15. Theorem (Doob's inequality). If \((\xi_n, \mathcal{F}_n), n = 0, 1, \ldots\), is a nonnegative submartingale and \(p > 1\), then

\[
E\left[ \sup_n \xi_n \right]^p \leq q^p \sup_n E\xi_n^p,
\]

where \(q = p/(p - 1)\). In particular,

\[
E\left[ \sup_n \xi_n \right]^2 \leq 4 \sup_n E\xi_n^2.
\]

Proof. Without losing generality we assume that the right-hand side of (3) is finite. Then for any integer \(N\)

\[
\left[ \sup_{n \leq N} \xi_n \right]^p \leq \left[ \sum_{n \leq N} \xi_n \right]^p \leq N^p \sum_{n \leq N} E\xi_n^p, \quad E\left[ \sup_{n \leq N} \xi_n \right]^p < \infty.
\]

Next, by the Doob-Kolmogorov inequality, for \(c > 0\),

\[
P\{\sup_{n \leq N} \xi_n \geq c\} \leq \frac{1}{c} E\xi_N I_{\sup_{n \leq N} \xi_n \geq c}.
\]

We multiply both sides by \(pc^{p-1}\), integrate with respect to \(c \in (0, \infty)\), and use

\[
P(\eta \geq c) = EI_{\eta \geq c}, \quad \eta^p = p \int_0^\infty c^{p-1} I_{\eta \geq c} dc,
\]

where \(\eta\) is any nonnegative random variable. We also use Hölder’s inequality. Then we find that

\[
E\left[ \sup_{n \leq N} \xi_n \right]^p \leq qE\xi_N \left[ \sup_{n \leq N} \xi_n \right]^{p-1} \leq q\left( E\xi_N^p \right)^{1/p} \left( E\left[ \sup_{n \leq N} \xi_n \right]^p \right)^{1-1/p}.
\]

Upon dividing through by the last factor (which is finite by the above) we conclude that

\[
E\left[ \sup_{n \leq N} \xi_n \right]^p \leq q^p \sup_n E\xi_n^p.
\]

It only remains to use Fatou’s theorem and let \(N \to \infty\). The theorem is proved.
4. Limit theorems for martingales

Let \((\xi_n, \mathcal{F}_n), n = 0, 1, ..., N\), be a submartingale, and let \(a\) and \(b\) be fixed numbers such that \(a < b\). Define consecutively the following:

\[
\tau_1 = \inf(n \geq 0 : \xi_n \leq a) \land N, \quad \sigma_1 = \inf(n \geq \tau_1 : \xi_n \geq b) \land N,
\]

\[
\tau_n = \inf(n \geq \sigma_{n-1} : \xi_n \leq a) \land N, \quad \sigma_n = \inf(n \geq \tau_n : \xi_n \geq b) \land N.
\]

Clearly \(0 \leq \tau_1 \leq \sigma_1 \leq \tau_2 \leq \sigma_2 \leq ... \) and \(\tau_{N+i} = \sigma_{N+i} = N\) for all \(i \geq 0\). We have seen before that \(\tau_1\) is a stopping time.

1. Exercise*. Prove that all \(\tau_n\) and \(\sigma_n\) are stopping times.

The points \((n, \xi_n)\) belong to \(\mathbb{R}^2\). We join the points \((n, \xi_n)\) and \((n + 1, \xi_{n+1})\) for \(n = 0, ..., N - 1\) by straight segments. Then we obtain a piecewise linear function, say \(\xi\). Let us say that if \(\xi \leq a\) and \(\xi \geq b\), then on \([\tau_m, \sigma_m]\) the function \(l\) upcrosses \((a, b)\). Denote \(\beta(a, b)\) the number of upcrossings of the interval \((a, b)\) by \(l\). It is seen that \(\beta(a, b) = m\) if and only if \(\xi \leq a\), \(\xi \geq b\) and either \(\xi_{m+1} > a\) or \(\xi_{m+1} < b\).

The following theorem is the basis for obtaining limit theorems for martingales.

2. Theorem (Doob’s upcrossing inequality). If \((\xi_n, \mathcal{F}_n), n = 0, 1, ..., N,\) is a submartingale and \(a < b\), then

\[
E\beta(a, b) \leq \frac{1}{b - a} E(\xi_N - a)_+.
\]

Proof. Notice that \(\beta(a, b)\) is also the number of upcrossing of \((0, b - a)\) by the piecewise linear function constructed from \((\xi_n - a)_+\). Furthermore, \(\xi_n - a\) and \((\xi_n - a)_+\) are submartingales along with \(\xi_n\). It follows that without loss of generality we may assume that \(\xi_n \geq 0\) and \(a = 0\). In that case notice that any upcrossing of \((0, b)\) can only occur on an interval of type \([\tau_i, \sigma_i]\) with \(\xi_{\sigma_i} - \xi_{\tau_i} \geq b\). Also in any case, \(\xi_{\sigma_n} - \xi_{\tau_n} \geq 0\). Hence,

\[
b\beta(a, b) \leq (\xi_{\sigma_1} - \xi_{\tau_1}) + (\xi_{\sigma_2} - \xi_{\tau_2}) + ... + (\xi_{\sigma_N} - \xi_{\tau_N}).
\]

Furthermore, \(\tau_{n+1} \geq \sigma_n\) and \(E\xi_{\tau_{n+1}} \geq E\xi_{\sigma_n}\). It follows that

\[
b E\beta(a, b) \\
\leq -E\xi_{\tau_1} + (E\xi_{\sigma_1} - E\xi_{\tau_2}) + (E\xi_{\sigma_2} - E\xi_{\tau_3}) + ... + (E\xi_{\sigma_{N-1}} - E\xi_{\tau_N}) + E\xi_{\sigma_N} \\
\leq E\xi_{\sigma_N} - E\xi_{\tau_1} \leq E\xi_{\sigma_N} = E\xi_N,
\]

thus proving the theorem.
3. **Exercise.** For $\xi_n \geq 0$ and $a = 0$ it seems that typically $\xi_{\sigma_n} \geq b$ and $\xi_{\tau_n+1} = 0$. Then why do we have $E\xi_{\tau_n+1} \geq E\xi_{\sigma_n}$?

If we have a submartingale $(\xi_n, F_n)$ defined for all $n = 0, 1, 2, \ldots$, then we can construct our piecewise linear function on $(0, \infty)$ and define $\beta_\infty(a, b)$ as the number of upcrossing of $(a, b)$ on $[0, \infty)$ by this function. Obviously $\beta_\infty(a, b)$ is the monotone limit of upcrossing numbers on $[0, N]$. By Fatou’s theorem we obtain the following.

4. **Corollary.** If $(\xi_n, F_n)$, $n = 0, 1, 2, \ldots$, is a submartingale, then

$$E\beta_\infty(a, b) \leq \frac{1}{b-a} \sup_n E(\xi_n - a)_+ \leq \frac{1}{b-a} (\sup_n E(\xi_n)_+ + |a|).$$

5. **Theorem.** Let one of the following conditions hold:

(i) $(\xi_n, F_n)$, $n = 0, 1, 2, \ldots$, is a submartingale and $\sup_n E(\xi_n)_+ < \infty$;

(ii) $(\xi_n, F_n)$, $n = 0, 1, 2, \ldots$, is a supermartingale and $\sup_n E(\xi_n)_- < \infty$;

(iii) $(\xi_n, F_n)$, $n = 0, 1, 2, \ldots$, is a martingale and $\sup_n E|\xi_n| < \infty$.

Then the limit $\lim_{n \to \infty} \xi_n$ exists with probability one.

Proof. Obviously we only need prove the assertion under condition (i). Define $\rho$ as the set of all rational numbers on $\mathbb{R}$, and notice that almost obviously

$$\{\omega: \lim_{n \to \infty} \xi_n(\omega) > \lim_{n \to \infty} \xi_n(\omega)\} = \bigcup_{a,b \in \rho, a < b} \{\omega: \beta_\infty(a, b) = \infty\}.$$

Then it only remains to notice that the events on the right have probability zero since

$$E\beta_\infty(a, b) \leq \frac{1}{b-a} (\sup_n E(\xi_n)_+ + |a|) < \infty,$$

so that $\beta_\infty(a, b) < \infty$ (a.s.). The theorem is proved.

6. **Corollary.** Any nonnegative supermartingale converges at infinity with probability one.

7. **Corollary** (cf. Exercise 2.2). If $(\xi_n, F_n)$, $n = 0, 1, 2, \ldots$, is a martingale and $\xi$ is a random variable such that $|\xi_n| \leq \xi$ for all $n$ and $E\xi < \infty$, then $\xi_n = E(\xi_\infty| F_n)$ (a.s.), where $\xi_\infty = \lim_{n \to \infty} \xi_n$. 
Indeed, by the dominated convergence theorem for martingales
\[ \xi_n = E(\xi_{n+m}|\mathcal{F}_n) = \lim_{m \to \infty} E(\xi_{n+m}|\mathcal{F}_n) = E(\xi_{\infty}|\mathcal{F}_n). \]

Corollary 7 describes all bounded martingales. The situation with unbounded, even nonnegative, martingales is much more subtle.

8. Exercise. Let \( \xi_n = \exp(w_n - n/2) \), where \( w_t \) is a Wiener process. By using Corollary 2.4.3, show that \( \xi_{\infty} = 0 \), so that \( \xi_n > E(\xi_{\infty}|\mathcal{F}_n) \). Conclude that \( E(\sup_n \xi_n|\mathcal{F}_n) = \infty \) and, moreover, that for every nonrandom sequence \( n(k) \to \infty \), no matter how sparse it is, \( E(\sup_k \xi_{n(k)}|\mathcal{F}_n) = \infty \).

In the case of reverse martingales one does not need any additional conditions for its limit to exist.

9. Theorem. Let \((\xi_n, \mathcal{F}_n)\), \(n = 0, 1, 2, \ldots\), be a reverse martingale. Then \(\lim_{n \to \infty} \xi_n \) exists with probability one.

Proof. By definition \((\xi_{-n}, \mathcal{F}_{-n})\), \(n = \ldots, -2, -1, 0\), is a martingale. Denote by \( \beta_N(a, b) \) the number of upcrossing of \((a, b)\) by the piecewise linear function constructed from \( \xi_{-n} \) restricted to \([-N, 0]\). By Doob’s theorem, \( E\beta_N(a, b) \leq (E|\xi_0| + |a|)/(b - a) \). Hence \( E \lim_{N \to \infty} \beta_N(a, b) < \infty \), and we get the result as in the proof of Theorem 5.

10. Theorem (Lévy-Doob). Let \( \xi \) be a random variable such that \( E|\xi| < \infty \), and let \( \mathcal{F}_n \) be \( \sigma \)-fields defined for \( n = 0, 1, 2, \ldots \) and satisfying \( \mathcal{F}_n \subset \mathcal{F} \).

(i) Assume \( \mathcal{F}_n \subset \mathcal{F}_{n+1} \) for each \( n \), and denote by \( \mathcal{F}_\infty \) the smallest \( \sigma \)-field containing all \( \mathcal{F}_n \) (\( \mathcal{F}_\infty = \bigvee_n \mathcal{F}_n \)). Then

\[
\lim_{n \to \infty} E(\xi|\mathcal{F}_n) = E(\xi|\mathcal{F}_\infty) \quad (a.s.),
\]

(ii) Assume \( \mathcal{F}_n \supset \mathcal{F}_{n+1} \) for all \( n \) and denote \( \mathcal{F}_\infty = \bigcap_n \mathcal{F}_n \). Then (1) and (2) hold again.

To prove the theorem we need the following remarkable result.

11. Lemma (Scheffé). Let \( \xi, \xi_n, n = 1, 2, \ldots \), be nonnegative random variables such that \( \xi_n \overset{P}{\to} \xi \) and \( E\xi_n \to E\xi \) as \( n \to \infty \). Then \( E|\xi_n - \xi| \to 0 \).