[1] List 1 – deadline 22.02.21

1. Let $(X := \mathbb{R}, \mathcal{B} := \mathrm{Bor}, \mathbf{P})$ be a probability space, $\xi : \Omega \to X$ random variable, and let $f : \mathbb{R} \to \mathbb{R}$ be an absolutely continuous function with $f(\pm\infty) = a$, $\int_{-\infty}^{\infty} f(x)\ d\mathbf{P}(x) = b$. Calculate $Q := \int_{-\infty}^{\infty} f'(x)\ \mathbf{P}(\xi \geq x)\ dx$.
2. Let $\{\xi_i\}_{i=1}^{n}$ be independent r.v. with $E\xi_i = 0, D\xi_i < \infty$ and let $\eta_k := \sum_{i=1}^{k} \xi_i$. Prove/disprove that $\mathbf{P}(\max_{k \leq n} |\eta_k| \geq \varepsilon) \leq 2\mathbf{P}(\eta_n \geq \varepsilon - \sqrt{2D\eta_n})\ \ \forall \varepsilon \in \mathbb{R}$.
3. Let $\xi_t : \Omega \to \mathbb{R}$ be a homogeneous random process (i.e. the distribution $(\xi_{t+s} - \xi_t)$ does not depend on t) with independent increments. Is it true that the r.v. ξ_t is infinitely divisible $\forall t$? (ξ is infinitely divisible if $\xi = \sum_i^n \eta_i\ \ \forall n$, where $\{\eta_i\}$ are iid.)
4. Let $\{\xi_i\}_{i=1}^{n}$ be independent r.v. with the same distribution function $F(x)$. Let $\xi_- := \min_i \xi_i, \xi_+ := \max_i \xi_i$. Find the distribution function of the vector (ξ_-, ξ_+).
5. Let ξ be a r.v. with median m_ξ. Prove/disprove $m_{\xi^{2022}} = (m_\xi)^{2022}$.
6. Let $\{\xi_i\}_{i=1}^{n}$ be iid r.v. with $0 < D\xi_i < \infty$. Find all possible values of the function $\varphi(x) := \lim_{n\to\infty} \mathbf{P}(\sum_{i=1}^{n} \xi_i < x),\ \ x \in \mathbb{R}$.