8. Girsanov's theorem

Itô's formula allows one to obtain an extremely important theorem about change of probability measure. We consider here a d-dimensional Wiener process $(w_t, \mathcal{F}_t)$ given on a complete probability space $(\Omega, \mathcal{F}, P)$ and assume that the $\mathcal{F}_t$ are complete.

We need the following lemma in which, in particular, we show how one can do Exercises 3.2.5 and 4.3 by using Itô's formula.

1. Lemma. *Let $b \in \mathcal{S}$ be an $\mathbb{R}^d$-valued process. Denote*

$$\rho_t = \rho_t(b) = \exp\big(\int_0^t b_s\,dw_s - (1/2)\int_0^t (b_s)^2\,ds\big)$$

$$= \exp\big(\sum_{i=1}^d \int_0^t b_s^i\,dw_s^i - (1/2)\sum_{i=1}^d \int_0^t (b_s^i)^2\,ds\big). \tag{1}$$

Then

(i) $d\rho_t = b_t\rho_t\,dw_t$;

(ii) ρ_t *is a supermartingale;*

(iii) *if the process b_t is bounded, then ρ_t is a martingale and, in particular, $E\rho_t = 1$;*

(iv) *if $T \in [0,\infty)$ and $E\rho_T = 1$, then $(\rho_t, \mathcal{F}_t)$ is a martingale for $t \in [0,T]$, and also for any sequence of* bounded $b^n \in \mathcal{S}$ *such that* $\int_0^T |b_s^n - b_s|^2\,ds \to 0$ *(a.s.) we have*

$$E|\rho_T(b^n) - \rho_T(b)| \to 0. \tag{2}$$

Proof. Assertion (i) follows at once from Itô's formula. To prove (ii) define

$$\tau_n = \inf\{t \geq 0 : \int_0^t |b_s|^2\rho_s^2\,ds \geq n\}.$$

Then $I_{t<\tau_n} b_t\rho_t \in H$ (see the beginning of Sec. 3), and so $\int_0^t I_{s<\tau_n} b_s\rho_s\,dw_s$ is a martingale. By adding that

$$\rho_{t\wedge\tau_n} = 1 + \int_0^{t\wedge\tau_n} b_s\rho_s\,dw_s = 1 + \int_0^t I_{s<\tau_n} b_s\rho_s\,dw_s,$$

we see that $\rho_{t\wedge\tau_n}$ is a martingale. Consequently, for $t_1 \geq t_2$ (a.s.)

$$E(\rho_{t_2\wedge\tau_n}|\mathcal{F}_{t_1}) = \rho_{t_1\wedge\tau_n}.$$

As $n \to \infty$, we have $\tau_n \to \infty$ and $t_i \wedge \tau_n \to t_i$, so that by Fatou's theorem (a.s.)

$$E(\rho_{t_2}|\mathcal{F}_{t_1}) \le \rho_{t_1}.$$

This proves (ii) and implies that

$$E \exp\big(\int_0^t b_s\,dw_s - (1/2)\int_0^t |b_s|^2\,ds\big) \le 1. \tag{3}$$

To prove (iii) let $|b_s| \le K$, where K is a constant, and notice that by virtue of (3)

$$E\int_0^t |b_s|^2\rho_s^2\,ds \le K^2 E\int_0^t \rho_s^2\,ds$$

$$= K^2\int_0^t E\rho_s^2(2b)\exp\big(\int_0^t |b_s|^2\,ds\big)\,ds \le K^2\int_0^t e^{K^2 s}\,ds < \infty.$$

Hence

$$\int_0^t b_s\rho_s\,dw_s \quad \text{and} \quad \rho_t = 1 + \int_0^t b_s\rho_s\,dw_s$$

are martingales.

To prove (iv), first notice that $E\rho_T(b^n) = 1$ by (iii), $E\rho_T(b) = 1$ by the assumption, and $\rho_T(b^n) \to \rho_T(b)$ in probability by properties of stochastic integrals. This implies (2) by Scheffé's theorem. Furthermore, for $t \le T$ (a.s.)

$$\rho_t(b^n) = E(\rho_T(b^n)|\mathcal{F}_t).$$

Letting $n \to \infty$ here and using Corollary 3.1.10 lead to a similar equality for b in place of b^n, and the martingale property of $\rho_t(b)$ for $t \le T$ now follows from Exercise 3.2.2. The lemma is proved.

2. Remark. Notice again that ρ_t is a solution of $d\rho_t = b_t\rho_t\,dw_t$. We know that in the usual calculus solutions of $df_t = \alpha f_t\,dt$ (that is, exponential functions) play a very big role. As big a role in stochastic calculus is played by *exponential martingales* $\rho_t(b)$.

Inequality (3) implies the following.

3. Corollary. *If b_s is a bounded process or $\int_0^t |b_s|^2\,ds$ is bounded, then*

$$E\exp\int_0^t b_s\,dw_s < \infty.$$

4. Exercise. (i) By following the argument in the proof of Lemma 1 (ii), prove that if $E\sup_{t\leq T}\rho_t<\infty$, then $(\rho_t,\mathcal{F}_t)$ is a martingale for $t\in[0,T]$.

(ii) Use the result of (i) to prove that, if $p>1$ and $N<\infty$ and $E\rho^p_\tau\leq N$ for every stopping time $\tau\leq T$, then $(\rho_t,\mathcal{F}_t)$ is a martingale for $t\in[0,T]$.

5. Exercise. Use Hölder's inequality and Exercise 4 (ii) to prove that if

$$E\exp\int_0^T c|b_t|^2\,dt<\infty$$

for a constant $c>1/2$, then $E\rho_T(b)=1$.

6. Exercise. By using Exercise 5 and inspecting the inequality

$$1=E\rho_T((1-\varepsilon)b)\leq\big[E\rho_T(b)\big]^{1-\varepsilon}\big[E\exp\frac{1-\varepsilon}{2}\int_0^T|b_t|^2\,dt\big]^\varepsilon,$$

improve the result of Exercise 5 and show that it holds if

$$\lim_{\varepsilon\downarrow 0}\varepsilon\ln E\exp\frac{1-\varepsilon}{2}\int_0^T|b_t|^2\,dt=0,\tag{4}$$

which is true if, for instance, $E\exp(1/2)\int_0^T|b_t|^2\,dt<\infty$ (A. Novikov). It turns out that condition (4) can be relaxed even further by replacing $=0$ with $<\infty$ on the right and lim with $\underline{\lim}$ on the left.

The next lemma treats $\rho_t(b)$ for complex-valued d-dimensional b_t. In this situation we introduce $\rho_t(b)$ by the same formula (1) and for d-vectors $f=(f_1,...,f_d)$ with complex entries f_k denote $(f)^2=\sum_k f_k^2$.

7. Lemma. *If b_t is a bounded d-dimensional* complex-valued *process of class $\mathcal{S}$, then $\rho_t(b)$ is a (complex-valued) martingale and, in particular, $E\rho_t(b)=1$ for any t.*

Proof. Take $t_2>t_1\geq 0$ and $A\in\mathcal{F}_{t_1}$. To prove the lemma it suffices to prove that, if f_t and g_t are bounded $\mathbb{R}^d$-valued processes of class $\mathcal{S}$, then for all complex z

$$EI_A\exp\Big(\int_0^{t_2}(f_s+zg_s)\,dw_s-(1/2)\int_0^{t_2}(f_s+zg_s)^2\,ds\Big)$$

$$=EI_A\exp\Big(\int_0^{t_1}(f_s+zg_s)\,dw_s-(1/2)\int_0^{t_1}(f_s+zg_s)^2\,ds\Big).\tag{5}$$

Observe that (5) holds for real z by Lemma 1 (iii). Therefore we will prove (5) if we prove that both sides are analytic functions of z. In turn to prove this it suffices to show that both sides are continuous and their integrals along closed bounded paths vanish. Finally, due to the analyticity of the expressions under expectation signs and Fubini's theorem we only need to show that, for every $R \in [0,\infty)$ and all $|z| \le R$, these expressions are bounded by a summable function independent of z. This boundedness follows easily from Corollary 3, boundedness of f, g, and the fact that

$$\big|\exp\int_0^{t_j}(f_s + zg_s)\,dw_s\big| = \exp\int_0^{t_j}(f_s + g_s\mathrm{Re}\,z)\,dw_s$$

$$\le \exp\int_0^{t_j}(f_s + Rg_s)\,dw_s + \exp\int_0^{t_j}(f_s - Rg_s)\,dw_s,$$

where we have used the inequality

$$e^{\alpha} \le e^{\alpha} + e^{-\alpha} \le e^{\beta} + e^{-\beta}$$

if $|\alpha| \le |\beta|$. The lemma is proved.

8. Theorem (Girsanov)**.** *Let $T \in [0,\infty)$, and let b be an $\mathbb{R}^d$-valued process of class $\mathcal{S}$ satisfying*

$$E\rho_T(b) = 1.$$

On the measurable space $(\Omega,\mathcal{F})$ introduce the measure $\tilde{P}$ by

$$\tilde{P}(d\omega) = \rho_T(b)(\omega)\,P(d\omega).$$

Then $(\Omega,\mathcal{F},\tilde{P})$ is a probability space and w_t is a d-dimensional Wiener process on $(\Omega,\mathcal{F},\tilde{P})$ for $t \le T$.

Proof. That $(\Omega,\mathcal{F},\tilde{P})$ is a probability space follows from

$$\tilde{P}(\Omega) = \int_\Omega \rho_T(b)\,P(d\omega) = E\rho_T(b) = 1.$$

Next denote $\xi_t = w_t - \int_0^t b_s\,ds$. Since $\xi_0 = 0$ and ξ_t is continuous in t, to prove that ξ_t is a Wiener process, it suffices to show that relative to $(\Omega,\mathcal{F},\tilde{P})$ the joint distributions of the increments of the ξ_t, $t \le T$, are the same as for w_t relative to $(\Omega,\mathcal{F},P)$.

Let $0 \le t_0 \le t_1 \le ... \le t_n = T$. Fix $\lambda_j \in \mathbb{R}^d$, $j = 0,...,n-1$, and define the function λ_s as $i\lambda_j$ on $[t_j, t_{j+1})$, $j = 0,...,n-1$. Also denote by $\tilde{E}$ the expectation sign relative to $\tilde{P}$. By Lemma 7, if b is bounded, then

$$\tilde{E}\exp i\sum_{j=0}^{n-1}\lambda_j(\xi_{t_{j+1}}-\xi_{t_j})=E\exp\big(\int_0^T\lambda_s\,dw_s-\int_0^T\lambda_s\cdot b_s\,ds\big)\rho_T(b)$$

$$=E\rho_T(\lambda+b)e^{(1/2)\int_0^T(\lambda_s)^2\,ds}=e^{(1/2)\int_0^T(\lambda_s)^2\,ds}.$$

It follows that

$$\tilde{E}\exp i\sum_{j=0}^{n-1}\lambda_j(\xi_{t_{j+1}}-\xi_{t_j})=\exp\big(-(1/2)\sum_{j=0}^{n-1}|\lambda_j|^2(t_{j+1}-t_j)\big). \tag{6}$$

This proves the theorem if b is bounded. In the general case take a sequence of bounded $b^n\in\mathcal{S}$ such that (a.s.) $\int_0^T|b^n_s-b_s|^2\,ds\to 0$ (for instance, cutting off large values of $|b_s|$). Then

$$E\rho_T(\lambda+b)=\lim_{n\to\infty}E\rho_T(\lambda+b^n),$$

since by Lemma 1 (iv) and the dominated convergence theorem (remember λ_s is imaginary)

$$E|\rho_T(\lambda+b^n)-\rho_T(\lambda+b)|$$

$$=e^{(1/2)\int_0^T|\lambda_s|^2\,ds}E\big|\rho_T(b^n)e^{-\int_0^T\lambda_s\cdot b^n_s\,ds}-\rho_T(b)e^{-\int_0^T\lambda_s\cdot b_s\,ds}\big|$$

$$\le e^{(1/2)\int_0^T|\lambda_s|^2\,ds}\big(E|\rho_T(b^n)-\rho_T(b)|$$

$$+E\big|e^{-\int_0^T\lambda_s\cdot b_s\,ds}-e^{-\int_0^T\lambda_s\cdot b^n_s\,ds}\big|\rho_T(b)\big)\to 0.$$

This and (6) yield the result in the general case. The theorem is proved.

Girsanov's theorem and the lemmas proved before it have numerous applications. We discuss only few of them.

From the theory of ODE's it is known that the equation $dx_t=b(t,x_t)\,dt$ need not have a solution for any bounded Borel b. In contrast with this it turns out that, for almost any trajectory of the Wiener process, the equation $dx_t=b(t,x_t+w_t)\,dt$ does have a solution whenever b is Borel and bounded. This fact is obtained from the following theorem after replacing x_t with ξ_t-w_t.

9. Theorem. *Let $b(t,x)$ be an $\mathbb{R}^d$-valued Borel bounded function on $(0,\infty)\times\mathbb{R}^d$. Then there exist a probability space $(\Omega,\mathcal{F},P)$, a d-dimensional continuous process ξ_t and a d-dimensional Wiener process w_t defined on that space for $t\in[0,T]$ such that*

$$\xi_t=\int_0^t b(s,\xi_s)\,ds+w_t \tag{7}$$

for all $t\in[0,T]$ and $\omega\in\Omega$.

Proof. Take any complete probability space $(\Omega,\mathcal{F},\tilde{P})$ carrying a d-dimensional Wiener process, say ξ_t. Define

$$w_t=\xi_t-\int_0^t b(s,\xi_s)\,ds$$

and on $(\Omega,\mathcal{F})$ introduce a new measure P by the formula

$$P(d\omega)=\exp\big(\int_0^T b(s,\xi_s)\,d\xi_s-(1/2)\int_0^T |b(s,\xi_s)|^2\,ds\big)\tilde{P}(d\omega).$$

Then $(\Omega,\mathcal{F},P)$ is a probability space, w_t is a Wiener process on $(\Omega,\mathcal{F},P)$ for $t\in[0,T]$, and, by definition, ξ_t solves (7). The theorem is proved.

The proof of this theorem looks like a trick and usually leaves the reader unsatisfied. Indeed firstly, no real method is given such as Picard's method of successive approximations or Euler's method allowing one to find solutions. Secondly, the question remains as to whether one can find solutions on a *given* probability space without changing it, so that ξ_t would be defined by the Wiener process w_t and not conversely. Theorem 9 was proved by I. Girsanov around 1965. Only in 1978 did A. Veretennikov prove that indeed the solutions can be found on any probability space, and only in 1996 did it become clear that Euler's method allows one to construct the solutions effectively.

Let us also show the application of Girsanov's theorem to finding

$$P(\max_{t\le 1}(w_t+t)\ge 1),$$

where w_t is a one-dimensional Wiener process. Let $b=-1$ and

$$\tilde{P}(d\omega)=e^{-w_t-1/2}P(d\omega).$$

By Girsanov's theorem $\bar{w}_t:=w_t+t$ is a Wiener process for $t\in[0,1]$. Since the distributions of Wiener processes in the space of continuous functions are all the same and are given by Wiener measure, we conclude

$$P(\max_{t\leq 1}(w_t+t)\geq 1)=\int_\Omega I_{\max_{t\leq 1}\bar{w}_t\geq 1}e^{\bar{w}_1-1/2}e^{-w_1-1/2}P(d\omega)$$

$$=\int_\Omega I_{\max_{t\leq 1}\bar{w}_t\geq 1}e^{\bar{w}_1-1/2}\tilde{P}(d\omega)=EI_{\max_{t\leq 1}w_t\geq 1}e^{w_1-1/2}.$$

Now remember the result of Exercise 2.2.10, which is

$$P(\max_{t\leq 1}w_t\geq 1, w_1\leq x)=\begin{cases}P(w_1\geq 2-x) & \text{if} \quad x\leq 1,\\ 2P(w_1\geq 1)-P(w_1\geq x) & \text{if} \quad x\geq 1.\end{cases}$$

Then by using the hint to Exercise 2.2.12, we get

$$P(\max_{t\leq 1}(w_t+t)\geq 1)=\int_1^\infty e^{x-1/2}\frac{1}{\sqrt{2\pi}}\,e^{-x^2/2}\,dx$$

$$+\int_{-\infty}^1 e^{x-1/2}\frac{1}{\sqrt{2\pi}}\,e^{-(2-x)^2/2}\,dx=\frac{1}{\sqrt{2\pi e}}\int_1^\infty (e^x+e^{2-x})e^{-x^2/2}\,dx.$$

In the following exercise we suggest the reader derive a particular case of the Burkholder-Davis-Gundy inequalities.

10. Exercise. Let τ be a bounded stopping time. Then for any real λ we have

$$Ee^{\lambda w_\tau-\lambda^2\tau/2}=1.$$

By using Corollary 3, prove that we can differentiate this equality with respect to λ as many times as we wish, bringing all derivatives inside the expectation sign. Then, for any integer $k\geq 1$, prove that

$$E(a_0w_\tau^{2k}+a_2w_\tau^{2k-2}\tau+a_4w_\tau^{2k-4}\tau^2+...+a_{2k}\tau^k)=0,$$

where $a_0,...,a_{2k}$ are certain absolute constants (depending on k) and $a_0\neq 0$ and $a_{2k}\neq 0$. Finally, remembering Hölder's inequality, prove that

$$Ew_\tau^{2k}\leq NE\tau^k,\quad E\tau^k\leq NEw_\tau^{2k},$$

where the constant N depends only on k.