Ex. A1. Find the explicitly solution of the differential problem:

$$\begin{cases} \partial_{xx}u + 10\,\partial_{xy}u + 16\,\partial_{yy}u = 0 \\ u(0,y) = \cos(y) \\ (\partial_x u)(0,y) = \cos(y) + 5\sin(y) \end{cases}$$

Ex. A2. Consider the problem:

$$\begin{cases} \partial_{tt}u - (\partial_x u)^2 = 0 \\ u(0,x) = 0 \\ (\partial_t u)(0,x) = cx \end{cases}$$

Does it admit analytic solutions? If so, is it unique? In such a case, find it explicitly as a function of the arbitrary constant $c \in \mathbb{R}$.

Ex. A3. Let Ω be a domain of $\mathbb{R}^n$ not containing the origin, and let $f \colon \Omega \to \mathbb{R}$.
Given a function $u \colon \Omega \to \mathbb{R}$, its Kelvin transform v is defined by

$$v(x) := |x|^{2-n} u\big(\tfrac{x}{|x|^2}\big)$$

Prove that, if u satisfies the equation $\Delta u = f$ in Ω, then v satisfies $(\Delta v)(x) = |x|^{-n-2} f\big(\frac{x}{|x|^2}\big)$.

Ex. A4. Consider the Cauchy problem for the string equation

$$\begin{cases} \partial_{tt}u = a^2 \partial_{xx}u \\ u(0,x) = \varphi(x) \\ (\partial_t u)(0,x) = \psi(x), \quad x \in \mathbb{R}. \end{cases}$$

where the functions $\varphi(x)$ and $\psi(x)$ are odd. Prove that the solution $u(x,t)$ of this problem constructed by the d'Alambert formula satisfies the boundary condition $u(0,t) = 0,\ \forall t \geq 0$.

Sol. A1. The characteristic equation is $y'^2 - 10y' + 16 = 0$, so that $D = 9$ and the equation is hyperbolic. The characteristics are given by $y - 8x = c$, $y - 2x = c$ which induces a smooth, bijective change of coordinates $\xi = x$, $\eta = y - 5x$. Thus we set

$$u(x,y) = v(x, y - 5x)$$

to get

$$\begin{aligned} &\partial_x u = \partial_\xi v - 5\partial_\eta v, \quad \partial_y u = \partial_\eta v \\ &\partial_{xx}u = \partial_{\xi\xi}v - 10\partial_{\xi\eta}v + 25\partial_{\eta\eta}v, \quad \partial_{xy}u = \partial_{\xi\eta}v - 5\partial_{\eta\eta}v, \quad \partial_{yy}u = \partial_{\eta\eta}v \end{aligned}$$

which entails

$$\begin{cases} \partial_{\xi\xi}v - 9\partial_{\eta\eta}v = 0 \\ v(0,\eta) = \cos(\eta) \\ (\partial_\xi v)(0,\eta) = \cos(\eta) \end{cases}$$

Therefore

$$\begin{aligned} v(\xi,\eta) &= \tfrac{1}{2}\cos(\eta - 3\xi) + \tfrac{1}{2}\cos(\eta + 3\xi) + \tfrac{1}{6}\int_{\eta-3\xi}^{\eta+3\xi} \cos(\zeta)d\zeta \\ &= \tfrac{1}{2}\cos(\eta - 3\xi) + \tfrac{1}{2}\cos(\eta + 3\xi) + \tfrac{1}{6}\sin(\eta + 3\xi) - \tfrac{1}{6}\sin(\eta - 3\xi) \end{aligned}$$

and finally

$$u(x,y) = \tfrac{1}{2}\cos(y - 8x) + \tfrac{1}{2}\cos(y - 2x) + \tfrac{1}{6}\sin(y - 2x) - \tfrac{1}{6}\sin(y - 8x)$$

As an alternative, once it was determined that this is a hyperbolic equation, one could have equivalently found a solution via the ansatz $u(x,y) = f(y - 8x) + g(y - 2x)$.

Sol. A2. There exists a unique analytic solution by C-K's Theorem. We have

$$\begin{aligned}
&(\partial_{x^n} u)(0,x) = 0,\ n \ge 0\\
&(\partial_{t,x} u)(0,x) = c, \quad (\partial_{t,x^n} u)(0,x) = 0,\ n \ge 2\\
&(\partial_{t,t} u)(0,x) = 0\\
&(\partial_{t^3} u)(0,x) = 2(\partial_x u)(\partial_{t,x} u)(0,x) = 0\\
&(\partial_{t^4} u)(0,x) = 2(\partial_{t,x} u)^2 + 2\partial_x u \partial_{t,t,x} u = 2c^2\\
&(\partial_{t^m,x^n} u)(0,x) = 0, \quad m \ge 5, n \ge 0
\end{aligned}$$

And indeed $u(t,x) = c\,t\,x + \frac{c^2}{12} t^4$ solves the problem.

Sol. A3. Set for the sake of clear notation, $r(x) = |x|$ and $\bar{u} = u(x/|x|^2)$. Then, understanding that u, ∇u and Δu are always calculated at the point $x/|x|^2$ on the right hand side

$$\begin{aligned}
(\nabla r^m)(x) &= m|x|^{m-2} x\\
(\nabla \bar{u})(x) &= |x|^{-2} \nabla u - \frac{2x \cdot \nabla u}{|x|^4} x\\
\nabla v(x) &= \nabla(r^{2-n})(x)\, \bar{u} + |x|^{2-n} (\nabla \bar{u})(x)\\
&= |x|^{-n} \big((2-n) x\, u + \nabla u - 2\tfrac{x \cdot \nabla u}{|x|^2} x \big)
\end{aligned}$$

where in the last line we understand that u is calculated at $x/|x|^2$. Recalling $\mathrm{div}(a\mathbf{b}) = \nabla a \cdot \mathbf{b} + a\,\mathrm{div}(\mathbf{b})$ and noticing $\mathrm{div}(x) = n$

$$(\Delta v)(x) = \mathrm{div}(\nabla v)(x) = |x|^{-n-2}((\Delta u)(x/|x|^2) + 0) = |x|^{-n-2} f(x/|x|^2)$$

Sol. A4. Since

$$u(t,x) = \frac{\varphi(x-at) + \varphi(x+at)}{2} + \frac{1}{2a} \int_{x-at}^{x+at} \psi(y) dy$$

we have that $u(0,t) = (\varphi(-at) + \varphi(at))/2 + \frac{1}{2a} \int_0^{at} \psi(y) dy - \frac{1}{2a} \int_0^{at} \psi(-y) dy$. All the terms vanish, since φ and ψ are odd.