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Lévy Processes and Stochastic Calculus

Second Edition

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Stochastic differential equations

Summary After a review of first-order differential equations and their associated flows, we investigate stochastic differential equations (SDEs) driven by Brownian motion and an independent Poisson random measure. We establish the existence and uniqueness of solutions under the standard Lipschitz and growth conditions, using the Picard iteration technique. We then turn our attention to investigating properties of the solution. These are exhibited as stochastic flows and as multiplicative cocycles. The interlacing structure is established, and we prove the continuity of solutions as a function of their initial conditions. We then show that solutions of SDEs are Feller processes and compute their generators. Perturbations are studied via the Feynman–Kac formula. We briefly survey weak solutions and associated martingale problems. The existence of Lyapunov exponents for solutions of SDES will be investigated.

Finally, we study solutions of Marcus canonical equations and discuss the respective conditions under which these yield stochastic flows of homeomorphisms and diffeomorphisms.

One of the most important applications of Itô's stochastic integral is in the construction of *stochastic differential equations* (SDEs). These are important for a number of reasons.

(1) Their solutions form an important class of Markov processes where the infinitesimal generator of the corresponding semigroup can be constructed explicitly. Important subclasses that can be studied in this way include diffusion and jump-diffusion processes.
(2) Their solutions give rise to stochastic flows, and hence to interesting examples of random dynamical systems.
(3) They have many important applications to, for example, filtering, control, finance and physics.

Before we begin our study of SDEs, it will be useful to remind ourselves of some of the key features concerning the construction and elementary properties of ordinary differential equations (ODEs).

6.1 Differential equations and flows

Our main purpose in this section is to survey some of those aspects of ODEs that recur in the study of SDEs. We aim for a simple pedagogic treatment that will serve as a useful preparation and we do not attempt to establish optimal results. We mainly follow Abraham *et al*. [1], section 4.1.

Let $b : \mathbb{R}^d \to \mathbb{R}^d$, so that $b = (b^1, \dots, b^{\mathrm{d}})$ where $b^i : \mathbb{R}^d \to \mathbb{R}$ for $1 \leq i \leq d$.

We study the vector-valued differential equation

$$\frac{dc(t)}{dt} = b(c(t)), \tag{6.1}$$

with fixed initial condition $c(0) = c_0 \in \mathbb{R}^d$, whose solution, if it exists, is a curve $(c(t), t \in \mathbb{R})$ in $\mathbb{R}^d$.

Note that (6.1) is equivalent to the system of ODEs

$$\frac{dc^i(t)}{dt} = b^i(c(t))$$

for each $1 \leq i \leq d$.

To solve (6.1), we need to impose some structure on b. We say that b is *(globally) Lipschitz* if there exists $K > 0$ such that, for all $x, y \in \mathbb{R}^d$,

$$|b(x) - b(y)| \leq K|x - y|. \tag{6.2}$$

The expression (6.2) is called a *Lipschitz condition* on b and the constant K appearing therein is called a *Lipschitz constant*. Clearly if b is Lipschitz then it is continuous.

Exercise 6.1.1 Show that if b is differentiable with bounded partial derivatives then it is Lipschitz.

Exercise 6.1.2 Deduce that if b is Lipschitz then it satisfies a linear growth condition

$$|b(x)| \leq L(1 + |x|)$$

for all $x \in \mathbb{R}^d$, where $L = \max\{K, |b(0)|\}$.

The following existence and uniqueness theorem showcases the important technique of *Picard iteration*. We first rewrite (6.1) as an integral equation,

$$c(t) = c(0) + \int_0^t b(c(s))ds,$$

for each $t \in \mathbb{R}$. Readers should note that we are adopting the convention whereby $\int_0^t$ is understood to mean $\int_t^0$ when $t < 0$.

Theorem 6.1.3 *If $b : \mathbb{R}^d \to \mathbb{R}^d$ is (globally) Lipschitz, then there exists a unique solution $c : \mathbb{R} \to \mathbb{R}^d$ of the initial value problem* (6.1).

Proof Define a sequence $(c_n, n \in \mathbb{N} \cup \{0\})$, where $c_n : \mathbb{R} \to \mathbb{R}^d$ is defined by

$$c_0(t) = c_0, \quad c_{n+1}(t) = c_0 + \int_0^t b(c_n(s))ds,$$

for each $n \geq 0, t \in \mathbb{R}$. Using induction and Exercise 6.1.2, it is straightforward to deduce that each c_n is integrable on $[0, t]$, so that the sequence is well defined.

Define $\alpha_n = c_n - c_{n-1}$ for each $n \in \mathbb{N}$. By Exercise 6.1.2, for each $t \in \mathbb{R}$ we have

$$|\alpha_1(t)| \leq |b(c_0)|\,|t| \leq Mt, \tag{6.3}$$

where $M = L(1 + |c_0|)$.

Using the Lipschitz condition (6.2), for each $t \in \mathbb{R}$, $n \in \mathbb{N}$, we obtain

$$|\alpha_{n+1}(t)| \leq \int_0^t |b(c_n(s)) - b(c_{n-1}(s))|ds \leq K \int_0^t |\alpha_n(s)|ds \tag{6.4}$$

and a straightforward inductive argument based on (6.3) and (6.4) yields the estimate

$$|\alpha_n(t)| \leq \frac{MK^{n-1}|t|^n}{n!}$$

for each $t \in \mathbb{R}$. Hence for all $t > 0$ and $n, m \in \mathbb{N}$ with $n > m$, we have

$$\sup_{0 \leq s \leq t} |c_n(s) - c_m(s)| \leq \sum_{r=m+1}^{n} \sup_{0 \leq s \leq t} |\alpha_r(s)| \leq \sum_{r=m+1}^{n} \frac{MK^{r-1}|t|^r}{r!}.$$

Hence $(c_n, n \in \mathbb{N})$ is uniformly Cauchy and so uniformly convergent on finite intervals $[0, t]$ (and also on intervals of the form $[-t, 0]$ by a similar argument.)

Define $c = (c(t), t \in \mathbb{R})$ by

$$c(t) = \lim_{n \to \infty} c_n(t) \quad \text{for each } t \in \mathbb{R}.$$

To see that c solves (6.1), note first that by (6.2) and the uniformity of the convergence we have, for each $t \in \mathbb{R}, n \in \mathbb{N}$,

$$\begin{aligned} \left| \int_0^t b(c(s))ds - \int_0^t b(c_n(s))ds \right| &\le \int_0^t |b(c(s)) - b(c_n(s))|ds \\ &\le Kt \sup_{0 \le s \le t} |c(s) - c_n(s)| \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Hence, for each $t \in \mathbb{R}$,

$$\begin{aligned} c(t) - c(0) + \int_0^t b(c(s))ds &= \lim_{n \to \infty} \left[c_{n+1}(t) - c(0) + \int_0^t b(c_n(s))ds \right] \\ &= 0. \end{aligned}$$

Finally, we show that the solution is unique. Assume that c' is another solution of (6.1) and, for each $n \in \mathbb{N}$, $t \in \mathbb{R}$, define

$$\beta_n(t) = c_n(t) - c'(t),$$

so that $\beta_{n+1}(t) = \int_0^t b(\beta_n(s))ds$.

Arguing as above, we obtain the estimate

$$|\beta_n(t)| \le \frac{MK^{n-1}|t|^n}{n!},$$

from which we deduce that each $\lim_{n\to\infty} \beta_n(t) = 0$, so that $c(t) = c'(t)$ as required. □

Note that by the uniformity of the convergence in the proof of Theorem 6.1.3 the map $t \to c(t)$ is continuous from $\mathbb{R}$ to $\mathbb{R}^d$.

Now that we have constructed unique solutions to equations of the type (6.1), we would like to explore some of their properties. A useful tool in this regard is *Gronwall's inequality*, which will also play a major role in the analysis of solutions to SDEs.

Proposition 6.1.4 (Gronwall's inequality) *Let $[a, b]$ be a closed interval in $\mathbb{R}$ and $\alpha, \beta : [a, b] \to \mathbb{R}$ be non-negative with α locally bounded and β integrable. If there exists $C \geq 0$ such that, for all $t \in [a, b]$,*

$$\alpha(t) \leq C + \int_a^t \alpha(s)\beta(s)ds, \tag{6.5}$$

then we have

$$\alpha(t) \leq C \exp\left[\int_a^t \beta(s)ds\right]$$

for all $t \in [a, b]$.

Proof First assume that $C > 0$ and let $h : [a, b] \to (0, \infty)$ be defined by

$$h(t) = C + \int_a^t \alpha(s)\beta(s)ds$$

for all $t \in [a, b]$. By Lebesgue's differentiation theorem (see e.g. Cohn [80], p. 187), h is differentiable on (a, b), with

$$h'(t) = \alpha(t)\beta(t) \leq h(t)\beta(t)$$

by (6.5), for (Lebesgue) almost all $t \in (a, b)$.

Hence $h'(t)/h(t) \leq \beta(t)$ (a.e.) and the required result follows on integrating both sides between a and b.

Now suppose that $C = 0$; then, by the above analysis, for each $t \in [a, b]$ we have $\alpha(t) \leq (1/n) \exp\left[\int_a^b \beta(s)ds\right]$ for each $n \in \mathbb{N}$, hence $\alpha(t) = 0$ as required. □

Note that in the case where equality holds in (6.5), Gronwall's inequality is (essentially) just the familiar integrating factor method for solving first-order linear differential equations.

Now let us return to our consideration of the solutions to (6.1). There are two useful perspectives from which we can regard these.

- If we fix the initial condition $c_0 = x \in \mathbb{R}$ then the solution is a curve $(c(t), t \in \mathbb{R})$ in $\mathbb{R}^d$ passing through x when $t = 0$.
- If we allow the initial condition to vary, we can regard the solution as a function of two variables $(c(t, x), t \in \mathbb{R}, x \in \mathbb{R}^d)$ that generates a family of curves.

It is fruitful to introduce some notation that allows us to focus more clearly on our ability to vary the initial conditions. To this end we define for each $t \in \mathbb{R}$, $x \in \mathbb{R}^d$,

$$\xi_t(x) = c(t, x),$$

so that each $\xi_t : \mathbb{R}^d \to \mathbb{R}^d$.

Lemma 6.1.5 *For each $t \in \mathbb{R}$, $x, y \in \mathbb{R}^d$,*

$$|\xi_t(x) - \xi_t(y)| \leq e^{K|t|}|x - y|,$$

so that, in particular, each $\xi_t : \mathbb{R}^d \to \mathbb{R}^d$ is continuous.

Proof Fix t, x and y and let $\gamma_t = |\xi_t(x) - \xi_t(y)|$. By (6.1) and (6.2) we obtain

$$\gamma_t \ \leq \ |x - y| + \int_0^t |b(\xi_s(x)) - b(\xi_s(y))|ds \ \leq \ |x - y| + K \int_0^t \gamma_s ds,$$

and the result follows by Gronwall's inequality. □

Suppose now that b is C^1; then we may differentiate b at each $x \in \mathbb{R}^d$, and its derivative $Db(x) : \mathbb{R}^d \to \mathbb{R}^d$ is the Jacobian matrix of b. We will now investigate the implications of the smoothness of b for the solution $(\xi_t, t \in \mathbb{R})$.

Exercise 6.1.6 Let $(\xi_t, t \geq 0)$ be the solution of (6.1) and suppose that $b \in C_b^1(\mathbb{R}^d)$. Deduce that for each $x \in \mathbb{R}^d$ there is a unique solution to the $d \times d$-matrix-valued differential equation

$$\frac{d}{dt}\gamma(t, x) = Db(\xi_t(x))\gamma(t, x)$$

with initial condition $\gamma(0, x) = I$.

Theorem 6.1.7 *If $b \in C_b^k(\mathbb{R}^d)$ for some $k \in \mathbb{N}$, then $\xi_t \in C^k(\mathbb{R}^d)$ for each $t \in \mathbb{R}$.*

Proof We begin by considering the case $k = 1$.

Let γ be as in Exercise 6.1.6. We will show that ξ_t is differentiable and that $D\xi_t(x) = \gamma(t, x)$ for each $t \in \mathbb{R}$, $x \in \mathbb{R}^d$.

Fix $h \in \mathbb{R}^d$ and let $\theta(t,h) = \xi_t(x+h) - \xi_t(x)$. Then, by (6.1),

$$\begin{aligned}\theta(t,h) - \gamma(t,x)(h) &= \int_0^t \big[b(\xi_s(x+h)) - b(\xi_s(x))\big]ds \\ &\quad - \int_0^t Db(\xi_s(x))\gamma(s,x)(h)ds \\ &= I_1(t) + I_2(t), \end{aligned} \tag{6.6}$$

where

$$I_1(t) = \int_0^t \big[b(\xi_s(x+h)) - b(\xi_s(x)) - Db(\xi_s(x))\theta(s,h)\big]ds$$

and

$$I_2(t) = \int_0^t Db(\xi_s(x))\big(\theta(s,h) - \gamma(s,x)(h)\big)ds.$$

By the mean value theorem,

$$|b(\xi_s(x+h)) - b(\xi_s(x))| \leq C|\theta(s,h)|,$$

where $C = d\ \sup_{y \in \mathbb{R}^d} \max_{1 \leq i,j \leq d} |Db(y)_{ij}|$. Hence, by Lemma 6.1.5,

$$|I_1(t)| \leq 2Ct \sup_{0 \leq s \leq t} |\theta(s,h)| \leq 2Ct|h|e^{K|t|}, \tag{6.7}$$

while

$$|I_2(t)| \leq C' \int_0^t |\theta(s,h) - \gamma(s,x)(h)|ds,$$

where $C' = Cd^{1/2}$.

Substitute (6.7) and (6.8) in (6.6) and apply Gronwall's inequality to deduce that

$$|\theta(t,h) - \gamma(t,x)(h)| \leq 2Ct|h|e^{(K+C')|t|},$$

from which the required result follows. From the result of Exercise 6.1.6, we also have the 'derivative flow' equation

$$\frac{dD\xi_t(x)}{dt} = Db(\xi_t(x))D\xi_t(x).$$

The general result is proved by induction using the argument given above. □

Exercise 6.1.8 Under the conditions of Theorem 6.1.7, show that, for all $x \in \mathbb{R}^d$, the map $t \to \xi_t(x)$ is C^{k+1}.

We recall that a bijection $\phi : \mathbb{R}^d \to \mathbb{R}^d$ is a *homeomorphism* if ϕ and ϕ^{-1} are both continuous and a C^k*-diffeomorphism* if ϕ and ϕ^{-1} are both C^k.

A family $\phi = \{\phi_t, t \in \mathbb{R}\}$ of homeomorphisms of $\mathbb{R}^d$ is called a *flow* if

$$\phi_0 = I \quad \text{and} \quad \phi_s \phi_t = \phi_{s+t} \tag{6.8}$$

for all $s, t \in \mathbb{R}$. If each ϕ_t is a C^k-diffeomorphism, we say that ϕ is a *flow of* C^k*-diffeomorphisms*.

Equation (6.8) is sometimes called the *flow property*. Note that an immediate consequence of it is that

$$\phi_t^{-1} = \phi_{-t}$$

for all $t \in \mathbb{R}$, so that (6.8) tells us that ϕ is a one-parameter group of homeomorphisms of $\mathbb{R}^d$.

Lemma 6.1.9 *If* $\phi = \{\phi_t, t \geq 0\}$ *is a family of* C^k*-mappings from* $\mathbb{R}^d$ *to* $\mathbb{R}^d$ *such that* $\phi_0 = I$ *and* $\phi_s \phi_t = \phi_{s+t}$ *for all* s, $t \in \mathbb{R}$ *then* ϕ *is a flow of* C^k*-diffeomorphisms.*

Proof It is enough to observe that, for all $t \in \mathbb{R}$, we have $\phi_{-t}\phi_t = \phi_t \phi_{-t} = I$, so that each ϕ_t has a two-sided C^k-inverse and thus is a C^k-diffeomorphism. □

Theorem 6.1.10 *Let* $\xi = (\xi_t, t \in \mathbb{R})$ *be the unique solution of* (6.1). *If* $b \in C_{\mathrm{b}}^k(\mathbb{R}^d)$*, then* ξ *is a flow of* C^k*-diffeomorphisms.*

Proof We seek to apply Lemma 6.1.9. By Theorem 6.1.7 we see that each $\xi_t \in C^k(\mathbb{R}^d)$, so we must establish the flow property.

The fact that $\xi_0 = I$ is immediate from (6.1). Now, for each $x \in \mathbb{R}^d$ and $s, t \in \mathbb{R}$,

$$\begin{aligned}
\xi_{t+s}(x) &= x + \int_0^{t+s} b(\xi_u(x))du \\
&= x + \int_0^s b(\xi_u(x))du + \int_s^{t+s} b(\xi_u(x))du \\
&= \xi_s(x) + \int_s^{t+s} b(\xi_u(x))du \\
&= \xi_s(x) + \int_0^t b(\xi_{u+s}(x))du.
\end{aligned}$$

However, we also have

$$\xi_t(\xi_s(x)) = \xi_s(x) + \int_0^t b(\xi_u(\xi_s(x)))du,$$

and it follows that $\xi_{t+s}(x) = \xi_t(\xi_s(x))$ by the uniqueness of solutions to (6.1). □

Exercise 6.1.11 Deduce that if b is Lipschitz then the solution $\xi = (\xi(t), t \in \mathbb{R})$ is a flow of homeomorphisms.

Exercise 6.1.12 Let ξ be the solution of (6.1) and let $f \in C^k(\mathbb{R}^d)$; show that

$$\frac{df}{dt}(\xi_t(x)) = b^i(\xi_t(x))\frac{\partial f}{\partial x_i}(\xi_t(x)). \tag{6.9}$$

If $b \in C^k(\mathbb{R}^d)$, it is convenient to consider the linear mapping $Y : C^{k+1}(\mathbb{R}^d) \to C^k(\mathbb{R}^d)$ defined by

$$(Yf)(x) = b^i(x)\frac{\partial f}{\partial x_i}(x)$$

for each $f \in C^k(\mathbb{R}^d),\ x \in \mathbb{R}^d$. The mapping Y is called a C^k-*vector field*. We denoted as $\mathcal{L}_k(\mathbb{R}^d)$ the set of all C^k-vector fields on $\mathbb{R}^d$.

Exercise 6.1.13 Let X, Y and $Z \in \mathcal{L}_k(\mathbb{R}^d)$.

(1) Show that $\alpha X + \beta Y \in \mathcal{L}_k(\mathbb{R}^d)$ for all $\alpha, \beta \in \mathbb{R}$.
(2) Show that the commutator $[X, Y] \in \mathcal{L}_k(\mathbb{R}^d)$, where

$$([X, Y]f)(x) = (X(Y(f))(x) - \big(Y(X(f)\big)(x)$$

for each $f \in C^k(\mathbb{R}^d),\ x \in \mathbb{R}^d$.
(3) Establish the *Jacobi identity*

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

We saw in the last exercise that $\mathcal{L}_k(\mathbb{R}^d)$ is a *Lie algebra*, i.e. it is a real vector space equipped with a binary operation $[\cdot, \cdot]$ that satisfies the Jacobi identity and the condition $[X, X] = 0$ for all $X \in \mathcal{L}_k(\mathbb{R}^d)$. Note that a Lie algebra is not an 'algebra' in the usual sense since the commutator bracket is not associative (we have the Jacobi identity instead).

In general, a vector field $Y = b^i\partial_i$ is said to be *complete* if the associated differential equation (6.1) has a unique solution $(\xi(t)(x), t \in \mathbb{R})$ for all initial conditions $x \in \mathbb{R}^d$. The vector field Y fails to be complete if the solution only exists locally, e.g. for all $t \in (a, b)$ where $-\infty < a < b < \infty$, and 'blows up' at a and b.

Exercise 6.1.14 Let $d = 1$ and $Y(x) = x^2 d/dx$ for each $x \in \mathbb{R}$. Show that Y is not complete.

If Y is complete, each $(\xi(t)(x), t \in \mathbb{R})$ is called the *integral curve* of Y through the point x, and the notation $\xi(t)(x) = \exp(Y)(x)$ is often employed to emphasise that, from an infinitesimal viewpoint, Y is the fundamental object from which all else flows. We call 'exp' the *exponential map*. These ideas all extend naturally to the more general set-up where $\mathbb{R}^d$ is replaced by a differentiable manifold.

6.2 Stochastic differential equations – existence and uniqueness

We now turn to the main business of this chapter. Let $(\Omega, \mathcal{F}, P)$ be a probability space equipped with a filtration $\{\mathcal{F}_t, t \geq 0\}$ that satisfies the usual hypotheses. Let $B = (B(t), t \geq 0)$ be an r-dimensional standard Brownian motion and N an independent Poisson random measure on $\mathbb{R}^+ \times (\mathbb{R}^d - \{0\})$ with associated compensator $\tilde{N}$ and intensity measure ν, where we assume that ν is a Lévy measure. We always assume that B and N are independent of $\mathcal{F}_0$.

In the last section, we considered ODEs of the form

$$\frac{dy(t)}{dt} = b(y(t)), \tag{6.10}$$

whose solution $(y(t), t \in \mathbb{R})$ is a curve in $\mathbb{R}^d$.

We begin by rewriting this 'Itô-style' as

$$dy(t) = b(y(t))dt. \tag{6.11}$$

Now restrict the parameter t to the non-negative half-line $\mathbb{R}^+$ and consider $y = (y(t), t \geq 0)$ as the evolution in time of the state of a system from some initial value $y(0)$. We now allow the system to be subject to random noise effects, which we introduce additively in (6.11). In general, these might be described in terms of arbitrary semimartingales (see e.g. Protter [298]), but in line with the usual philosophy of this book, we will use the 'noise' associated with a Lévy process.

We will focus on the following SDE:

$$\begin{aligned} dY(t) &= b(Y(t-))dt + \sigma(Y(t-))dB(t) \\ &\quad + \int_{|x|<c} F(Y(t-),x)\tilde{N}(dt,dx) \\ &\quad + \int_{|x|\geq c} G(Y(t-),x)N(dt,dx), \end{aligned} \tag{6.12}$$

which is a convenient shorthand for the system of SDEs

$$\begin{aligned} dY^i(t) &= b^i(Y(t-))dt + \sigma^i_j(Y(t-))dB^j(t) \\ &\quad + \int_{|x|<c} F^i(Y(t-),x)\tilde{N}(dt,dx) \\ &\quad + \int_{|x|\geq c} G^i(Y(t-),x)N(dt,dx), \end{aligned} \tag{6.13}$$

where each $1 \leq i \leq d$. Here the mappings $b^i : \mathbb{R}^d \to \mathbb{R}, \sigma^i_j : \mathbb{R}^d \to \mathbb{R}, F^i : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ and $G^i : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ are all assumed to be measurable for $1 \leq i \leq d$, $1 \leq j \leq r$. Further conditions on these mappings will follow later. The convenient parameter $c \in [0, \infty]$ allows us to specify what we mean by 'large' and 'small' jumps in specific applications. Quite often, it will be convenient to take $c = 1$. If we want to put both 'small' and 'large' jumps on the same footing we take $c = \infty$ (or 0), so that the term involving G (or F, respectively) is absent in (6.12)).

We will always consider (6.12), or equivalently (6.13), as a random initial-value problem with a fixed initial condition $Y(0) = Y_0$, where Y_0 is a given $\mathbb{R}^d$-valued random vector. Sometimes we may want to fix $Y_0 = y_0$ (a.s.), where $y_0 \in \mathbb{R}^d$.

In order to give (6.13) a rigorous meaning we rewrite it in integral form, for each $t \geq 0$, $1 \leq i \leq d$, as

$$\begin{aligned} Y^i(t) &= Y^i(0) + \int_0^t b^i(Y(s-))ds + \int_0^t \sigma^i_j(Y(s-))dB^j(s) \\ &\quad + \int_0^t \int_{|x|<c} F^i(Y(s-),x)\tilde{N}(ds,dx) \\ &\quad + \int_0^t \int_{|x|\geq c} G^i(Y(s-),x)N(ds,dx) \quad \text{a.s.} \end{aligned} \tag{6.14}$$

The solution to (6.14), when it exists, will be an $\mathbb{R}^d$-valued stochastic process $(Y(t), t \geq 0)$ with each $Y(t) = (Y^1(t), \ldots, Y^{\mathrm{d}}(t))$. Note that we are implicitly assuming that Y has left-limits in our formulation of (6.14), and we will in fact be seeking càdlàg solutions so that this is guaranteed.

As we have specified the noise B and N in advance, any solution to (6.14) is sometimes called a *strong solution* in the literature. There is also a notion of a *weak solution*, which we will discuss in Section 6.7.3. We will require solutions to (6.14) to be unique, and there are various notions of uniqueness available. The strongest of these, which we will look for here, is to require our solutions to be *pathwise unique*, i.e. if $Y_1 = (Y_1(t), t \geq 0)$ and $Y_2 = (Y_2(t), t \geq 0)$ are both solutions to (6.14) then $P\big(Y_1(t) = Y_2(t) \text{ for all } t \geq 0\big) = 1$.

The term in (6.14) involving large jumps is that controlled by G. This is easy to handle using interlacing, and it makes sense to begin by omitting this term and concentrate on the study of the equation driven by continuous noise interspersed with small jumps. To this end, we introduce the *modified* SDE

$$dZ(t) = b(Z(t-))dt + \sigma(Z(t-))dB(t) + \int_{|x|<c} F(Z(t-), x)\tilde{N}(dt, dx), \tag{6.15}$$

with initial condition $Z(0) = Z_0$.

We now impose some conditions on the mappings b, σ and F that will enable us to solve (6.15). First, for each $x, y \in \mathbb{R}^d$ we introduce the $d \times d$ matrix

$$a(x, y) = \sigma(x)\sigma(y)^{\mathrm{T}},$$

so that $a^{ik}(x, y) = \sum_{j=1}^{r} \sigma_j^i(x)\sigma_j^k(y)$ for each $1 \leq i, k \leq d$.

We will have need of the matrix seminorm on $d \times d$ matrices, given by

$$||a|| = \sum_{i=1}^{d} |a_i^i|.$$

We impose the following two conditions.

(C1) Lipschitz condition There exists $K_1 > 0$ such that, for all $y_1, y_2 \in \mathbb{R}^d$,

$$|b(y_1) - b(y_2)|^2 + ||a(y_1, y_1) - 2a(y_1, y_2) + a(y_2, y_2)|| + \int_{|x|<c} |F(y_1, x) - F(y_2, x)|^2 \nu(dx) \leq K_1 |y_1 - y_2|^2. \tag{6.16}$$

(C2) Growth condition There exists $K_2 > 0$ such that, for all $y \in \mathbb{R}^d$,

$$|b(y)|^2 + ||a(y,y)|| + \int_{|x|<c} |F(y,x)|^2 \nu(dx) \;\leq\; K_2(1 + |y|^2). \tag{6.17}$$

We make some comments on these.

First, the condition $||a(y_1, y_1) - 2a(y_1, y_2) + a(y_2, y_2)|| \leq L|y_1 - y_2|^2$, for some $L > 0$, is sometimes called *bi-Lipschitz continuity*. It may seem at odds with the other terms on the left-hand side of (6.16) but this is an illusion. A straightforward calculation yields

$$||a(y_1, y_1) - 2a(y_1, y_2) + a(y_2, y_2)|| = \sum_{i=1}^{d} \sum_{j=1}^{r} \left[\sigma_j^i(y_1) - \sigma_j^i(y_2)\right]^2,$$

and if you take $d = r = 1$ then

$$|a(y_1, y_1) - 2a(y_1, y_2) + a(y_2, y_2)| = |\sigma(y_1) - \sigma(y_2)|^2.$$

Exercise 6.2.1 If a is bi-Lipschitz continuous, show that there exists $L_1 > 0$ such that

$$||a(y,y)|| \leq L_1(1 + ||y||^2)$$

for all $y \in \mathbb{R}^d$.

Our second comment on the conditions is this: if you take $F = 0$, it follows from Exercises 6.1.2 and 6.2.1 that the growth condition (C2) is a consequence of the Lipschitz condition (C1). Hence in the case of non-zero F, in the presence of (C1), (C2) is equivalent to the requirement that there exists $M > 0$ such that, for all $y \in \mathbb{R}^d$,

$$\int_{|x|<c} |F(y,x)|^2 \nu(dx) \leq M(1 + |y|^2).$$

Exercise 6.2.2

(1) Show that if ν is finite, then the growth condition is a consequence of the Lipschitz condition.

(2) Show that if $F(y,x) = H(y)f(x)$ for all $y \in \mathbb{R}^d$, $|x| \leq c$, where H is Lipschitz continuous and $\int_{|x|\leq c} |f(x)|^2 \nu(dx) < \infty$, then the growth condition is a consequence of the Lipschitz condition.

Having imposed conditions on our coefficients, we now discuss the initial condition. Throughout this chapter, we will always deal with the *standard initial condition* $Y(0) = Y_0$ (a.s.), for which Y_0 is $\mathcal{F}_0$-measurable. Hence $Y(0)$ is independent of the noise B and N.

Throughout the remainder of this chapter, we will frequently employ the following inequality for $n \in \mathbb{N}$ and $x_1, x_2, \ldots, x_n \in \mathbb{R}$:

$$|x_1 + x_2 + \cdots + x_n|^2 \leq n(|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2). \tag{6.18}$$

This is easily verified by using induction and the Cauchy–Schwarz inequality.

Our existence and uniqueness theorem will employ the technique of Picard iteration, which served us well in the ODE case (Theorem 6.1.3); cf. Ikeda and Watanabe [167], chapter 4, section 9.

Theorem 6.2.3 *Assume the Lipschitz and growth conditions. There exists a unique solution $Z = (Z(t), t \geq 0)$ to the modified SDE (6.15) with the standard initial condition. The process Z is adapted and càdlàg.*

Our strategy is to first carry out the proof of existence and uniqueness in the case $\mathbb{E}(|Z_0|^2) < \infty$ and then consider the case $\mathbb{E}(|Z_0|^2) = \infty$.

Proof of existence for $\mathbb{E}(|Z_0|^2) < \infty$ Define a sequence of processes $(Z_n, n \in \mathbb{N} \cup \{0\})$ by $Z_0(t) = Z_0$ and, for all $n \in \mathbb{N} \cup \{0\}$, $t \geq 0$,

$$\begin{aligned} dZ_{n+1}(t) &= b(Z_n(t-))dt + \sigma(Z_n(t-))dB(t) \\ &\quad + \int_{|x|<c} F(Z_n(t-), x)\tilde{N}(dt, dx). \end{aligned}$$

A simple inductive argument and use of Theorem 4.2.12 demonstrates that each Z_n is adapted and càdlàg.

For each $1 \leq i \leq d$, $n \in \mathbb{N} \cup \{0\}$, $t \geq 0$, we have

$$\begin{aligned} &Z^i_{n+1}(t) - Z^i_n(t) \\ &\quad = \int_0^t \big[b^i(Z_n(s-)) - b^i(Z_{n-1}(s-)\big]ds \\ &\qquad + \int_0^t \big[\sigma^i_j(Z_n(s-)) - \sigma^i_j(Z_{n-1}(s-))\big]dB^j(s) \\ &\qquad + \int_0^t \int_{|x|<c} \big[F^i(Z_n(s-), x) - F^i(Z_{n-1}(s-), x)\big]\tilde{N}(ds, dx). \end{aligned}$$

We need to obtain some inequalities, and we begin with the case $n = 0$.

First note that on using the inequality (6.18), with $n = 3$, we have

$$
\begin{aligned}
&|Z_1(t) - Z_0(t)|^2 \\
&= \sum_{i=1}^{d} \bigg[\int_0^t b^i(Z(0))ds + \int_0^t \sigma_j^i(Z(0))dB^j(s) \\
&\quad + \int_0^t \int_{|x|<c} F^i(Z(0), x)\tilde{N}(ds, dx) \bigg]^2 \\
&\leq 3 \sum_{i=1}^{d} \Bigg\{ \bigg[\int_0^t b^i(Z(0))ds \bigg]^2 + \bigg[\int_0^t \sigma_j^i(Z(0))dB^j(s) \bigg]^2 \\
&\quad + \bigg[\int_0^t \int_{|x|<c} F^i(Z(0), x)\tilde{N}(ds, dx) \bigg]^2 \Bigg\} \\
&= 3 \sum_{i=1}^{d} \Bigg\{ t^2 \big[b^i(Z(0)) \big]^2 + \big[\sigma_j^i(Z(0))B^j(t) \big]^2 \\
&\quad + \bigg[\int_{|x|<c} F^i(Z(0), x)\tilde{N}(t, dx) \bigg]^2 \Bigg\}
\end{aligned}
$$

for each $t \geq 0$. We now take expectations and apply Doob's martingale inequality to obtain

$$
\begin{aligned}
&\mathbb{E}\bigg(\sup_{0 \leq s \leq t} |Z_1(s) - Z_0(s)|^2 \bigg) \\
&\quad \leq 3t^2\, \mathbb{E}(|b(Z(0))|^2) + 12t\, \mathbb{E}(||a(Z(0), Z(0))||) \\
&\qquad + 12t \int_{|x|<c} \mathbb{E}(|F(Z(0), x)|)^2\, \nu(dx).
\end{aligned}
$$

On applying the growth condition (C2), we can finally deduce that

$$
\mathbb{E}\bigg(\sup_{0 \leq s \leq t} |Z_1(s) - Z_0(s)|^2 \bigg) \leq C_1(t)tK_2(1 + \mathbb{E}(|Z(0)|^2)), \tag{6.19}
$$

where $C_1(t) = \max\{3t, 12\}$.

We now consider the case for general $n \in \mathbb{N}$. Arguing as above, we obtain

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\leq s\leq t} |Z_{n+1}(s) - Z_n(s)|^2\right) \\
&\leq \sum_{i=1}^{d} \left[3\,\mathbb{E}\left(\sup_{0\leq s\leq t}\left\{\int_0^s \left[b^i\big(Z_n(u-)\big) - b^i(Z_{n-1}(u-))\right]du\right\}^2\right)\right. \\
&\quad + 12\,\mathbb{E}\left(\left\{\int_0^t \left[\sigma_j^i(Z_n(s-)) - \sigma_j^i(Z_{n-1}(s-))\right]dB^j(s)\right\}^2\right) \\
&\quad + 12\,\mathbb{E}\left(\left\{\int_0^t \int_{|x|<c} \left[F^i(Z_n(s-),x)\right.\right.\right. \\
&\quad \left.\left.\left.\left. -F^i(Z_{n-1}(s-),x)\right]\tilde{N}(ds,dx)\right\}^2\right)\right].
\end{aligned}$$

By the Cauchy–Schwarz inequality, for all $s \geq 0$,

$$\begin{aligned}
&\left\{\int_0^s \left[b^i(Z_n(u-)) - b^i(Z_{n-1}(u-))\right]du\right\}^2 \\
&\quad \leq s\int_0^s \left[b^i(Z_n(u-)) - b^i(Z_{n-1}(u-))\right]^2 du
\end{aligned}$$

and so, by Itô's isometry, we obtain

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\leq s\leq t} |Z_{n+1}(s) - Z_n(s)|^2\right) \\
&\leq C_1(t)\left[\int_0^t \mathbb{E}\big(|b(Z_n(s-)) - b(Z_{n-1}(s-))|^2\big)ds\right. \\
&\quad + \int_0^t \mathbb{E}\big(\big|a(Z_n(s-),Z_n(s-)) - 2a(Z_n(s-),Z_{n-1}(s-)) \\
&\quad + a(Z_{n-1}(s-),Z_{n-1}(s-))\big|\big)ds \\
&\quad \left.+ \int_0^t\int_{|x|<c} \mathbb{E}\big(\big|F(Z_n(s-),x) - F(Z_{n-1}(s-),x)\big|^2\big)\nu(dx)ds\right].
\end{aligned}$$

We now apply the Lipschitz condition (C1) to find that

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\le s\le t}|Z_{n+1}(s)-Z_n(s)|^2\right)\\
&\quad\le C_1(t)K_1\int_0^t \mathbb{E}\left(\sup_{0\le u\le s}|Z_n(u)-Z_{n-1}(u)|^2\right)ds \qquad (6.20)
\end{aligned}$$

By induction based on (6.19) and (6.20), we thus deduce the key estimate

$$\mathbb{E}\left(\sup_{0\le s\le t}|Z_n(s)-Z_{n-1}(s)|^2\right)\le \frac{C_2(t)^n K_3^n}{n!} \qquad (6.21)$$

for all $n\in\mathbb{N}$, where $C_2(t)=tC_1(t)$ and

$$K_3=\max\left\{K_1,\ K_2\left[1+\mathbb{E}(|Z(0)|^2)\right]\right\}.$$

Our first observation is that $(Z_n(t), t\ge 0)$ is convergent in L^2 for each $t\ge 0$. Indeed, for each $m,n\in\mathbb{N}$ we have (using $||\cdot||_2=[\mathbb{E}(|\cdot|^2)]^{1/2}$ to denote the L^2-norm), for each $0\le s\le t$,

$$||Z_n(s)-Z_m(s)||_2\le\sum_{r=m+1}^{n}||Z_r(s)-Z_{r-1}(s)||_2\le\sum_{r=m+1}^{n}\frac{C_2(t)^{r/2}K_3^{r/2}}{(r!)^{1/2}},$$

and, since the series on the right converges, we have that each $(Z_n(s), n\in\mathbb{N})$ is Cauchy and hence convergent to some $Z(s)\in L^2(\Omega,\mathcal{F},P)$. We denote as Z the process $(Z(t), t\ge 0)$. A standard limiting argument yields the useful estimate

$$||Z(s)-Z_n(s)||_2\le\sum_{r=n+1}^{\infty}\frac{C_2(t)^{r/2}K_3^{r/2}}{(r!)^{1/2}}, \qquad (6.22)$$

for each $n\in\mathbb{N}\cup\{0\}$, $0\le s\le t$.

We also need to establish the almost sure convergence of $(Z_n, n\in\mathbb{N})$. Applying the Chebyshev–Markov inequality in (6.21), we deduce that

$$P\left(\sup_{0\le s\le t}|Z_n(s)-Z_{n-1}(s)|\ge\frac{1}{2^n}\right)\le\frac{[4K_3C_2(t)]^n}{n!},$$

from which we see that

$$P\left(\limsup_{n\to\infty} \sup_{0\le s\le t} |Z_n(s) - Z_{n-1}(s)| \ge \frac{1}{2^n}\right) = 0,$$

by Borel's lemma. Arguing as in Theorem 2.6.2, we deduce that $(Z_n, n \in \mathbb{N}\cup\{0\})$ is almost surely uniformly convergent on finite intervals $[0, t]$ to Z, from which it follows that Z is adapted and càdlàg.

Now we must verify that Z really satisfies the SDE. Define a stochastic process $\tilde{Z} = (\tilde{Z}(t), t \ge 0)$ by

$$\begin{aligned}\tilde{Z}^i(t) = Z_0^i &+ \int_0^t b^i(Z(s-))ds + \int_0^t \sigma_j^i(Z(s-))dB^j(s) \\ &+ \int_0^t \int_{|x|<c} F^i(Z(s-), x)\tilde{N}(ds, dx)\end{aligned}$$

for each $1 \le i \le d$, $t \ge 0$. Hence, for each $n \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned}\tilde{Z}^i(t) - Z_n^i(t) = &\int_0^t \big[b^i(Z(s-)) - b^i(Z_n(s-))\big]ds \\ &+ \int_0^t \big[\sigma_j^i(Z(s-)) - \sigma_j^i(Z_n(s-))\big]dB^j(s) \\ &+ \int_0^t \int_{|x|<c} \big[F^i(Z(s-), x) - F^i(Z_n(s-), x)\big]\tilde{N}(ds, dx).\end{aligned}$$

Now using the same argument with which we derived (6.20) and then applying (6.22), we obtain for all $0 \le s \le t < \infty$,

$$\begin{aligned}\mathbb{E}(|\tilde{Z}(s) - Z_n(s)|^2) &\le C_1(t)K_1 \int_0^t \mathbb{E}(|Z(u) - Z_n(u)|^2)du \\ &\le C_2(t)K_1 \sup_{0\le u\le t} \mathbb{E}(|Z(u) - Z_n(u)|^2) \\ &\le C_2(t)K_1 \left(\sum_{r=n+1}^{\infty} \frac{C_2(t)^{r/2}K_3^{r/2}}{(r!)^{1/2}}\right)^2 \\ &\to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Hence each $Z(s) = L^2 - \lim_{n\to\infty} Z_n(s)$ and so, by uniqueness of limits, $\tilde{Z}(s) = Z(s)$ (a.s.) as required.

Proof of uniqueness for $\mathbb{E}(|Z_0|^2) < \infty$ Let Z_1 and Z_2 be two distinct solutions to (6.15). Hence, for each $t \geq 0$, $1 \leq i \leq d$,

$$
\begin{aligned}
&Z_1^i(t) - Z_2^i(t) \\
&\quad = \int_0^t \big[b^i(Z_1(s-)) - b^i(Z_2(s-))\big]ds \\
&\qquad + \int_0^t \big[\sigma_j^i(Z_1(s-)) - \sigma_j^i(Z_2(s-))\big]dB^j(s) \\
&\qquad + \int_0^t \int_{|x|<c} \big[F^i(Z_1(s-), x) - F^i(Z_2(s-), x)\big]\tilde{N}(ds, dx).
\end{aligned}
$$

We again follow the same line of argument as used in deducing (6.20), to find that

$$
\begin{aligned}
&\mathbb{E}\left(\sup_{0\leq s\leq t} |Z_1(s) - Z_2(s)|^2\right) \\
&\qquad \leq C_1(t)K_1 \int_0^t \mathbb{E}\left(\sup_{0\leq u\leq s} |Z_1(u) - Z_2(u)|^2\right) ds.
\end{aligned}
$$

Thus, by Gronwall's inequality, $\mathbb{E}\big(\sup_{0\leq s\leq t} |Z_1(s) - Z_2(s)|^2\big) = 0$. Hence $Z_1(s) = Z_2(s)$ for all $0 \leq s \leq t$ (a.s.). By continuity of probability, we obtain, as required,

$$
\begin{aligned}
&P\big(Z_1(t) = Z_2(t) \text{ for all } t \geq 0\big) \\
&\qquad = P\left(\bigcap_{N\in\mathbb{N}} \big(Z_1(t) = Z_2(t) \text{ for all } 0 \leq t \leq N\big)\right) = 1.
\end{aligned}
$$

Proof of existence and uniqueness for $\mathbb{E}(|Z_0|^2) = \infty$ (cf. Itô [174]). For each $n \in \mathbb{N}$, define $\Omega_N = \{\omega \in \Omega; |Z_0| \leq N\}$. Then $\Omega_N \subseteq \Omega_M$ whenever $n \leq M$ and $\Omega = \bigcup_{n\in\mathbb{N}} \Omega_N$. Let $Z_0^N = Z_0\chi_{\Omega_N}$. By the above analysis, the equation (6.15) with initial condition Z_0^N has a unique solution $(Z_N(t), t \geq 0)$. Clearly, for $M > N, Z_M(t)(\omega) = Z_{M-1}(t)(\omega) = \cdots = Z_N(t)(\omega)$ for all $t \geq 0$ and almost all $\omega \in \Omega_N$.

By continuity of probability, given any $\epsilon > 0$ there exists $n \in N$ such that $n > N \Rightarrow P(\Omega_n) > 1 - \epsilon$. Then given any $\delta > 0$, for all $m, n > N$,

$$
P\left(\sup_{t\geq 0} |Z_n(t) - Z_m(t)| > \delta\right) < \epsilon.
$$

Hence $(Z_n, n \in \mathbb{N})$ is uniformly Cauchy in probability and so is uniformly convergent in probability to a process $Z = (Z(t), t \geq 0)$. We can extract a subsequence for which the convergence holds uniformly (a.s.) and from this it follows that Z is adapted, càdlàg and solves (6.15).

For uniqueness, suppose that $Z' = (Z'(t), t \geq 0)$ is another solution to (6.15); then, for all $M \geq N$, $Z'(t)(\omega) = Z_M(t)(\omega)$ for all $t \geq 0$ and almost all $\omega \in \Omega_N$. For, suppose this fails to be true for some $M \geq N$. Define $Z''_M(t)(\omega) = Z'_M(t)(\omega)$ for $\omega \in \Omega_N$ and $Z''_M(t)(\omega) = Z_M(t)(\omega)$ for $\omega \in \Omega_N^c$. Then Z''_M and Z_M are distinct solutions to (6.15) with the same initial condition Z_0^M, and our earlier uniqueness result gives the required contradiction. That $P(Z(t) = Z'(t)$ for all $t \geq 0) = 1$ follows by a straightforward limiting argument, as above. □

Corollary 6.2.4 *Let Z be the unique solution of* (6.15) *as constructed in Theorem* 6.2.3. *If* $\mathbb{E}(|Z_0|^2) < \infty$ *then* $\mathbb{E}(|Z(t)|^2) < \infty$ *for each* $t \geq 0$ *and there exists a constant* $D(t) > 0$ *such that*

$$\mathbb{E}(|Z(t)|^2) \leq D(t)\big[1 + \mathbb{E}(|Z_0|^2)\big].$$

Proof By (6.22) we see that, for each $t \geq 0$, there exists $C(t) \geq 0$ such that

$$||Z(t) - Z_0||_2 \leq \sum_{n=0}^{\infty} ||Z_n(t) - Z_{n-1}(t)||_2 \leq C(t).$$

Now

$$\mathbb{E}(|Z(t)|^2) \leq 2\,\mathbb{E}(|Z(t) - Z(0)|^2) + 2\,\mathbb{E}(|Z(0)|^2),$$

and the required result follows with $D(t) = 2\max\{1, C(t)^2\}$. □

Exercise 6.2.5 Consider the SDE

$$dZ(t) = \sigma(Z(t-))dB(t) + \int_{|x|<c} F(Z(t-), x)\tilde{N}(dt, dx)$$

satisfying all the conditions of Corollary 6.2.4. Deduce that Z is a square-integrable martingale. Hence deduce that the discounted stock price $\tilde{S}_1$ discussed in Section 5.6.3 is indeed a martingale, as was promised.

Exercise 6.2.6 Deduce that $Z = (Z(t), t \geq 0)$ has continuous sample paths, where

$$dZ(t) = b(Z(t))dt + \sigma(Z(t))dB(t).$$

(Hint: Use the uniformity of the convergence in Theorem 6.2.3 and recall the discussion of Section 4.3.1)

Exercise 6.2.7 Show that the following Lipschitz condition on the matrix-valued function $\sigma(\cdot)$ is a sufficient condition for the bi-Lipschitz continuity of $a(\cdot) = \sigma(\cdot)\sigma(\cdot)^{\mathrm{T}}$: there exists $K > 0$ such that, for each $1 \leq i \leq d$, $1 \leq j \leq r$, $y_1, y_2 \in \mathbb{R}^d$,

$$\left|\sigma_j^i(y_1) - \sigma_j^i(y_2)\right| \leq K|y_1 - y_2|.$$

Having dealt with the modified equation, we can now apply a standard interlacing procedure to construct the solution to the original equation (6.12). We impose the following assumption on the coefficient G, which ensures that the integrands in Poisson integrals are predictable.

Assumption 6.2.8 From now on we will assume that $c > 0$. We also require that the mapping $y \to G(y, x)$ is continuous for all $x \geq c$.

Theorem 6.2.9 *There exists a unique càdlàg adapted solution to* (6.12).

Proof Let $(\tau_n, n \in \mathbb{N})$ be the arrival times for the jumps of the compound Poisson process $(P(t), t \geq 0)$, where each $P(t) = \int_{|x|\geq c} xN(t, dx)$. We then construct a solution to (6.12) as follows:

$$\begin{aligned}
Y(t) &= Z(t) && \text{for } \ 0 \leq t < \tau_1,\\
Y(\tau_1) &= Z(\tau_1-) + G(Z(\tau_1-), \Delta P(\tau_1)) && \text{for } \ t = \tau_1,\\
Y(t) &= Y(\tau_1) + Z_1(t) - Z_1(\tau_1) && \text{for } \tau_1 < t < \tau_2,\\
Y(\tau_2) &= Y(\tau_2-) + G(Y(\tau_2-), \Delta P(\tau_2)) && \text{for } \ t = \tau_1,
\end{aligned}$$

and so on, recursively. Here Z_1 is the unique solution to (6.15) with initial condition $Z_1(0) = Y(\tau_1)$. Y is clearly adapted, càdlàg and solves (6.12). Uniqueness follows by the uniqueness in Theorem 6.2.3 and the interlacing structure. □

Note Theorem 6.2.9 may be generalised considerably. More sophisticated techniques were developed by Protter [298], chapter 5, and Jacod [186], pp. 451ff., in the case where the driving noise is a general semimartingale with jumps.

In some problems we might require time-dependent coefficients and so we study the inhomogeneous SDE

$$
\begin{aligned}
dY(t) = b(t, Y(t-))dt + \sigma(t, Y(t-))dB(t) \\
+ \int_{|x|<c} F(t, Y(t-), x)\tilde{N}(dt, dx) \\
+ \int_{|x|>c} G(t, Y(t-), x)N(dt, dx). \qquad (6.23)
\end{aligned}
$$

We can again reduce this problem by interlacing to the study of the modified SDE with small jumps. In order to solve the latter we can impose the following (crude) Lipschitz and growth conditions.

For each $t > 0$, there exists $K_1(t) > 0$ such that, for all $y_1, y_2 \in \mathbb{R}^d$,

$$
\begin{aligned}
|b(t, y_1) - b(t, y_2)| + ||a(t, y_1, y_1) - 2a(t, y_1, y_2) + a(t, y_2, y_2)|| \\
+ \int_{|x|<c} |F(t, y_1, x) - F(t, y_2, x)|^2 \nu(dx) \leq K_1(t)|y_1 - y_2|^2.
\end{aligned}
$$

There exists $K_2(t) > 0$ such that, for all $y \in \mathbb{R}^d$,

$$
|b(t, y)|^2 + ||a(t, y, y)|| + \int_{|x|<c} |F(t, y, x)|^2 \leq K_2(t)(1 + |y|^2),
$$

where $a(t, y_1, y_2) = \sigma(t, y_1)\sigma(t, y_2)^{\mathrm{T}}$ for each $t \geq 0$, $y_1, y_2 \in \mathbb{R}^d$. We assume that the mappings $t \to K_i(t)(i = 1, 2)$ are locally bounded and measurable.

Exercise 6.2.10 Show that (6.23) has a unique solution under the above conditions.

The final variation which we will examine in this chapter involves local solutions. Let T_∞ be a stopping time and suppose that $Y = (Y(t), 0 \leq t < T_\infty)$ is a solution to (6.12). We say that Y is a *local solution* if $T_\infty < \infty$ (a.s.) and a *global solution* if $T_\infty = \infty$ (a.s.). We call T_∞ the *explosion time* for the SDE (6.12). So far in this chapter we have looked at global solutions. If we want to allow local solutions to (6.12) we can weaken our hypotheses to allow *local* Lipschitz and growth conditions on our coefficients. More precisely we impose:

(C3) Local Lipschitz condition For all $n \in \mathbb{N}$ and $y_1, y_2 \in \mathbb{R}^d$ with $\max\{|y_1|, |y_2|\} \leq n$, there exists $K_1(n) > 0$ such that

$$|b(y_1) - b(y_2)| + ||a(y_1, y_1) - 2a(y_1, y_2) + a(y_2, y_2)|| \\ + \int_{|x|<c} |F(y_1, x) - F(y_2, x)|^2 \nu(dx) \leq K_1(n)|y_1 - y_2|^2.$$

(C4) Local Growth condition For all $n \in \mathbb{N}$ and for all $y \in \mathbb{R}^d$ with $|y| \leq n$, there exists $K_2(n) > 0$ such that

$$|b(y)|^2 + ||a(y, y)|| + \int_{|x|<c} |F(y, x)|^2 \leq K_2(n)(1 + |y|^2).$$

We then have

Theorem 6.2.11 *If we assume* (C3) *and* (C4) *and impose the standard initial condition, then there exists a unique local solution* $Y = (Y(t), 0 \leq t < T_\infty)$ *to the SDE* (6.12)*.*

Proof Once again we can reduce the problem by interlacing to the solution of the modified SDE. The proof in this case is almost identical to the case of equations driven by Brownian motion, and we refer the reader to the account of Durrett [99] for the details. □

We may also consider *backwards stochastic differential equations* on a time interval $[0, T]$. We write these as follows (in the time-homogeneous case):

$$\begin{aligned} dY(t) = -b(Y(t))dt - \sigma(Y(t)) \cdot_{\mathrm{b}} dB(t) \\ - \int_{|x|<c} F(t, Y(t-), x) \cdot_{\mathrm{b}} \tilde{N}(dt, dx) \\ - \int_{|x|>c} G(t, Y(t-), x) N(dt, dx), \end{aligned}$$

where $\cdot_{\mathrm{b}}$ denotes the backwards stochastic integral. The solution (when it exists) is a backwards adapted process $(Y(s);\ 0 \leq s \leq T)$. Instead of an initial condition $Y(0) = Y_0$ (a.s.) we impose a final condition $Y(T) = Y_T$ (a.s.). We then have the integral form

$$\begin{aligned} Y^i(s) = Y_T^i &- \int_s^T b^i(Y(u))du - \int_s^T \sigma_j^i(Y(u)) \cdot_b dB^j(u) \\ &- \int_s^T \int_{|x|<c} F^i(Y(u), x) \cdot_b \tilde{N}(dt, dx) \\ &- \int_s^T \int_{|x|\geq c} G^i(Y(u)), x) N(dt, dx) \quad \text{a.s.} \end{aligned}$$

for each $0 \leq s \leq T$, $1 \leq i \leq d$.

The theory of backwards SDEs can now be developed just as in the forward case, so we can obtain the existence and uniqueness of solutions by imposing Lipschitz and growth conditions on the coefficients and the usual independence condition on Y_T.

Backwards SDEs with discontinuous noise have not been developed so thoroughly as the forward case. This may change in the future as more applications are found; see e.g. Nualart and Schoutens [279], where backwards SDEs driven by the Teugels martingales of Exercise 2.4.24 are applied to option pricing. Other articles on backwards SDEs include Situ [336] and Ouknine [285].

6.3 Examples of SDEs

SDEs driven by Lévy processes

Let $X = (X(t), t \geq 0)$ be a Lévy process taking values in $\mathbb{R}^m$. We denote its Lévy–Itô decomposition as

$$X^i(t) = \lambda^i t + \tau_j^i B^j(t) + \int_0^t \int_{|x|<1} x^i \tilde{N}(ds, dx) + \int_0^t \int_{|x|\geq 1} x^i N(ds, dx)$$

for each $1 \leq i \leq m$, $t \geq 0$. Here, as usual, $\lambda \in \mathbb{R}^m$ and (τ_j^i) is a real-valued $m \times r$ matrix.

For each $1 \leq i \leq d$, $1 \leq j \leq m$, let $L_j^i : \mathbb{R}^d \to \mathbb{R}^d$ be measurable, and form the $d \times m$ matrix $L(x) = (L_j^i(x))$ for each $x \in \mathbb{R}^d$. We consider the SDE

$$dY(t) = L(Y(t-))dX(t), \tag{6.24}$$

with standard initial condition $Y(0) = Y_0$ (a.s.), so that, for each $1 \leq i \leq d$,

$$dY^i(t) = L_j^i(Y(t-))dX^j(t).$$

This is of the same form as (6.12), with coefficients given by $b(\cdot) = L(\cdot)\lambda$, $\sigma(\cdot) = L(\cdot)\tau$, $F(\cdot, x) = L(\cdot)x$ for $|x| < 1$ and $G(\cdot, x) = L(\cdot)x$ for $|x| \geq 1$.

To facilitate discussion of the existence and uniqueness of solutions of (6.24), we introduce two new matrix-valued functions, N, a $d \times d$ matrix given by $N(x) = L(x)\tau\tau^{\mathrm{T}}L(x)^{\mathrm{T}}$ for each $x \in \mathbb{R}^d$ and M, a $m \times m$ matrix defined by $M(x) = L(x)^{\mathrm{T}}L(x)$.

We impose the following Lipschitz-type conditions on M and N: there exist $D_1, D_2 > 0$ such that, for all $y_1, y_2 \in \mathbb{R}^d$,

$$||N(y_1, y_1) - 2N(y_1, y_2) + N(y_2, y_2)|| \leq D_1|y_1 - y_2|^2, \tag{6.25}$$

$$\max_{1 \leq p,q \leq m} |M_q^p(y_1, y_1) - 2M_q^p(y_1, y_2) + M_q^p(y_2, y_2)| \leq D_2|y_1 - y_2|^2. \tag{6.26}$$

Note that (6.25) is just the usual bi-Lipschitz condition, which allows control of the Brownian integral terms within SDEs.

Tedious but straightforward algebra then shows that (6.25) and (6.26) imply the Lipschitz and growth conditions (C1) and (C2) and hence, by Theorems 6.2.3 and 6.2.9, equation (6.24) has a unique solution. In applications, we often meet the case $m = d$ and $L = \operatorname{diag}(L_1, \ldots, L_d)$. In this case, readers can check that a sufficient condition for (6.25) and (6.26) is the single Lipschitz condition that there exists $D_3 > 0$ such that, for all $y_1, y_2 \in \mathbb{R}^d$,

$$|L(y_1) - L(y_2)| \leq D_3|y_1 - y_2|, \tag{6.27}$$

where we are regarding L as a vector-valued function.

Another class of SDEs that are often considered in the literature take the form

$$dY(t) = b(Y(t-)dt + L(Y(t-))dX(t),$$

and these clearly have a unique solution whenever L is as in (6.27) and b is globally Lipschitz. The important case where X is α-stable was studied by Janicki *et al*. [188].

Stochastic exponentials

We consider the equation

$$dY(t) = Y(t-)dX(t),$$

so that, for each $1 \leq i \leq d$, $dY^i(t) = Y^i(t-)dX^i(t)$.

This trivially satisfies the Lipschitz condition (6.27) and so has a unique solution. In the case $d = 1$ with $Y_0 = 1$ (a.s.), we saw in Section 5.1 that the solution is given by the *stochastic exponential*

$$Y(t) = \mathcal{E}_X(t) = \exp\Big\{X(t) - \tfrac{1}{2}[X_c, X_c](t)\Big\} \prod_{0 \le s \le t} \big[1 + \Delta X(s)\big]e^{-\Delta X(s)},$$

for each $t \ge 0$.

The Langevin equation and Ornstein–Uhlenbeck process revisited

The process $B = (B(t), t \ge 0)$ that we have been calling 'Brownian motion' throughout this book is not the best possible description of the physical phenomenon of Brownian motion.

A more realistic model was proposed by Ornstein and Uhlenbeck [284] in the 1930s; see also chapter 9 of Nelson [277] and Chandrasekar [76]. Let $x = (x(t), t \ge 0)$, where $x(t)$ is the displacement after time t of a particle of mass m executing Brownian motion, and let $v = (v(t), t \ge 0)$, where $v(t)$ is the velocity of the particle. Ornstein and Uhlenbeck argued that the total force on the particle should arise from a combination of random bombardments by the molecules of the fluid and also a macroscopic frictional force, which acts to dampen the motion. In accordance with Newton's laws, this total force equals the rate of change of momentum and so we write the formal equation

$$m\frac{dv}{dt} = -\beta m v + m\frac{dB}{dt},$$

where β is a positive constant (related to the viscosity of the fluid) and the formal derivative 'dB/dt' describes random velocity changes due to molecular bombardment. This equation acquires a meaning as soon as we interpret it as an Itô-style SDE. We thus obtain the *Langevin equation*, named in honour of the French physicist Paul Langevin,

$$dv(t) = -\beta v(t)dt + dB(t). \tag{6.28}$$

It is more appropriate for us to generalise this equation and replace B by a Lévy process $X = (X(t), t \ge 0)$, to obtain

$$dv(t) = -\beta v(t)dt + dX(t), \tag{6.29}$$

which we continue to call the Langevin equation. It has a unique solution by Theorem 6.2.9. We can in fact solve (6.29) by multiplying both sides by the integrating factor $e^{-\beta t}$ and using Itô's product formula. This yields our old friend the *Ornstein–Uhlenbeck process* (4.9),

$$v(t) = e^{-\beta t} v_0 + \int_0^t e^{-\beta(t-s)} dX(s)$$

for each $t \geq 0$. Recall from Exercise 4.3.18 that when X is a Brownian motion, v is Gaussian. In this latter case, the integrated Ornstein–Uhlenbeck process also has a physical interpretation. It is nothing but the displacement of the Brownian particle

$$x(t) = \int_0^t v(s) ds,$$

for each $t \geq 0$.

An interesting generalisation of the Langevin equation is obtained when the number β is replaced by a matrix Q, all of whose eigenvalues have a positive real part. We thus obtain the equation

$$dY(t) = -QY(t)dt + dX(t),$$

whose unique solution is the *generalised Ornstein–Uhlenbeck process*, $Y = (Y(t), t \geq 0)$, where, for each $t \geq 0$,

$$Y(t) = e^{-Qt} Y_0 + \int_0^t e^{-Q(t-s)} dX(s).$$

For further details see Sato and Yamazoto [322] and Barndorff-Nielsen, Jensen and Sørensen [24].

Diffusion processes

The most intensively studied class of SDEs is the class of those that lead to diffusion processes. These generalise the Ornstein–Uhlenbeck process for Brownian motion, but now the aim is to describe all possible random motions that are due to 'diffusion'. A hypothetical particle that diffuses should move continuously and be characterised by two functions, a 'drift coefficient' b that describes the deterministic part of the motion and a 'diffusion coefficient a' that corresponds to the random part. Generalising the Langevin equation, we model diffusion as a stochastic process $Y = (Y(t), t \geq 0)$, starting at $Y(0) = Y$ (a.s.) and solving the SDE

$$dY(t) = b(Y(t))dt + \sigma(Y(t))dB(t), \tag{6.30}$$

where $a(\cdot) = \sigma(\cdot)\sigma(\cdot)^{\mathrm{T}}$. We impose the usual Lipschitz conditions on b and the stronger one given in Exercise 6.2.7 on σ; these ensure that (6.30) has a unique strong solution. In this case, $Y = (Y(t), t \geq 0)$ is sometimes called an *Itô diffusion*.

A more general approach was traced back to Kolmogorov [206] by David Williams in [357]. A *diffusion process* in $\mathbb{R}^d$ is a path-continuous Markov process $Y = (Y(t), t \geq 0)$ starting at $Y_0 = x$ (a.s) for which there exist continuous functions $\beta : \mathbb{R}^d \to \mathbb{R}^d$ and $\alpha : \mathbb{R}^d \to M_d(\mathbb{R})$ such that

$$\frac{d}{dt}\mathbb{E}(Y(t))\bigg|_{t=0} = \beta(x) \quad \text{and} \quad \frac{d}{dt}\mathrm{Cov}\,(Y(t), Y(t))_{ij}\bigg|_{t=0} = \alpha_{ij}(x). \tag{6.31}$$

We call β and α the *infinitesimal mean* and *infinitesimal covariance*, respectively. The link between this more general definition and SDEs is given in the following result.

Theorem 6.3.1 *Every Itô diffusion is a diffusion with $\beta = b$ and $\alpha = a$.*

Proof Every Itô diffusion $Y = (Y(t), t \geq 0)$ has continuous sample paths. To see this, return to the proof of Theorem 6.2.3 and put $F \equiv 0$ therein, then each Picard iterate Z_n has continuous paths (see Section 4.3.1). Y then inherits this property through the a.s. uniform convergence of the sequence on finite intervals. The Markov property will be discussed later in this chapter. We now turn our attention to the mappings b and a. Continuity of these follows from the Lipschitz conditions. For the explicit calculations below, we follow Durrett [99], pp. 178–9.

Writing the Itô diffusion in integral form we have, for each $t \geq 0$,

$$Y(t) = x + \int_0^t b(Y(s))ds + \int_0^t \sigma(Y(s))dB(s).$$

Since the Brownian integral is a centred L^2-martingale, we have that

$$\mathbb{E}(Y(t)) = x + \int_0^t \mathbb{E}(b(X(s))\, ds,$$

and $\beta(x) = b(x)$ now follows on differentiating.

For each $1 \leq i, j \leq d$, $t \geq 0$,

$$\mathrm{Cov}\,(Y(t), Y(t))_{ij} = \mathbb{E}(Y_i(t)Y_j(t)) - \mathbb{E}(Y_i(t))\,\mathbb{E}(Y_j(t)).$$

By Itô's product formula,

$$\begin{aligned} d(Y_i(t)Y_j(t)) &= dY_i(t)Y_j(t) + Y_i(t)dY_j(t) + d[Y_i, Y_j](t) \\ &= dY_i(t)Y_j(t) + Y_i(t)dY_j(t) + a_{ij}(Y(t))dt. \end{aligned}$$

Hence

$$\begin{aligned} &\mathbb{E}(Y_i(t)Y_j(t)) \\ &\quad = x_i x_j + \int_0^t \mathbb{E}\big(Y_i(s)b_j(Y(s)) + Y_j(s)b_i(Y(s)) + a_{ij}(Y(s))\big)ds \end{aligned}$$

and so

$$\frac{d}{dt}\mathbb{E}(Y_i(t)Y_j(t))\bigg|_{t=0} = x_i b_j(x) + x_j b_i(x) + a_{ij}(x).$$

We can easily verify that

$$\frac{d}{dt}\big[\mathbb{E}(Y_i(t))\ \mathbb{E}(Y_j(t))\big]\bigg|_{t=0} = x_i b_j(x) + x_j b_i(x),$$

and the required result follows. □

Diffusion processes have a much wider scope than physical models of diffusing particles; for example, the Black–Scholes model for stock prices $(S(t), t \geq 0)$ is an Itô diffusion taking the form

$$dS(t) = \beta S(t)dt + \sigma S(t)dt,$$

where $\beta \in \mathbb{R}$ and $\sigma > 0$ denote the usual stock drift and volatility parameters.

We will not make a detailed investigation of diffusions in this book. For more information on this extensively studied topic, see e.g. Durrett [99], Ikeda and Watanabe [167], Itô and McKean [170], Krylov [212], Rogers and Williams [308], [309] and Stroock and Varadhan [340].

When a particle diffuses in accordance with Brownian motion, its standard deviation at time t is $\sqrt{t}$. In *anomalous diffusion*, particles diffuse through a non-homogeneous medium that either slows the particles down (*subdiffusion*) or speeds them up (*superdiffusion*). The standard deviation behaves like t^{ν}, where $\nu < 1/2$ for subdiffusion and $\nu > 1/2$ for superdiffusion. A survey of some of these models is given in chapter 12 of Uchaikin and Zolotarev [350];

compound Poisson processes and symmetric stable laws play a key role in the analysis.

Jump-diffusion processes

By a *jump-diffusion* process, we mean the strong solution $Y = (Y(t), t \geq 0)$ of the SDE

$$dY(t) = b(Y(t-))dt + \sigma(Y(t-))dB(t)$$
$$+ int_{\mathbb{R}^d - \{0\}} G(Y(t-), x)N(dt, dx),$$

where N is a Poisson random measure that is independent of the Brownian motion B having finite intensity measure ν [so we have effectively taken $c = 0$ in (6.12)]. It then follows, by the construction in the proof of Theorem 6.2.9, that the paths of Z simply consist of that of an Itô diffusion process interlaced by jumps at the arrival times of the compound Poisson process $P = (P(t), t \geq 0)$, where each $P(t) = \int_0^t \int_{\mathbb{R}^d - \{0\}} xN(dt, dx)$.

We note that there is by no means universal agreement about the use of the phrase 'jump-diffusion process', and some authors use it to denote the more general processes arising from the solution to (6.12). The terminology may also be used when N is the random measure counting the jumps of a more general point process.

6.4 Stochastic flows, cocycle and Markov properties of SDEs

6.4.1 Stochastic flows

Let $Y_y = (Y_y(t), t \geq 0)$ be the strong solution of the SDE (6.12) with fixed deterministic initial condition $Y_0 = y$ (a.s.). Just as in the case of ordinary differential equations, we would like to study the properties of $Y_y(t)$ as y varies. Imitating the procedure of Section 6.1, we define $\Phi_t : \mathbb{R}^d \times \Omega \to \mathbb{R}^d$ by

$$\Phi_t(y, \omega) = Y_y(t)(\omega)$$

for each $t \geq 0$, $y \in \mathbb{R}^d$, $\omega \in \Omega$. We will find it convenient below to fix $y \in \mathbb{R}^d$ and regard these mappings as random variables. We then employ the notation $\Phi_t(y)(\cdot) = \Phi_t(y, \cdot)$.

Based on equation (6.8), we might expect that

$$\text{`}\Phi_{s+t}(y, \omega) = \Phi_t(\Phi_s(y, \omega), \omega),\text{'}$$

for each $s, t \geq 0$. In fact this is not the case, as the following example shows.

Example 6.4.1 (Random translation) Consider the simplest SDE driven by a Lévy process $X = (X(t), t \geq 0)$,

$$dY_y(t) = dX(t), \quad Y_y(0) = y, \quad \text{a.s.},$$

whose solution is the random translation $\Phi_t(y) = y + X(t)$. Then

$$\Phi_{t+s}(y) = y + X(t+s).$$

But $\Phi_t(\Phi_s(y)) = y + X(t) + X(s)$ and these are clearly not the same (except in the trivial case where $X(t) = mt$, for all $t \geq 0$, with $m \in \mathbb{R}$). However, if we define the two-parameter motion

$$\Phi_{s,t}(y) = y + X(t) - X(s),$$

where $0 \leq s \leq t < \infty$, then it is easy to check that, for all $0 \leq r < s < t < \infty$,

$$\Phi_{r,t}(y) = \Phi_{s,t}(\Phi_{r,s}(y)),$$

and this gives us a valuable clue as to how to proceed in general.

Example6.4.1 suggests that if we want to study the flow property for random dynamical systems then we need a two-parameter family of motions. The interpretation of the random mapping $\Phi_{s,t}$ is that it describes motion commencing at the 'starting time' s and ending at the 'finishing time' t. We now give some general definitions.

Let $\Phi = \{\Phi_{s,t}, 0 \leq s \leq t < \infty\}$ be a family of measurable mappings from $\mathbb{R}^d \times \Omega \to \mathbb{R}^d$. For each $\omega \in \Omega$, we have associated mappings $\Phi^{\omega}_{s,t} : \mathbb{R}^d \to \mathbb{R}^d$, given by $\Phi^{\omega}_{s,t}(y) = \Phi_{s,t}(\omega, y)$ for each $y \in \mathbb{R}^d$.

We say that Φ is a *stochastic flow* if there exists $\mathcal{N} \subset \Omega$, with $P(\mathcal{N}) = 0$, such that for all $\omega \in \Omega - \mathcal{N}$:

(1) $\Phi^{\omega}_{r,t} = \Phi^{\omega}_{s,t} \circ \Phi^{\omega}_{r,s}$ for all $0 \leq r < s < t < \infty$;
(2) $\Phi^{\omega}_{s,s}(y) = y$ for all $s \geq 0$, $y \in \mathbb{R}^d$.

If, in addition, each $\Phi^{\omega}_{s,t}$ is a homeomorphism (C^k-diffeomorphism) of $\mathbb{R}^d$, for all $\omega \in \Omega - \mathcal{N}$, we say that Φ is a *stochastic flow of homeomorphisms* (*C^k-diffeomorphisms*, respectively).

If, in addition to properties (1) and (2), we have that

(3) for each $n \in \mathbb{N}, 0 \leq t_1 < t_2 < \cdots < t_n < \infty, y \in \mathbb{R}^d$, the random variables $\{\Phi_{t_j,t_{j+1}}(y); 1 \leq j \leq n-1\}$ are independent,
(4) the mappings $t \to \Phi_{s,t}(y)$ are càdlàg for each $y \in \mathbb{R}^d, 0 \leq s < t$,

we say that Φ is a *Lévy flow*.

If (4) can be strengthened from 'càdlàg' to 'continuous', we say that Φ is a *Brownian flow*.

The reason for the terminology 'Lévy flow' and 'Brownian flow' is that when property (3) holds we can think of Φ as a Lévy process on the group of all diffeomorphisms from $\mathbb{R}^d$ to itself (see Baxendale [33], Fujiwara and Kunita [125] and Applebaum and Kunita [6] for more about this viewpoint).

Brownian flows of diffeomorphisms were studied extensively by Kunita in [215]. It was shown in Section 4.2 therein that they all arise as solutions of SDEs driven by (a possibly infinite number of) standard Brownian motions. The programme for Lévy flows is less complete, see section 3 of Fujiwara and Kunita [125] for some partial results.

Here we will study flows driven by the SDEs studied in Section 6.2. We consider two-parameter versions of these, i.e.

$$\begin{aligned} d\Phi_{s,t}(y) &= b(\Phi_{s,t-}(y))dt + \sigma(\Phi_{s,t-}(y))dB(t) \\ &\quad + \int_{|x|<c} F(\Phi_{s,t-}(y), x)\tilde{N}(dt, dx) \\ &\quad + \int_{|x|\geq c} G(\Phi_{s,t-}(y), x)N(dt, dx) \end{aligned} \tag{6.32}$$

with initial condition $\Phi_{s,s}(y) = y$ (a.s.), so that, for each $1 \leq i \leq d$,

$$\begin{aligned} \Phi_{s,t}(y)^i = y^i &+ \int_0^t b^i(\Phi_{s,u-}(y))du + \int_0^t \sigma^i_j(\Phi_{s,u-}(y))dB^j(u) \\ &+ \int_0^t \int_{|x|<c} F^i(\Phi_{s,u-}(y), x)\tilde{N}(du, dx) \\ &+ \int_0^t \int_{|x|\geq c} G^i(\Phi_{s,u-}(y), x)N(du, dx). \end{aligned}$$

The fact that (6.32) has a unique strong solution under the usual Lipschitz and growth conditions is achieved by a minor modification to the proofs of Theorems 6.2.3 and 6.2.9.

Theorem 6.4.2 Φ *is a Lévy flow.*

Proof The measurability of each $\Phi_{s,t}$ and the càdlàg property (4) follow from the constructions of Theorems 6.2.3 and 6.2.9. Property (2) is immediate. To establish the flow property (1), we follow similar reasoning to that in the proof of Theorem 6.1.10.

To simplify the form of expressions appearing below, we will omit, without loss of generality, all except the compensated Poisson terms in (6.32).

For all $0 \leq r < s < t < \infty$, $1 \leq i \leq d$, $y \in \mathbb{R}^d$, we have

$$\begin{aligned}\Phi_{r,t}(y)^i &= y^i + \int_r^t \int_{|x|<c} F^i(\Phi_{r,u-}(y), x)\tilde{N}(du, dx)\\ &= y^i + \int_r^s \int_{|x|<c} F^i(\Phi_{r,u-}(y), x)\tilde{N}(du, dx)\\ &\quad + \int_s^t \int_{|x|<c} F^i(\Phi_{r,u-}(y), x)\tilde{N}(du, dx)\\ &= \Phi_{r,s}(y)^i + \int_s^t \int_{|x|<c} F^i(\Phi_{r,u-}(y), x)\tilde{N}(du, dx).\end{aligned}$$

However,

$$\Phi_{s,t}(\Phi_{r,s}(y))^i = \Phi_{r,s}(y)^i + \int_s^t \int_{|x|<c} F^i(\Phi_{s,u-}(\Phi_{r,s}(y)), x)\tilde{N}(du, dx),$$

and the required result follows by the uniqueness of solutions to SDEs.

For the independence (3), consider the sequence of Picard iterates $(\Phi_{s,t}^{(n)}, n \in \mathbb{N} \cup \{0\})$ constructed in the proof of Theorem 6.2.3. Using induction and arguing as in the proof of Lemma 4.3.12, we see that each $\Phi_{s,t}^{(n)}$ is measurable with respect to $\sigma\{N(v, A) - N(u, A),\ 0 \leq s \leq u < v \leq t,\ A \in \mathcal{B}(B_c(0))\}$, from which the required result follows. $\square$

Exercise 6.4.3 Extend Theorem 6.4.2 to the case of the general standard initial condition.

Example 6.4.4 (Randomising deterministic flows) We assume that $b \in C_b^k(\mathbb{R})$ and consider the one-dimensional ODE

$$\frac{d\xi(a)}{da} = b(\xi(a)).$$

By Theorem 6.1.10, its unique solution is a flow of C^k-diffeomorphisms $\xi = (\xi(a), a \in \mathbb{R})$. We randomise the flow ξ by defining

$$\Phi_{s,t}(y) = \xi(X(t) - X(s))(y)$$

for all $0 \leq s \leq t < \infty$, $y \in \mathbb{R}^d$, where X is a one-dimensional Lévy process with characteristics (m, σ^2, ν). It is an easy exercise to check that Φ is a

Lévy flow of C^k-diffeomorphisms. It is of interest to find the SDE satisfied by Φ. Thanks to Exercise 6.1.8, we can use Itô's formula to obtain

$$
\begin{aligned}
d\Phi_{s,t}(y) \\
&= mb(\Phi_{s,t-}(y))dt + \sigma b(\Phi_{s,t-}(y))dB(t) \\
&\quad + \tfrac{1}{2}\sigma^2 b'(\Phi_{s,t-}(y))b(\Phi_{s,t-}(y))dt \\
&\quad + \int_{|x|<1} \big[\xi(x)(\Phi_{s,t-}(y)) - \Phi_{s,t-}(y)\big]\tilde{N}(dt,dx) \\
&\quad + \int_{|x|\geq 1} \big[\xi(x)(\Phi_{s,t-}(y)) - \Phi_{s,t-}(y)\big]N(dt,dx) \\
&\quad + \int_{|x|<1} \big[\xi(x)(\Phi_{s,t-}(y)) - \Phi_{s,t-}(y) - xb(\Phi_{s,t-}(y))\big]\nu(dx)dt. \qquad (6.33)
\end{aligned}
$$

Here we have used the flow property for ξ in the jump term and the fact that

$$\frac{d^2}{da^2}\xi(a) = b'(\xi(a))b(\xi(a)).$$

The SDE (6.33) is the simplest example of a *Marcus canonical equation*. We will return to this theme in Section 6.10.

6.4.2 The Markov property

Here we will apply the flow property established above to prove that solutions of SDEs give rise to Markov processes.

Theorem 6.4.5 *The strong solution to* (6.12) *is a Markov process.*

Proof Let $t \geq 0$. Following Exercise 6.4.3, we can consider the solution $Y = (Y(t), t \geq 0)$ as a stochastic flow with random initial condition Y_0, and we will abuse notation to the extent of writing each

$$Y(t) = \Phi_{0,t}(Y_0) = \Phi_{0,t}.$$

Our aim is to prove that

$$\mathbb{E}(f(\Phi_{0,t+s})|\mathcal{F}_s) = \mathbb{E}(f(\Phi_{0,t+s})|\Phi_{0,s})$$

for all $s, t \geq 0, f \in B_{\mathrm{b}}(\mathbb{R}^d)$.

Now define $G_{f,s,t} \in B_b(\mathbb{R}^d)$ by

$$G_{f,s,t}(y) = \mathbb{E}(f(\Phi_{s,s+t}(y))),$$

for each $y \in \mathbb{R}^d$. By Theorem 6.4.2, and Exercise 6.4.3, we have that $\Phi_{0,t+s} = \Phi_{s,s+t} \circ \Phi_{0,s}$ (a.s.) and that $\Phi_{s,s+t}$ is independent of $\mathcal{F}_s$. Hence, by Lemma 1.1.9,

$$\mathbb{E}(f(\Phi_{0,t+s})|\mathcal{F}_s) = \mathbb{E}(f(\Phi_{s,s+t} \circ \Phi_{0,s})|\mathcal{F}_s) = \mathbb{E}(G_{f,s,t}(\Phi_{0,s})).$$

By the same argument, we also get $\mathbb{E}(f(\Phi_{0,t+s})|\Phi_{0,s}) = \mathbb{E}(G_{f,s,t}(\Phi_{0,s}))$, and the required result follows. □

As in Section 3.1, we can now define an associated stochastic evolution $(T_{s,t}, 0 \leq s \leq t < \infty)$, by the prescription

$$(T_{s,t}f)(y) = \mathbb{E}\big(f(\Phi_{s,t})|\Phi_{0,s} = y\big)$$

for each $f \in B_b(\mathbb{R}^d)$, $y \in \mathbb{R}^d$. We will now strengthen Theorem 6.4.5.

Theorem 6.4.6 *The strong solution to* (6.12) *is a homogeneous Markov process.*

Proof We must show that $T_{s,s+t} = T_{0,t}$ for all $s, t \geq 0$.

Without loss of generality, we just consider the compensated Poisson terms in (6.12). Using the stationary increments property of Lévy processes, we obtain for each $f \in B_b(\mathbb{R}^d)$, $y \in \mathbb{R}^d$,

$$\begin{aligned}
(T_{s,s+t}f)(y) &= \mathbb{E}\left(f(\Phi_{s,s+t}(y))\,\middle|\,\Phi_{0,s} = y\right) \\
&= \mathbb{E}\left(f\left(y + \int_s^{s+t}\int_{|x|<c} F(\Phi_{0,s+u-}(y), x)\tilde{N}(ds, du)\right)\right) \\
&= \mathbb{E}\left(f\left(y + \int_0^{t}\int_{|x|<c} F(\Phi_{0,u-}(y), x)\tilde{N}(ds, du)\right)\right) \\
&= \mathbb{E}\left(f(\Phi_{0,t}(y))\,\middle|\,\Phi_{0,0}(y) = y\right) \\
&= (T_{0,t}f)(y).
\end{aligned}$$

□

Referring again to Section 3.1, we see that we have a semigroup $(T_t, t \geq 0)$ on $B_b(\mathbb{R}^d)$, which is given by

$$(T_tf)(y) = \mathbb{E}\left(f(\Phi_{0,t}(y))\,\middle|\,\Phi_{0,0}(y) = y\right) = \mathbb{E}(f(\Phi_{0,t}(y)))$$

for each $t \geq 0$, $f \in B_b(\mathbb{R}^d)$, $y \in \mathbb{R}^d$. We would like to investigate the Feller property for this semigroup, but first we need to probe deeper into the properties of solution flows.

Exercise 6.4.7 Establish the strong Markov property for SDEs, i.e. show that

$$\mathbb{E}(f(\Phi_{0,t+S})|\mathcal{F}_S) = \mathbb{E}(f(\Phi_{0,t+S})|\Phi_{0,S})$$

for any $t \geq 0$, where S is a stopping time with $P(S < \infty) = 1$.

(Hint: Imitate the proof of Theorem 6.4.5, or see theorem 32 in Protter [298], chapter 5.)

6.4.3 Cocycles

As we will see below, the cocycle property of SDEs is quite closely related to the flow property. In this section, we will work throughout with the canonical Lévy process constructed in Section 1.4.1. So Ω is the path space $\{\omega : \mathbb{R}^+ \to \mathbb{R};\ \omega(0) = 0\}$, $\mathcal{F}$ is the σ-algebra generated by the cylinder sets and P is the unique probability measure given by Kolmogorov's existence theorem from the recipe (1.28) on cylinder sets. Hence $X = (X(t), t \geq 0)$ is a Lévy process on $(\Omega, \mathcal{F}, P)$, where $X(t)\omega = \omega(t)$ for each $\omega \in \Omega$, $t \geq 0$.

The space Ω comes equipped with a *shift* $\theta = (\theta_t, t \geq 0)$, each $\theta_t : \Omega \to \Omega$ being defined as follows (see Appendix 2.10). For each $s, t \geq 0$,

$$(\theta_t\omega)(s) = \omega(t+s) - \omega(t). \tag{6.34}$$

Exercise 6.4.8 Deduce the following:

(1) θ is a one-parameter semigroup, i.e. $\theta_{t+s} = \theta_t\theta_s$ for all $s, t \geq 0$;
(2) the measure P is θ-invariant, i.e. $P(\theta_t^{-1}(A)) = P(A)$ for all $A \in \mathcal{F}, t \geq 0$. (Hint: First establish this on cylinder sets, using (1.28).)

Lemma 6.4.9 *X is an additive cocycle for θ, i.e. for all $s, t \geq 0$,*

$$X(t+s) = X(s) + (X(t) \circ \theta(s)).$$

Proof For each $s, t \geq 0$, $\omega \in \Omega$,

$$X(t)(\theta_s(\omega)) = (\theta_s\omega)(t) = \omega(s+t) - \omega(s) = X(t+s)(\omega) - X(s)(\omega). \qquad \square$$

Additive cocycles were introduced into probability theory by Kolmogorov [207], who called them *helices*; see also de Sam Lazaro and Meyer [321] and Arnold and Scheutzow [15].

We now turn to the Lévy flow Φ that arises from solving (6.32).

Lemma 6.4.10 *For all* $0 \leq s \leq t < \infty$, $y \in \mathbb{R}^d$, $\omega \in \Omega$,

$$\Phi_{s,s+t}(y,\omega) = \Phi_{0,t}(y,\theta_s\omega) \quad \text{a.s.}$$

Proof (See proposition 24 of Arnold and Scheutzow [15].) We use the sequence of Picard iterates $(\Phi^{(n)}_{s,s+t}, n \in \mathbb{N} \cup \{0\})$ constructed in the proof of Theorem 6.4.2 and aim to show that

$$\Phi^{(n)}_{s,s+t}(y,\omega) = \Phi^{(n)}_{0,t}(y,\theta_s\omega) \quad \text{a.s.}$$

for all $n \in \mathbb{N} \cup \{0\}$, from which the result follows on taking limits as $n \to \infty$.

We proceed by induction. Clearly the result is true when $n = 0$. Suppose that it holds for some $n \in \mathbb{N}$. Just as in the proof of Theorem 6.4.2 we will consider a condensed SDE, without loss of generality, and this time we will retain only the Brownian motion terms. Using our usual sequence of partitions and the result of Lemma 6.4.9, for each $1 \leq i \leq d$, $s,t \geq 0$, $y \in \mathbb{R}^d$, $\omega \in \Omega$, the following holds with probability 1:

$$\begin{aligned}
&\Phi^{i,(n+1)}_{s,s+t}(y,\omega) \\
&\quad = y^i + \int_s^{t+s} \sigma^i_j\big(\Phi^{(n)}_{s,s+u}(y,\omega)\big)dB^j(u)(\omega) \\
&\quad = y^i + \lim_{n\to\infty} \sum_{k=0}^{m(n)} \sigma^i_j\big(\Phi^{(n)}_{s,s+t_k}(y,\omega)\big)\big(B^j(s+t_{k+1}) \\
&\qquad - B^j(s+t_k)\big)(\omega) \\
&\quad = y^i + \lim_{n\to\infty} \sum_{k=0}^{m(n)} \sigma^i_j\big(\Phi^{(n)}_{0,t_k}(y,\theta_s\omega)\big)\big(B^j(t_{k+1}) - B^j(t_k)\big)(\theta_s\omega) \\
&\quad = y^i + \int_0^t \sigma^i_j\big(\Phi^{(n)}_{0,u}(y,\theta_s\omega)\big)dB^j(u)(\theta_s\omega) \\
&\quad = \left[y^i + \int_0^t \sigma^i_j\big(\Phi^{(n)}_{0,u}(y)\big)dB^j(u)\right](\theta_s\omega) \\
&\quad = \Phi^{i,(n+1)}_{0,t}(y,\theta_s\omega),
\end{aligned}$$

where the limit is taken in the L^2 sense. □

Corollary 6.4.11 *Φ is a multiplicative cocycle, i.e.*

$$\Phi_{0,s+t}(y,\omega) = \Phi_{0,t}(\Phi_{0,s}(y), \theta_s(\omega))$$

for all $s, t \geq 0$, $y \in \mathbb{R}^d$ and almost all $\omega \in \Omega$.

Proof By the flow property and Lemma 6.4.10, we obtain

$$\Phi_{0,s+t}(y,\omega) = \Phi_{s,s+t}(\Phi_{0,s}(y),\omega) = \Phi_{0,t}(\Phi_{0,s}(y), \theta_s\omega) \quad \text{a.s.}$$

□

We can use the cocycle property to extend our two-parameter flow to a one-parameter family (as in the deterministic case) by including the action of the shift on Ω. Specifically, define $\Upsilon_t : \mathbb{R}^d \times \Omega \to \mathbb{R}^d \times \Omega$ by

$$\Upsilon_t(y,\omega) = \big(\Phi_{0,t}(y, \theta_t(\omega)),\ \theta_t(\omega)\big)$$

for each $t \geq 0$, $\omega \in \Omega$.

Corollary 6.4.12 *The following holds almost surely:*

$$\Upsilon_{t+s} = \Upsilon_t \circ \Upsilon_s$$

for all $s, t \geq 0$.

Proof By using the semigroup property of the shift (Exercise 6.4.8(1)) and Corollary 6.4.11, we have, for *all* $y \in \mathbb{R}^d$ and *almost all* $\omega \in \Omega$,

$$\begin{aligned}
\Upsilon_{t+s}(y,\omega) &= (\Phi_{0,t+s}(y, \theta_{t+s}(\omega)),\ \theta_{t+s}(\omega)) \\
&= (\Phi_{0,t}\left(\Phi_{0,s}\big((y, \theta_s\omega), (\theta_t\theta_s)(\omega)\big), (\theta_t\theta_s)(\omega)\right) \\
&= (\Upsilon_t \circ \Upsilon_s)(y,\omega).
\end{aligned}$$

□

Of course, it would be more natural for Corollary 6.4.12 to hold for all $\omega \in \Omega$, and a sufficient condition for this is that Corollary 6.4.11 is itself valid for all $\omega \in \Omega$. Cocycles that have this property are called *perfect*, and these are also important in studying ergodic properties of stochastic flows. For conditions under which cocycles arising from Brownian flows are perfect, see Arnold and Scheutzow [15].

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