A quick review of characteristics

First order PDE:. Consider the equation on $\mathbb{R}^n$ in the unknown $u \equiv u(x)$

$$\sum_{i=1}^{n} v_i(x)\partial_{x_i} u = 0 \tag{1}$$

We are thus looking for a function u such that the vector field $du = (\partial_{x_1} u, \dots, \partial_{x_n} u)$ is orthogonal at any point at the vector field $v = (v_1, \dots, v_n)$. Since du is orthogonal to the level sets of u (namely the subsets of $\mathbb{R}^n$ $\{u = c\}$), it holds that v is tangent to such level sets.

In particular, if we consider any curve $X \equiv X_t$ solution to

$$\dot{X} = v(X) \tag{2}$$

and any solution u to (1), then $u(X_t)$ is constant in t (as indeed $\frac{d}{dt}u(X_t) = v(X_t) \cdot du(X_t) = 0$).

This means that, if one wants to assign a boundary condition to (1) on a subset $\Omega \subset \mathbb{R}^n$, namely

$$u(x) = u_0(x) \qquad x \in \Omega$$

then u_0 must be constant on solutions to (2) as a necessary condition for existence of a solution u.

Geometric notation: One can write the previous statements more formally. If v is a tangent vector field on a smooth manifold M, one can consider the equation

$$\begin{cases} du\,(v) = 0 \\ u(x) = u_0(x) \qquad x \in \Omega \end{cases} \tag{3}$$

where $\Omega \subset M$. If v is smooth enough, one can also consider the ODE for all $x \in M$ and $s \in \mathbb{R}$

$$\begin{cases} \dot{X} = v(X) \\ X_s = x \end{cases} \tag{4}$$

Denote by $X^{s,x}$ the solution to (4), and set $x \sim y$ if there exists s, t such that $X^{s,x}_t = y$. This is an equivalence relation, and equivalence classes are called the orbits of (4). If v is smooth, the orbits foliate M and thus u must be constant on the leaves of such a foliation. Therefore, a condition for (3) to exist, is that $u_0\Omega \to \mathbb{R}$ is constant on leaves (this holds, for instance, if every orbit intersects Ω in exactly one point. Namely Ω is not characteristic).

Explicit solution: If this is the case, for each $x \in M$ there exists $\tau(x) \in \mathbb{R}$ and $y(x) \in \Omega$ such taht $X^{\tau(x),x} = y(x)$. The solution is then given by

$$u(x) = u_0(y(x)) \tag{5}$$

Change of coordinates: If we choose a system of coordinates associated to (4), say x_0 is a coordinate along the solution and $x_1, \dots$ are transversal, then (3) simply reads $\partial_{x_0} u = 0$. We can consider this as a canonical form of (4), and this is how one can generalize this idea to higher order PDEs.

Nonlinear PDEs: The same ideas hold if the vector field v depends on u, just it is more complicated to get sharp sufficient and necessary conditions for a solution to exists. However, the previous method can give us a way to find a solution. Since we know *a posteriori* that u is constant along the solutions to (4), we can thing that v depends on u as a parameter. Let's then write $v^u(x)$ the vector field, so that the PDE reads

$$du(v^u) = 0 \tag{6}$$

Then for each $u \in \mathbb{R}$ we have solutions to

$$\begin{cases} \dot{X} = v^u(X) \\ X_s = x \end{cases} \tag{7}$$

and as before (assuming that Ω is non-characteristic; this assumption is hard to check however, since characteristics depend on u), maps $\tau^u(x)$ and $y^u(x)$. The solution (5) now reads as an equation for u, however a non-differential one

$$u = u_0(y^u(x))$$

If we can solve this equation, we get a solution to (6) as long as u_0 is constant on orbits of (7).