta) = \eta. \tag{23}$$

The left-hand parts of the last equations can be easily evaluated, see, e.g., [16]. Up to the second order terms in the variables $\tau = T - 2\pi$ and λ, ξ, η they have the form

$$u(T, \lambda, \xi, \eta) = \xi + \lambda\xi + \tau\eta + \ldots, \quad u_t'(T, \lambda, \xi, \eta) = \eta - \tau\xi + \lambda\eta + \ldots.$$

Hence, the 2-truncation of (23) is the system of equations

$$\lambda\xi + \tau\eta = 0, \quad \tau\xi - \lambda\eta = 0.$$

The set of solutions for these equations consists of a pair of two-dimensional planes in the space of four-tuples $\{\tau, \lambda, \xi, \eta\}$ having the only common point, the zero point. One of these planes is specified by the equalities $\tau = \lambda = 0$ while the other is specified by the equalities $\xi = \eta = 0$.

Now, denote the vector $\{\tau, \lambda, \xi, \eta\}$ by x, introduce an auxiliary vector $y = \{y_1, y_2\}$ and set

$$f(x) := \{u(T, \lambda, \xi, \eta) - \xi,\ u_t'(T, \lambda, \xi, \eta) - \eta\}.$$

Then $f \in \mathscr{E}_{[2]}(4, 2)$ and its 2-truncation has the form $f^{(2)}(x) = \{\lambda\xi + \tau\eta,\ \tau\xi - \lambda\eta\}$ from which

$$\begin{gathered}\mathscr{R}_2(f^{(2)}; x, y) = \left((\lambda\xi + \tau\eta)^2 + (\tau\xi - \lambda\eta)^2\right)\left(y_1^2 + y_2^2\right) + \\ \left((\lambda y_1 + \tau y_2)^2 + (\tau y_1 - \lambda y_2)^2 + (\xi y_1 - \eta y_2)^2 + (\eta y_1 + \xi y_2)^2\right)\left(\tau^2 + \lambda^2 + \xi^2 + \eta^2\right).\end{gathered}$$

After collecting terms we get

$$\mathscr{R}_2(f^{(2)}; x, y) = \left((\tau^2 + \lambda^2)(\xi^2 + \eta^2) + |x|^4\right)|y|^2 \geq |x|^4|y|^2.$$

Therefore by Theorem 3.1 the jet $j^2 f(0)$ is sv-sufficient in $\mathscr{E}_{[2]}(4, 2)$. Then the set of small solutions of equations (23) consists of a pair of two-dimensional planes intersecting at the point $\tau = \lambda = \xi = \eta = 0$. Existence of one of the planes of solutions of equations (23) is obvious, it is the plane $\xi = \eta = 0$ corresponding to the trivial periodic solution $u(t) \equiv 0$ of equation (22). Existence of the second plane of solutions of equations (23), passing the point $\tau = \lambda = \xi = \eta = 0$ but different from the plane $\xi = \eta = 0$, testifies that equations (23) have nontrivial solutions with arbitrarily small $\tau = T - 2\pi$, λ and $\{\xi, \eta\} \neq 0$. Hence, equation (22) has small nonzero periodic solutions for some arbitrarily small values of the parameter λ, see [14, 16].

4. **Proofs.**

Throughout this section, $O(t^k)$ with $k \geq 0$ stands for values having an upper bound of the form $c|t|^k$ for small t with some $c < \infty$. Analogously, $o(t^k)$ denotes values of a higher order of smallness than $|t|^k$ for small t.

Before starting to prove Theorem 3.1 let us prove first Lemma 2.1.

4.1. **Proof of Lemma 2.1.**

Inequalities (16) are a straightforward consequence of the following two-sided form of the power mean inequality

$$\left(\frac{x+y}{2}\right)^p \leq \frac{x^p + y^p}{2} \leq (x+y)^p, \quad p \geq 1,\ x, y \geq 0.$$

So, we need only to prove inequalities (17) for a given $p \in \mathbb{N}$.

The right inequality in (17) is obvious. Therefore, it remains only to prove the left inequality in (17) which will be done by reductio ad absurdum.

If the left inequality in (17) is not valid then there are $x_i \to 0$ ($x_i \neq 0$), $y_i \neq 0$ and $\eta_i > 0$ such that $\eta_i^p \mathscr{R}_p(f; x_i, y_i) > \mathscr{T}_p(f; x_i, y_i)$. In particular, $\mathscr{R}_p(f; x_i, y_i) > 0$. Since the functions $\mathscr{T}_p(f; x, y)$ and $\mathscr{R}_p(f; x, y)$ are homogeneous in y with the same power of homogeneity $p \in \mathbb{N}$ then without loss of generality one may suppose that $|y_i| = 1$ and $y_i \to y_*$, $|y_*| = 1$. Let us write the following system of polynomial equalities and inequalities[1]:

$$\begin{gathered}
|f(x)|^{2p}|y|^{2p} = u^{2p} \quad |(df)^*(x)y|^{2p}|x|^{2p} = v^{2p}, \quad \langle (df)^*(x)y, x\rangle^{2p} = w^{2p}, \\
u^p + v^p = \varphi^p, \quad u^p + v^p - w^p = \psi^p, \quad \eta^p\varphi^p > \psi^p, \qquad (24) \\
|x|^2 > 0, \quad |y|^2 > 0, \quad \varphi > 0, \quad \psi \geq 0, \quad \eta > 0, \quad u \geq 0, \quad v \geq 0, \quad w \geq 0.
\end{gathered}$$

By the definition of the sequences $\{x_i\}, \{y_i\}$ and $\{\eta_i\}$, the set determined by the the relations (24) is not empty, and the point $x = \varphi = \psi = \eta = u = v = w = 0$, $y = y_*$ belongs to its closure. Hence, by the Curve Selection Lemma for semialgebraic sets (see, e.g., [24]) there are a number $\varepsilon > 0$ and real analytic around the origin functions $x(t), y(t), \varphi(t), \psi(t)$ and $\eta(t)$ satisfying the conditions

$$x(0) = \varphi(0) = \psi(0) = \eta(0) = 0, \quad y(0) = y_*$$

and

$$x(t) \neq 0, \quad \varphi(t) > 0, \quad \psi(t) \geq 0, \quad \eta(t) > 0 \quad \text{for} \quad 0 < t < \varepsilon$$

such that

$$\psi^p(t) < \eta^p(t)\varphi^p(t) \quad \text{for} \quad 0 < t < \varepsilon$$

or, what is the same by (12) and (13),

$$\begin{aligned}
\varphi^p(t) = u^p(t) + v^p(t) = \\
&|f(x(t))|^p|y(t)|^p + |(df)^*(x(t))y(t)|^p|x(t)|^p = \mathscr{R}_p(f; x(t), y(t)) \quad (25)
\end{aligned}$$

[1] Relations (24) are polynomial since p is integer and the norm $|\cdot|$ is Euclidean.

and

$$\eta^p(t)\varphi^p(t) > \psi^p(t) = u^p(t) + v^p(t) - w^p(t) = \\ |f(x(t))|^p|y(t)|^p + |(df)^*(x(t))y(t)|^p|x(t)|^p - |\langle (df)^*(x(t))y(t), x(t)\rangle|^p = \\ \mathscr{T}_p(f; x(t), y(t)). \quad (26)$$

The latter relations imply

$$|f(x(t))| \cdot |y(t)| \leq \eta(t)\varphi(t), \tag{27}$$

from which by (25)

$$|(df)^*(x(t))y(t)| \cdot |x(t)| \geq \varphi(t)\left(1 - \eta^p(t)\right)^{1/p}. \tag{28}$$

Relations (26) imply also

$$|(df)^*(x(t))y(t)|^p|x(t)|^p - |\langle (df)^*(x(t))y(t), x(t)\rangle|^p \leq \eta^p(t)\varphi^p(t).$$

By dividing the both sides of the last inequality on $|(df)^*(x(t))y(t)|^p|x(t)|^p$, we obtain by (28)

$$0 \leq 1 - \left(\frac{|\langle (df)^*(x(t))y(t), x(t)\rangle|}{|(df)^*(x(t))y(t)| \cdot |x(t)|}\right)^p \leq \frac{\eta^p(t)}{1 - \eta^p(t)}. \tag{29}$$

Because the functions $x(t), y(t), \varphi(t), \eta(t)$ are real analytic for small t then they can be represented in the following form:

$$x(t) = x_* t^q + o(t^q), \qquad x_* \neq 0,\ q \geq 1, \tag{30}$$

$$y(t) = y_* + O(t), \qquad |y_*| = 1, \tag{31}$$

$$\varphi(t) = \varphi_* t^r + o(t^r), \qquad \varphi_* > 0,\ r \geq 1, \tag{32}$$

$$\eta(t) = \eta_* t^s + o(t^s), \qquad \eta_* > 0,\ s \geq 1. \tag{33}$$

Since $f(x)$ is a polynomial and the functions $x(t), y(t)$ are analytic then the functions $(df)^*(x(t))y(t)$ and $f(x(t))$ are also analytic, and $f(x(t)) \to 0$ as $t \to 0$. Therefore by inequalities (27)–(29) there are integers $k \geq 1$, $l \geq 0$ such that

$$f(x(t)) = O(t^k) \qquad\qquad k \geq 1, \tag{34}$$

$$(df)^*(x(t))y(t) = h_* t^l + o(t^l), \quad h_* \neq 0,\ l \geq 0. \tag{35}$$

Substituting now representations (31)–(34) for the related functions in (27) we get

$$O(t^k) \cdot |y_* + O(t)| \leq (\varphi_* t^r + o(t^r))(\eta_* t^s + o(t^s)),$$

from which (since $y_* \neq 0$)

$$k \geq r + s. \tag{36}$$

Similarly, substituting representations (31)–(33) and (35) for the related functions in (28) we get

$$|h_* t^l + o(t^l)| \cdot |x_* t^q + o(t^q)| \geq (\varphi_* t^r + o(t^r))\left(1 - O(t)\right)^{1/p},$$

from which (since $h_*, x_*, \varphi_* \neq 0$)

$$r \geq l + q. \tag{37}$$

At last, substituting representations (30), (33) and (35) for the related functions in (29) we get

$$0 \leq 1 - \left(\frac{|\langle h_* t^l + o(t^l), x_* t^q + o(t^q)\rangle|}{|h_* t^l + o(t^l)| \cdot |x_* t^q + o(t^q)|}\right)^p \leq c\left(\eta_* t^s + o(t^s)\right)^p$$

with some constant $c < \infty$ from which

$$0 \le 1 - \left(\frac{|\langle h_*, x_*\rangle|}{|h_*| \cdot |x_*|}\right)^p + O(t) \le O(t).$$

Hence $|\langle h_*, x_*\rangle| = |h_*| \cdot |x_*|$ and therefore $h_* = \lambda x_*$ with some $\lambda \neq 0$ (since $h_*, x_* \neq 0$), and by equality (35)

$$(df)^*(x(t))y(t) = \lambda x_* t^l + o(t^l). \tag{38}$$

Let us evaluate now the function $z(t) = (f(x(t)), y(t))$. Because

$$z'(t) = \langle df(x(t))x'(t), y(t)\rangle + \langle f(x(t), y'(t)\rangle = \\ = \langle x'(t), (df)^*(x(t))y(t)\rangle + \langle f(x(t), y'(t)\rangle,$$

then formulae (30), (31), (34) and (38) imply the following equalities

$$z'(t) = \langle px_* t^{q-1} + O(t^q), \lambda x_* t^l + o(t^l)\rangle + \langle O(t^k), O(1)\rangle = \\ = \lambda p|x_*|^2 t^{q+l-1} + O(t^{q+l}) + O(t^k).$$

Here, by (36) and (37), $k \ge q + l + s$. Therefore $O(t^k) = o(t^{q+l})$, and then

$$z'(t) = \lambda p|x_*|^2 t^{q+l-1} + O(t^{q+l}).$$

By integrating the both sides of the last equality we get

$$\langle f(x(t)), y(t)\rangle = z(t) = \int_0^t z'(s)\, ds = \lambda \frac{q}{q+l}|x_*|^2 t^{q+l} + o(t^{q+l}). \tag{39}$$

Now, the obvious relation $\langle f(x(t)), y(t)\rangle \le |f(x(t))| \cdot |y(t)|$ and inequalities (39), (31) and (34) imply the estimate

$$\lambda \frac{q}{q+l}|x_*|^2 t^{q+l} + o(t^{q+l}) \le O(t^k) \cdot |y_* + O(t)|.$$

Since here $x_*, y_* \neq 0$ then $k \le q + l$. On the other hand, in view of (36) and (37) we have $k \ge q + l + s \ge q + l + 1$. A contradiction! Lemma 2.1 is proved.

4.2. **Proof of Theorem 3.1.** By Lemma 2.1 the conditions (18) for different $p \in \mathbb{N}$ and $\mathscr{K} = \mathscr{R}_p$ or $\mathscr{K} = \mathscr{T}_p$ are equivalent to each other, and the same is valid for the conditions (19). So, to prove Theorem 3.1 we need only to show that the Kuo condition (6) is equivalent to the condition (18) with $\mathscr{K} = \mathscr{R}_1$:

$$|f^{(r)}(x)| \cdot |y| + |(df^{(r)})^*(x)y| \cdot |x| \ge q|x|^r|y| \tag{40}$$

for small x and all y, while the Kuo condition (7) is equivalent to the condition (19) with $\mathscr{K} = \mathscr{R}_1$:

$$\frac{|f^{(r)}(x)| \cdot |y| + |(df^{(r)})^*(x)y| \cdot |x|}{|x|^{r+1}|y|} \to \infty \tag{41}$$

as $x \to 0$, $x \neq 0$, uniformly with respect to $y \neq 0$.

To prove equivalence between (6) and (40) introduce first, for a given set of vectors $v_1, v_2, \ldots, v_m \in \mathbb{R}^n$, the quantity $\widetilde{\mathscr{D}}(v_1, v_2, \ldots, v_m)$ as follows:

$$\widetilde{\mathscr{D}}(v_1, v_2, \ldots, v_m) = \min \left|\sum_{i=1}^m y_i v_i\right|, \quad v_1, v_2, \ldots, v_m \in \mathbb{R}^n, \tag{42}$$

where the minimum is taken over all m-tuples of real numbers $y_1, y_2, \ldots, y_m$ satisfying $\sum_{i=1}^m y_i^2 = 1$.

Represent now the vector $(df^{(r)})^*(x)y$ in (40) in the form

$$(df^{(r)})^*(x)y \equiv \sum_{i=1}^{m} y_i \operatorname{grad} f_i^{(r)}(x)$$

where $y_1, y_2, \dots y_m$ are the components of the vector y and $f_1^{(r)}, f_2^{(r)}, \dots f_m^{(r)}$ are the components of the map $f^{(r)}$. Then, taking the minimum in the left-hand part of (40) over all the vectors y satisfying $\sum_{i=1}^{m} y_i^2 = 1$, we obtain that

$$\min_{y\neq 0} \frac{\left|(df^{(r)})^*(x)y\right|}{|y|} = \widetilde{\mathscr{D}}(\operatorname{grad} f_1^{(r)}(x), \operatorname{grad} f_2^{(r)}(x), \dots, \operatorname{grad} f_m^{(r)}(x)). \tag{43}$$

Therefore (40), for small x, is equivalent to the condition:

$$|f^{(r)}(x)| + \widetilde{\mathscr{D}}(\operatorname{grad} f_1^{(r)}(x), \operatorname{grad} f_2^{(r)}(x), \dots, \operatorname{grad} f_m^{(r)}(x)) \cdot |x| \geq q|x|^r.$$

Then, taking into account that

$$\widetilde{\mathscr{D}}(v_1, v_2, \dots, v_m) \leq \mathscr{D}(v_1, v_2, \dots, v_m) \leq \sqrt{m}\widetilde{\mathscr{D}}(v_1, v_2, \dots, v_m) \tag{44}$$

where $\mathscr{D}$ is the function (8), see [25, p. 348], we may state that (40), for small x, is equivalent also to the condition:

$$|f^{(r)}(x)| + \mathscr{D}(\operatorname{grad} f_1^{(r)}(x), \operatorname{grad} f_2^{(r)}(x), \dots, \operatorname{grad} f_m^{(r)}(x)) \cdot |x| \geq \tilde{q}|x|^r \tag{45}$$

with an appropriate $\tilde{q} > 0$.

Now, let (6) be valid. Then for $x \in \mathscr{H}_r(f^{(r)}; \sigma)$, $|x| < \varepsilon$, the first summand in the left-hand side of (45) is greater than $\sigma|x|^r$. At the same time for $x \notin \mathscr{H}_r(f^{(r)}; \sigma)$, $|x| < \varepsilon$, by (6) the second summand in the left-hand side of (45) is greater than $C|x|^r$. So, for $|x| < \varepsilon$, (6) implies (45) with $\tilde{q} = \min\{\sigma, C\}$.

If (45) holds for $|x| < \varepsilon$ with some $\varepsilon > 0$ then clearly for $x \in \mathscr{H}_r(f^{(r)}; \frac{1}{2}\tilde{q})$ inequality (6) will be valid with $C = \frac{1}{2}\tilde{q}$. So, (45) implies (6) with $C = \frac{1}{2}\tilde{q}$.

Thus, conditions (6) and (45) are equivalent, and consequently the Kuo condition (6) is equivalent to (40).

The proof of equivalence between (7) and (41) is a bit more complicated. First, to prove that (7) implies (41) we will show that (7) is not valid provided that (41) is not valid. To do it, we will need the following lemma the proof of which is relegated to Section 4.3 below.

Lemma 4.1. *Let the map $f^{(r)}(x)$ do not satisfy (41). Then there are $x_i \to 0$ ($x_i \neq 0$), $y_i \to 0$ and a uniform polynomial $h : \mathbb{R}^n \to \mathbb{R}^m$ of degree $r+1$ such that for the map $g(x) = f^{(r)}(x) + h(x)$ the following estimates hold*

$$|g(x_i)| \leq c|x_i|^{r+1+\delta'}, \quad |(dg)^*(x_i)y_i| \leq c|y_i| \cdot |x_i|^{r+\delta'} \tag{46}$$

with some $\delta' > 0$ and $c < \infty$.

Now, let $\{x_i\}$ be a sequence defined by Lemma 4.1. Then by the first inequality (46) for any $\sigma > 0$ there is an $\varepsilon > 0$ such that

$$x_i \in \mathscr{H}_{r+1}(g; \sigma) \cap \{|x| < \varepsilon\} \tag{47}$$

for all sufficiently large indices i.

By Lemma 4.1, $f^{(r)}(x) = g(x) - h(x)$ where $h : \mathbb{R}^n \to \mathbb{R}^m$ is a uniform polynomial of degree $r+1$. Then $|(dh)^*(x)| \leq c_1|x|^r$ with some constant c_1, and by the second inequality (46)

$$|(df^{(r)})^*(x_i)y_i| \leq c_2|y_i| \cdot |x_i|^r, \quad i = 1, 2, \dots\ ,$$

with some constant c_2. Therefore by (42)

$$\widetilde{\mathscr{D}}(\operatorname{grad} f_1^{(r)}(x_i), \operatorname{grad} f_2^{(r)}(x_i), \ldots, \operatorname{grad} f_m^{(r)}(x_i)) \le c_3|x|^r, \quad i = 1, 2, \ldots\ ,$$

with some constant c_3, and by (44)

$$\mathscr{D}(\operatorname{grad} f_1^{(r)}(x_i), \operatorname{grad} f_2^{(r)}(x_i), \ldots, \operatorname{grad} f_m^{(r)}(x_i)) \le c_3|x|^r, \quad i = 1, 2, \ldots\ .$$

These last inequalities imply that for any $C, \delta > 0$

$$\mathscr{D}(\operatorname{grad} f_1^{(r)}(x_i), \operatorname{grad} f_2^{(r)}(x_i), \ldots, \operatorname{grad} f_m^{(r)}(x_i)) < C|x|^{r-\delta} \tag{48}$$

for all sufficiently large indices i.

Relations (47) and (48) show that for any choice of the numbers $C, \delta, \varepsilon, \sigma > 0$ condition (7) is not valid for the map $f^{(r)}$ so as for the map g determined by Lemma 4.1.

So, we completed the proof that non-validity of (41) implies non-validity of (7), and consequently the Kuo condition (7) implies (41).

It remains only to prove that (41) implies the Kuo condition (7). To do it, we will need the following lemma the proof of which is relegated to Section 4.4 below.

Lemma 4.2. *Let $g : \mathbb{R}^n \to \mathbb{R}^m$ be a polynomial map of degree $r+1$ such that $j^r g(0) = j^r f(0)$ where $f^{(r)}(x)$ satisfies the condition (41). Then*

$$\frac{|g(x)| \cdot |y| + |(df^{(r)})^*(x)y| \cdot |x|}{|x|^{r+1}|y|} \to \infty \tag{49}$$

as $x \to 0$, $x \ne 0$, uniformly with respect to $y \ne 0$.

Now, let condition (41) be valid. Take an arbitrary polynomial map $g : \mathbb{R}^n \to \mathbb{R}^m$ of degree $r+1$ satisfying $j^r g(0) = j^r f(0)$. Then by Lemma 4.2 relation (49) holds. In this case, by usual argumentation (see, e.g. [11, 22]) there are positive constants σ', δ' and $\varepsilon' < 1$ such that

$$|g(x)| \cdot |y| + |(df^{(r)})^*(x)y| \cdot |x| \ge \sigma'|x|^{r+1-\delta'}|y| \tag{50}$$

for $x \in \mathbb{R}^n$, $|x| < \varepsilon'$, and all $y \in \mathbb{R}^m$.

Let x, $|x| < \varepsilon'$, belong to the horn-neighbourhood $\mathscr{H}_{r+1}(g; \sigma'/2)$ of $g^{-1}(0)$. Then

$$|g(x)| < \frac{1}{2}\sigma'|x|^{r+1} \le \frac{1}{2}\sigma'|x|^{r+1-\delta'}$$

and by (50)

$$|(df^{(r)})^*(x)y| \ge \frac{1}{2}\sigma'|x|^{r-\delta'}|y|.$$

Hence, by (42), (43),

$$\widetilde{\mathscr{D}}(\operatorname{grad} f_1^{(r)}(x), \operatorname{grad} f_2^{(r)}(x), \ldots, \operatorname{grad} f_m^{(r)}(x)) \ge \frac{1}{2}\sigma'|x|^{r-\delta'},$$

and by (44),

$$\mathscr{D}(\operatorname{grad} f_1^{(r)}(x), \operatorname{grad} f_2^{(r)}(x), \ldots, \operatorname{grad} f_m^{(r)}(x)) \ge \frac{\sigma'}{2\sqrt{m}}|x|^{r-\delta'},$$

for $x \in \mathscr{H}_{r+1}(g; \sigma'/2) \cap \{|x| < \varepsilon'\}$, which is exactly the Kuo condition (7).

So, (41) implies the Kuo condition (7), and the proof of Theorem 3.1 is completed.

4.3. **Proof of Lemma 4.1.** Denote by H the class of polynomials in x of the form $\eta(x) = \langle x, v\rangle^p \langle x, w\rangle^q u$ where $p+q = r+1$ and $v, w \in \mathbb{R}^n$. Since for such polynomials

$$d\eta(x)z = p\langle x, v\rangle^{p-1}\langle x, w\rangle^q \langle z, v\rangle u + q\langle x, v\rangle^p \langle x, w\rangle^{q-1}\langle z, w\rangle u,$$

then, by the identity $\langle (d\eta)^*(x)y, z\rangle \equiv \langle y, d\eta(x)z\rangle$, it is valid also the identity

$$\langle (d\eta)^*(x)y, z\rangle \equiv p\langle x, v\rangle^{p-1}\langle x, w\rangle^q \langle z, v\rangle\langle y, u\rangle + q\langle x, v\rangle^p \langle x, w\rangle^{q-1}\langle z, w\rangle\langle y, u\rangle.$$

Therefore

$$(d\eta)^*(x)y = p\langle x, v\rangle^{p-1}\langle x, w\rangle^q \langle y, u\rangle v + q\langle x, v\rangle^p \langle x, w\rangle^{q-1}\langle y, u\rangle w.$$

The last formula will be needed below in two cases:

$$(d\eta)^*(x)y = (r+1)\langle x, v\rangle^r \langle y, u\rangle v, \tag{51}$$

if $\eta(x) = \langle x, v\rangle^{r+1} u$, and

$$(d\eta)^*(x)y = r\langle x, v\rangle^{r-1}\langle x, w\rangle\langle y, u\rangle v + q\langle x, v\rangle^r \langle x, w\rangle\langle y, u\rangle w, \tag{52}$$

if $\eta(x) = \langle x, v\rangle^r \langle x, w\rangle u$.

First, let us construct a polynomial $\eta_i(x) \in H$ such that for the map $g(x) = f^{(r)}(x) + \eta_i(x)$ the second inequality (46) be valid. Since $f^{(r)}$ does not satisfy (41) then by the Curve Selection Lemma [24] there are analytic around the origin functions

$$x(t) = ut^\alpha + o(t^\alpha), \qquad u \neq 0,\ \alpha \geq 1 \text{ is integer}, \tag{53}$$

$$y(t) = v + O(t), \qquad |v| = 1, \tag{54}$$

for which

$$|f^{(r)}(x(t))| \leq c|x(t)|^{r+1}, \qquad |(df^{(r)})^*(x(t))y(t)| \leq c|x(t)|^r. \tag{55}$$

Clearly, the function $(df^{(r)})^*(x(t))y(t)$ is also analytic. If it is identically zero then it suffices to set $\eta_1(x) \equiv 0$. In the opposite case let us represent it in the form

$$(df^{(r)})^*(x(t))y(t) = zt^\gamma + o(t^\gamma), \quad z \neq 0,\ \gamma \geq 1 \text{ is integer}. \tag{56}$$

Then relations (53) and (55) imply $\gamma \geq \alpha r$. If $\gamma > \alpha r$ then the second inequality (46) holds for $\eta_1(x) \equiv 0$ and $\delta' = (\gamma - \alpha r)\alpha$. So, it remains only to consider the case when

$$\gamma = \alpha r \tag{57}$$

Here we have two possibilities: $\langle u, z\rangle \neq 0$ and $\langle u, z\rangle = 0$.

a. Let $\langle u, z\rangle \neq 0$. Set

$$\eta_1(x) = \rho\langle x, z\rangle^{r+1} v$$

where $\rho \in \mathbb{R}^1$. By (51)

$$(d\eta_1)^*(x)y = \rho(r+1)\langle x, z\rangle^r \langle y, v\rangle z,$$

and, in view of (53), (54), (56) and (57), the following equalities hold:

$$\begin{aligned}(df^{(r)})^*(x(t))y(t) + (d\eta_1)^*(x(t))y(t) &= \\ = zt^{\alpha r} + o(t^{\alpha r}) + \rho(r+1)\langle ut^\alpha + o(t^\alpha), z\rangle^r \langle v + O(t), v\rangle z &= \\ = zt^{\alpha r} + \rho(r+1)\langle u, z\rangle^r \langle v, v\rangle z t^{\alpha r} + o(t^{\alpha r}).\end{aligned}$$

If to choose $\rho = \{(r+1)\langle u, z\rangle^r \langle v, v\rangle\}^{-1}$ then for the map $f^{(r)}(x) + \eta_1(x)$ the second estimate (46) with $\delta' = \delta'_1 = 1/(\alpha r)$ will be valid.

b. Let $\langle u, z\rangle = 0$. Set

$$\eta_1(x) = \rho\langle x, z\rangle\langle x, u\rangle^r v$$

where $\rho \in \mathbb{R}^1$. By (52)

$$(d\eta_1)^*(x)y = \rho\langle x,u\rangle^r \langle y,v\rangle z + \rho r\langle x,u\rangle^{r-1}\langle x,z\rangle\langle y,v\rangle u,$$

and, in view of the relations (53), (54), (56) and (57),

$$\begin{aligned}(df^{(r)})^*(x(t))y(t) + (d\eta_1)^*(x(t))y(t) ={}&\\ = zt^{\alpha r} + o(t^{\alpha r}) + \rho\langle ut^\alpha + o(t^\alpha),u\rangle^r\langle v+O(t),v\rangle z+&\\ + \rho r\langle ut^\alpha + o(t^\alpha),u\rangle^{r-1}\langle ut^\alpha + o(t^\alpha),z\rangle\langle v+O(t),v\rangle u.&\end{aligned}$$

By supposition, the multiplier $\langle ut^\alpha + o(t^\alpha), z\rangle$ in the last summand is of the order $O(t^{\alpha+1})$ and therefore the whole last summand has the order $O(t^{\alpha r+1})$. Hence

$$(df^{(r)})^*(x(t))y(t) + (d\eta_1)^*(x(t))y(t) = \{1+\rho\langle u,u\rangle^r\langle v,v\rangle\}t^{\alpha r}z + O(t^{\alpha r+1}).$$

If to choose now $\rho = -\{\langle u,u\rangle^r\langle v,v\rangle\}^{-1}$ then the map $f^{(r)}(x)+\eta_1(x)$ and any sequences of the elements $x_i = x(t_i)$, $y_i = y(t_i)$, where $t_i \to 0$, $t_i \neq 0$, will satisfy the second estimate (46) with $\delta' = \delta_1' = 1/(\alpha r)$.

So, the map $\eta_1(x)$ is constructed. The map $h(x)$ will be searched in the form $h(x) = \eta_1(x) + \eta_2(x)$, with an $\eta_2(x)$ such that not to break the second inequality (46) and to satisfy simultaneously the first of these inequalities. Denote the map $f^{(r)}(x)+\eta_1(x)$ by $g_1(x)$. Then, by construction,

$$|(dg_1)^*(x(t))y(t)| \le \tilde{c}|x(t)|^{r+\delta_1'}, \tag{58}$$

$$|g_1(x(t))| \le \tilde{c}|x(t)|^{r+1}. \tag{59}$$

The function $g_1(x(t))$ is analytic. If it is identically zero then it suffices to set $\eta_2(x) \equiv 0$. In the opposite case we let us write down the following representations:

$$g_1(x(t)) = wt^\mu + o(t^\mu), \quad w \neq 0,\ \mu \ge 1 \text{ is integer}, \tag{60}$$

$$(dg_1)^*(x(t))y(t) = O(t^\nu), \qquad \nu \ge 1 \text{ is integer}. \tag{61}$$

Relations (58), (61) and (53) imply

$$\nu \ge \alpha r + 1, \tag{62}$$

while relations (59), (60) and (53) imply $\mu \ge \alpha(r+1)$. If $\mu > \alpha(r+1)$ then inequalities (46) hold for $\eta_2(x) \equiv 0$, $\delta' = \min\{\delta_1', \delta_2'\}$ where $\delta_2' = \mu/\alpha - (r+1)$. Therefore we need only to consider the case when

$$\mu = \alpha(r+1). \tag{63}$$

Let us estimate the quantity $\langle g_1(x(t)), y(t)\rangle$. On the one hand, by (54) and (60),

$$\langle g_1(x(t)), y(t)\rangle = \langle wt^\mu + 0(t^\mu), v + O(t)\rangle = \langle w,v\rangle t^\mu + O(t^{\mu+1}). \tag{64}$$

On the other hand,

$$\begin{aligned}\langle g_1(x(t)), y(t)\rangle &= \int_0^t \langle dg_1(x(s))x'(s), y(s)\rangle\,ds + \int_0^t \langle g_1(x(s)), y'(s)\rangle\,ds =\\ &= \int_0^t \langle x'(s), (dg_1)^*(x(s))y(s)\rangle\,ds + \int_0^t \langle g_1(x(s)), y'(s)\rangle\,ds,\end{aligned}$$

from which, by using power series expansions in s of the integrands and by integrating the obtained relations, we get the equalities

$$\begin{aligned}\langle g_1(x(t)), y(t)\rangle &= \\ &= \int_0^t \langle \alpha u s^{\alpha-1} + O(s^\alpha), O(s^\nu)\rangle\, ds + \int_0^t \langle w s^\mu + o(s^\mu), O(1)\rangle\, ds = \\ &= O(t^{\alpha+\nu}) + O(t^{\alpha+\nu+1}) + O(t^{\mu+1}) + o(t^{\mu+1}).\end{aligned}$$

By (62) and (63), these last equalities imply $\langle g_1(x(t)), y(t)\rangle = O(t^{\mu+1})$. Therefore, in view of (64),

$$\langle w, v\rangle = 0. \tag{65}$$

Set now $\eta_2(x) = \rho\langle x, u\rangle^{r+1} w$ where $\rho = -\langle u, u\rangle^{-(r+1)}$. Then, by (53) and (60),

$$g_1(x(t)) + \eta_2(x(t)) = w t^{\alpha(r+1)} + o(t^{\alpha(r+1)}) - \langle u, u\rangle^{-(r+1)}\langle u t^\alpha + o(t^\alpha), u\rangle^{r+1} w,$$

from which $g_1(x(t)) + \eta_2(x(t)) = O(t^{\alpha(r+1)+1})$. Hence, for the map $g(x) = g_1(x) + \eta_2(x)$ and any sequence of elements $x_i = x(t_i)$ where $t_i \to 0$, $t_i \neq 0$, the first estimate (46) holds with $\delta' = 1/(\alpha(r+1))$.

It remains to verify validity of the second estimate (46). By (51)

$$(d\eta_2)^*(x)y = \rho(r+1)\langle x, u\rangle^r \langle y, w\rangle u,$$

and therefore (see (53), (54), (61))

$$\begin{aligned}(dg)^*(x(t))y(t) &= (dg_1)^*(x(t))y(t) + (d\eta_2)^*(x(t))y(t) = \\ &= O(t^\nu) + \rho(r+1)\langle u t^\alpha + o(t^\alpha), u\rangle^r \langle v + O(t), w\rangle u.\end{aligned}$$

Since, in view of (65), the multiplier $\langle v + O(t), w\rangle$ in the second summand is of the order $O(t)$ then the whole second summand has the order $O(t^{\alpha r+1})$. Then by (62) $(dg)^*(x(t))y(t) = O(t^{\alpha r+1})$.

So, for any sequence of pairs $\{x_i, y_i\}$ where $x_i = x(t_i)$, $y_i = y(t_i)$, $t_t \to 0$, $t_i \neq 0$, the inequalities (46) hold with $\delta' = \min\{1/(\alpha(r+1)), 1/(\alpha r)\}$. The proof of Lemma 4.1 is completed.

4.4. **Proof of Lemma 4.2.** Set $\theta(x) = g(x) - f^{(r)}(x)$. Then θ is a uniform polynomial of degree $r+1$. Therefore $|\theta(x)| \leq c|x|^{r+1}$ for sufficiently small values of $|x|$ where c is some constant. Then

$$|g(x)| \cdot |y| + |(df^{(r)})^*(x)y| \cdot |x| = |f^{(r)}(x) + \theta(x)| \cdot |y| + |(df^{(r)})^*(x)y| \cdot |x|,$$

and

$$\frac{|g(x)| \cdot |y| + |(df^{(r)})^*(x)y| \cdot |x|}{|x|^{r+1}|y|} \geq \frac{|f^{(r)}(x)| \cdot |y| + |(df^{(r)})^*(x)y| \cdot |x|}{|x|^{r+1}|y|} - \frac{|\theta(x)|}{|x|^{r+1}},$$

from which

$$\frac{|g(x)| \cdot |y| + |(df^{(r)})^*(x)y| \cdot |x|}{|x|^{r+1}|y|} \geq \frac{|f^{(r)}(x)| \cdot |y| + |(df^{(r)})^*(x)y| \cdot |x|}{|x|^{r+1}|y|} - c.$$

It remains to apply formula (41). Lemma 4.2 is proved.

REFERENCES

[1] O. M. Abderrahmane, *On the Lojasiewicz exponent and Newton polyhedron*, Kodai Math. J., **28** (2005), 106–110.

[2] A. A. Andronov, A. A. Vitt and S. È. Khaĭkin, "Theory of Oscillators," Dover Publications Inc., New York, 1987.

[3] K. Bekka and S. Koike, *The Kuo condition, Thom's type inequality and* (c)*-regularity*, Sūrikaisekikenkyūsho Kōkyūroku, (1996), 41–49.

[4] N. A. Bobylëv, V. G. Boltyanskiĭ, S. Y. Vsekhsvyatskiĭ et al., "Matematicheskaya Teoriya Sistem," Nauka, Moscow, 1986, in Russian.

[5] J. Bochnak and S. Łojasiewicz, *A converse of the Kuiper-Kuo theorem*, in: Proceedings of Liverpool Singularities—Symposium, I (1969/70), 254–261. Lecture Notes in Math., Vol. **192**, Springer, Berlin, 1971.

[6] J. Chądzyński and T. Krasiński, *A set on which the Łojasiewicz exponent at infinity is attained*, Ann. Polon. Math., **67** (1997), 191–197, arXiv:math/9802064.

[7] D. D'Acunto and K. Kurdyka, *Explicit bounds for the Łojasiewicz exponent in the gradient inequality for polynomials*, Ann. Polon. Math., **87** (2005), 51–61.

[8] D. B. Fuks and V. A. Rokhlin, "Beginner's Course in Topology," Universitext, Springer-Verlag, Berlin, 1984.

[9] E. García Barroso, T. Krasiński and A. Płoski, *On the Łojasiewicz numbers. II*, C. R. Math. Acad. Sci. Paris, **341** (2005), 357–360.

[10] E. García Barroso and A. Płoski, *On the Łojasiewicz numbers*, C. R. Math. Acad. Sci. Paris, **336** (2003), 585–588.

[11] E. A. Gorin, *Asymptotic properties of polynomials and algebraic functions of several variables*, Uspehi Mat. Nauk, **16** (1961), 91–118, in Russian, translation in Russ. Math. Surv., **16** (1961), 93–119.

[12] J. Gwoździewicz, *The Łojasiewicz exponent of an analytic function at an isolated zero*, Comment. Math. Helv., **74** (1999), 364–375.

[13] J. Kollár, *An effective Łojasiewicz inequality for real polynomials*, Period. Math. Hungar.,**38** (1999), 213–221, arXiv:math/9904161.

[14] V. S. Kozjakin and M. A. Krasnosel′skiĭ, *A method of parameter functionalization in the problem of bifurcation points*, Dokl. Akad. Nauk SSSR, **254** (1980), 1061–1064, in Russian, translation in Soviet Math. Dokl., **22** (1980), 513–517.

[15] V. S. Kozyakin, *On neglecting small terms in the study of nonlinear systems*, Avtomat. i Telemekh., (1984), 38–43, in Russian, translation in Autom. Remote Control, **45** (1984), 1275–1280.

[16] M. A. Krasnosel′skiĭ, "The Operator of Translation Along the Trajectories of Differential Equations," Translations of Mathematical Monographs, Vol. **19**, Translated from the Russian by Scripta Technica, American Mathematical Society, Providence, R.I., 1968.

[17] N. H. Kuiper, C^1*-equivalence of functions near isolated critical points*, in: Symposium on Infinite-Dimensional Topology (Louisiana State Univ., Baton Rouge, La., 1967), 199–218. Ann. of Math. Studies, No. 69, Princeton Univ. Press, Princeton, N. J., 1972.

[18] T.-C. Kuo, *On* C^0*-sufficiency of jets of potential functions*, Topology, **8** (1969), 167–171.

[19] T.-C. Kuo, *Characterizations of* v*-sufficiency of jets*, Topology, **11** (1972), 115–131.

[20] A. Lenarcik, *On the Łojasiewicz exponent of the gradient of a polynomial function*, Ann. Polon. Math., **71** (1999), 211–239, arXiv:math/9703224.

[21] S. Łojasiewicz, *Sur le problème de la division*, Studia Math., **18** (1959), 87–136.

[22] S. Łojasiewicz, "Introduction to Complex Analytic Geometry," Birkhäuser Verlag, Basel, 1991.

[23] B. Malgrange, "Ideals of Differentiable Functions," Tata Institute of Fundamental Research Studies in Mathematics, No. 3, Tata Institute of Fundamental Research, Bombay, 1967.

[24] J. Milnor, "Singular Points of Complex Hypersurfaces," Annals of Mathematics Studies, No. 61, Princeton University Press, Princeton, N.J., 1968.

[25] D. J. A. Trotman and L. C. Wilson, *Stratifications and finite determinacy*, Proc. London Math. Soc. (3), **78** (1999), 334–368.

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