15. Theorem (Doob's inequality)**.** *If* $(\xi_n, \mathcal{F}_n)$*,* $n = 0, 1, ...,$ *is a nonnegative submartingale and* $p > 1$*, then*

$$E\big[\sup_n \xi_n\big]^p \le q^p \sup_n E\xi_n^p, \tag{3}$$

where $q = p/(p-1)$*. In particular,*

$$E\big[\sup_n \xi_n\big]^2 \le 4 \sup_n E\xi_n^2.$$

Proof. Without losing generality we assume that the right-hand side of (3) is finite. Then for any integer N

$$\big[\sup_{n \le N} \xi_n\big]^p \le \big[\sum_{n \le N} \xi_n\big]^p \le N^p \sum_{n \le N} \xi_n^p, \quad E\big[\sup_{n \le N} \xi_n\big]^p < \infty.$$

Next, by the Doob-Kolmogorov inequality, for $c > 0$,

$$P\{\sup_{n \le N} \xi_n \ge c\} \le \frac{1}{c} E\xi_N I_{\sup_{n \le N} \xi_n \ge c}.$$

We multiply both sides by pc^{p-1}, integrate with respect to $c \in (0, \infty)$, and use

$$P(\eta \ge c) = EI_{\eta \ge c}, \quad \eta^p = p \int_0^\infty c^{p-1} I_{\eta \ge c}\, dc,$$

where η is any nonnegative random variable. We also use Hölder's inequality. Then we find that

$$E\big[\sup_{n \le N} \xi_n\big]^p \le qE\xi_N \big[\sup_{n \le N} \xi_n\big]^{p-1} \le q\big(E\xi_N^p\big)^{1/p}\big(E\big[\sup_{n \le N} \xi_n\big]^p\big)^{1-1/p}.$$

Upon dividing through by the last factor (which is finite by the above) we conclude that

$$E\big[\sup_{n \le N} \xi_n\big]^p \le q^p \sup_n E\xi_n^p.$$

It only remains to use Fatou's theorem and let $N \to \infty$. The theorem is proved.

4. Limit theorems for martingales

Let $(\xi_n, \mathcal{F}_n)$, $n = 0, 1, ..., N$, be a submartingale, and let a and b be fixed numbers such that $a < b$. Define consecutively the following:

$$\tau_1 = \inf(n \geq 0 : \xi_n \leq a) \wedge N, \quad \sigma_1 = \inf(n \geq \tau_1 : \xi_n \geq b) \wedge N,$$

$$\tau_n = \inf(n \geq \sigma_{n-1} : \xi_n \leq a) \wedge N, \quad \sigma_n = \inf(n \geq \tau_n : \xi_n \geq b) \wedge N.$$

Clearly $0 \leq \tau_1 \leq \sigma_1 \leq \tau_2 \leq \sigma_2 \leq ...$ and $\tau_{N+i} = \sigma_{N+i} = N$ for all $i \geq 0$. We have seen before that τ_1 is a stopping time.

1. Exercise*. Prove that all τ_n and σ_n are stopping times.

The points (n, ξ_n) belong to $\mathbb{R}^2$. We join the points (n, ξ_n) and $(n + 1, \xi_{n+1})$ for $n = 0, ..., N - 1$ by straight segments. Then we obtain a piecewise linear function, say l. Let us say that if $\xi_{\tau_m} \leq a$ and $\xi_{\sigma_m} \geq b$, then on $[\tau_m, \sigma_m]$ the function l *upcrosses* (a, b). Denote $\beta(a, b)$ the number of upcrossings of the interval (a, b) by l. It is seen that $\beta(a, b) = m$ if and only if $\xi_{\tau_m} \leq a$, $\xi_{\sigma_m} \geq b$ and either $\xi_{\tau_{m+1}} > a$ or $\xi_{\sigma_{m+1}} < b$.

The following theorem is the basis for obtaining limit theorems for martingales.

2. Theorem (Doob's upcrossing inequality)**.** *If* $(\xi_n, \mathcal{F}_n)$*,* $n = 0, 1, ..., N$*, is a submartingale and* $a < b$*, then*

$$E\beta(a, b) \leq \frac{1}{b - a} E(\xi_N - a)_+.$$

Proof. Notice that $\beta(a, b)$ is also the number of upcrossing of $(0, b - a)$ by the piecewise linear function constructed from $(\xi_n - a)_+$. Furthermore, $\xi_n - a$ and $(\xi_n - a)_+$ are submartingales along with ξ_n. It follows that without loss of generality we may assume that $\xi_n \geq 0$ and $a = 0$. In that case notice that any upcrossing of $(0, b)$ can only occur on an interval of type $[\tau_i, \sigma_i]$ with $\xi_{\sigma_i} - \xi_{\tau_i} \geq b$. Also in any case, $\xi_{\sigma_n} - \xi_{\tau_n} \geq 0$. Hence,

$$b\beta(a, b) \leq (\xi_{\sigma_1} - \xi_{\tau_1}) + (\xi_{\sigma_2} - \xi_{\tau_2}) + ... + (\xi_{\sigma_N} - \xi_{\tau_N}).$$

Furthermore, $\tau_{n+1} \geq \sigma_n$ and $E\xi_{\tau_{n+1}} \geq E\xi_{\sigma_n}$. It follows that

$$\begin{aligned} &bE\beta(a, b) \\ &\leq -E\xi_{\tau_1} + (E\xi_{\sigma_1} - E\xi_{\tau_2}) + (E\xi_{\sigma_2} - E\xi_{\tau_3}) + ... + (E\xi_{\sigma_{N-1}} - E\xi_{\tau_N}) + E\xi_{\sigma_N} \\ &\qquad\qquad \leq E\xi_{\sigma_N} - E\xi_{\tau_1} \leq E\xi_{\sigma_N} = E\xi_N, \end{aligned}$$

thus proving the theorem.

3. Exercise. For $\xi_n \geq 0$ and $a = 0$ it seems that typically $\xi_{\sigma_n} \geq b$ and $\xi_{\tau_{n+1}} = 0$. Then why do we have $E\xi_{\tau_{n+1}} \geq E\xi_{\sigma_n}$?

If we have a submartingale $(\xi_n, \mathcal{F}_n)$ defined for all $n = 0, 1, 2, ...$, then we can construct our piecewise linear function on $(0, \infty)$ and define $\beta_\infty(a, b)$ as the number of upcrossing of (a, b) on $[0, \infty)$ by this function. Obviously $\beta_\infty(a, b)$ is the monotone limit of upcrossing numbers on $[0, N]$. By Fatou's theorem we obtain the following.

4. Corollary. *If $(\xi_n, \mathcal{F}_n)$, $n = 0, 1, 2, ...$, is a submartingale, then*

$$E\beta_\infty(a, b) \leq \frac{1}{b-a} \sup_n E(\xi_n - a)_+ \leq \frac{1}{b-a} (\sup_n E(\xi_n)_+ + |a|).$$

5. Theorem. *Let one of the following conditions hold:*

(i) $(\xi_n, \mathcal{F}_n)$, $n = 0, 1, 2, ...$, is a submartingale and $\sup_n E(\xi_n)_+ < \infty$;

(ii) $(\xi_n, \mathcal{F}_n)$, $n = 0, 1, 2, ...$, is a supermartingale and $\sup_n E(\xi_n)_- < \infty$;

(iii) $(\xi_n, \mathcal{F}_n)$, $n = 0, 1, 2, ...$, is a martingale and $\sup_n E|\xi_n| < \infty$.

Then the limit $\lim_{n\to\infty} \xi_n$ exists with probability one.

Proof. Obviously we only need prove the assertion under condition (i). Define ρ as the set of all rational numbers on $\mathbb{R}$, and notice that almost obviously

$$\{\omega : \overline{\lim_{n\to\infty}}\, \xi_n(\omega) > \underline{\lim}_{n\to\infty}\, \xi_n(\omega)\} = \bigcup_{a,b\in\rho, a<b} \{\omega : \beta_\infty(a, b) = \infty\}.$$

Then it only remains to notice that the events on the right have probability zero since

$$E\beta_\infty(a, b) \leq \frac{1}{b-a} (\sup_n E(\xi_n)_+ + |a|) < \infty,$$

so that $\beta_\infty(a, b) < \infty$ (a.s.). The theorem is proved.

6. Corollary. *Any nonnegative supermartingale converges at infinity with probability one.*

7. Corollary (cf. Exercise 2.2)**.** *If $(\xi_n, \mathcal{F}_n)$, $n = 0, 1, 2, ...$, is a martingale and ξ is a random variable such that $|\xi_n| \leq \xi$ for all n and $E\xi < \infty$, then $\xi_n = E(\xi_\infty|\mathcal{F}_n)$ (a.s.), where $\xi_\infty = \lim_{n\to\infty} \xi_n$.*

Indeed, by the dominated convergence theorem for martingales

$$\xi_n = E(\xi_{n+m}|\mathcal{F}_n) = \lim_{m\to\infty} E(\xi_{n+m}|\mathcal{F}_n) = E(\xi_\infty|\mathcal{F}_n).$$

Corollary 7 describes all bounded martingales. The situation with unbounded, even nonnegative, martingales is much more subtle.

8. Exercise. Let $\xi_n = \exp(w_n - n/2)$, where w_t is a Wiener process. By using Corollary 2.4.3, show that $\xi_\infty = 0$, so that $\xi_n > E(\xi_\infty|\mathcal{F}_n)$. Conclude that $E\sup_n \xi_n = \infty$ and, moreover, that for every nonrandom sequence $n(k) \to \infty$, no matter how sparse it is, $E\sup_k \xi_{n(k)} = \infty$.

In the case of reverse martingales one does not need any additional conditions for its limit to exist.

9. Theorem. *Let $(\xi_n, \mathcal{F}_n)$, $n = 0, 1, 2, ...$, be a reverse martingale. Then $\lim_{n\to\infty} \xi_n$ exists with probability one.*

Proof. By definition $(\xi_{-n}, \mathcal{F}_{-n})$, $n = ..., -2, -1, 0$, is a martingale. Denote by $\beta_N(a, b)$ the number of upcrossing of (a, b) by the piecewise linear function constructed from ξ_{-n} restricted to $[-N, 0]$. By Doob's theorem, $E\beta_N(a, b) \le (E|\xi_0| + |a|)/(b - a)$. Hence $E \lim_{N\to\infty} \beta_N(a, b) < \infty$, and we get the result as in the proof of Theorem 5.

10. Theorem (Lévy-Doob)**.** *Let ξ be a random variable such that $E|\xi| < \infty$, and let $\mathcal{F}_n$ be σ-fields defined for $n = 0, 1, 2, ...$ and satisfying $\mathcal{F}_n \subset \mathcal{F}$.*

(i) *Assume $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ for each n, and denote by $\mathcal{F}_\infty$ the smallest σ-field containing all $\mathcal{F}_n$ ($\mathcal{F}_\infty = \bigvee_n \mathcal{F}_n$). Then*

$$\lim_{n\to\infty} E(\xi|\mathcal{F}_n) = E(\xi|\mathcal{F}_\infty) \quad (a.s.), \tag{1}$$

$$\lim_{n\to\infty} E|E(\xi|\mathcal{F}_n) - E(\xi|\mathcal{F}_\infty)| = 0. \tag{2}$$

(ii) *Assume $\mathcal{F}_n \supset \mathcal{F}_{n+1}$ for all n and denote $\mathcal{F}_\infty = \bigcap_n \mathcal{F}_n$. Then* (1) *and* (2) *hold again.*

To prove the theorem we need the following remarkable result.

11. Lemma (Scheffé)**.** *Let ξ, ξ_n, $n = 1, 2, ...$, be nonnegative random variables such that $\xi_n \xrightarrow{P} \xi$ and $E\xi_n \to E\xi$ as $n \to \infty$. Then $E|\xi_n - \xi| \to 0$.*