Ex. B1. Let Ω be a domain of $\mathbb{R}^n$ not containing the origin, and let $f\colon \Omega \to \mathbb{R}$.
Given a function $u\colon \Omega \to \mathbb{R}$, its Kelvin transform v is defined by

$$v(x) := |x|^{2-n} u\big(\tfrac{x}{|x|^2}\big)$$

Prove that, if u satisfies the equation $\Delta u = f$ in Ω, then v satisfies $(\Delta v)(x) = |x|^{-n-2} f\big(\tfrac{x}{|x|^2}\big)$.

Ex. B2. Find the explicitly solution of the differential problem:

$$\begin{cases} \partial_{tt}u - 10\,\partial_{tx}u + 16\,\partial_{xx}u = 0 \\ u(0,x) = \cos(x) \\ (\partial_t u)(0,x) = 4\,\sin(x) \end{cases}$$

Find the solution.

Ex. B3. Consider the problem:

$$\begin{cases} \partial_{tt}u - (\partial_x u)^2 = 0 \\ u(0,x) = x \\ (\partial_t u)(0,x) = x \end{cases}$$

Does it admit analytic solutions? If so, is it unique? In such a case, find it explicitly.

Ex. B4. Consider the Cauchy problem for the string equation

$$\begin{cases} \partial_{tt}u = a^2 \partial_{xx}u \\ u(0,x) = \varphi(x) \\ (\partial_t u)(0,x) = \psi(x), \quad x \in \mathbb{R}. \end{cases}$$

where the functions $\varphi(x)$ and $\psi(x)$ are even. Prove that the solution $u(x,t)$ of this problem constructed by the d'Alambert formula satisfies the boundary condition $(\partial_x u)(0,t) = 0$, $\forall t \geq 0$.

Sol. B1. Set for the sake of clear notation, $r(x) = |x|$ and $\bar{u} = u(x/|x|^2)$. Then, understanding that u, ∇u and Δu are always calculated at the point $x/|x|^2$ on the right hand side

$$\begin{aligned} (\nabla r^m)(x) &= m|x|^{m-2}x \\ (\nabla \bar{u})(x) &= |x|^{-2}\nabla u - \frac{2x\cdot\nabla u}{|x|^4}x \\ \nabla v(x) &= \nabla(r^{2-n})(x)\,\bar{u} + |x|^{2-n}(\nabla\bar{u})(x) \\ &= |x|^{-n}\big((2-n)x\,u + \nabla u - 2\tfrac{x\cdot\nabla u}{|x|^2}x\big) \end{aligned}$$

where in the last line we understand that u is calculated at $x/|x|^2$. Recalling $\operatorname{div}(a\mathbf{b}) = \nabla a \cdot \mathbf{b} + a\operatorname{div}(\mathbf{b})$ and noticing $\operatorname{div}(x) = n$

$$(\Delta v)(x) = \operatorname{div}(\nabla v)(x) = |x|^{-n-2}((\Delta u)(x/|x|^2) + 0) = |x|^{-n-2} f(x/|x|^2)$$

Sol. B2. The characteristic equation is $x'^2 + 10x' + 16 = 0$, so that $D = 9$ and the equation is hyperbolic. The characteristics are given by $x + 8t = c$, $x + 2t = c$ which induces a smooth, bijective change of coordinates $\xi = t$, $\eta = x + 5t$. Thus we set

$$u(t,x) = v(t, x+5t)$$

to get

$$\begin{aligned} &\partial_t u = \partial_\xi v + 5\partial_\eta v, \quad \partial_x u = \partial_\eta v \\ &\partial_{tt}u = \partial_{\xi\xi}v + 10\partial_{\xi\eta}v + 25\partial_{\eta\eta}v, \quad \partial_{xt}u = \partial_{\xi\eta}v + 5\partial_{\eta\eta}v, \quad \partial_{xx}u = \partial_{\eta\eta}v \end{aligned}$$

which entails

$$\begin{cases} \partial_{\xi\xi}v - 9\partial_{\eta\eta}v = 0 \\ v(0,\eta) = \cos(\eta) \\ (\partial_\xi v)(0,\eta) = 9\sin(\eta) \end{cases}$$

Therefore

$$v(\xi, \eta) = -\cos(\eta + 3\xi) + 2\cos(\eta - 3\xi)$$

and finally

$$u(x, y) = 2\cos(x + 2t) - \cos(x + 8t)$$

Sol. B3. There exists a unique analytic solution by C-K's Theorem. We have

$$\begin{aligned}
&(\partial_{t,t}u)(0, x) = (\partial_x u)^2(0, x) = 1 \\
&(\partial_{t,t,t}u)(0, x) = 2(\partial_x u)(0, x)\, \partial_{t,x}u(0, x) = 2 \\
&(\partial_{t,t,t,t}u)(0, x) = 2(\partial_{t,x}u)(0, x) + 2(\partial_x u)(0, x)\, (\partial_{t,t,x}u)(0, x) = 2 \\
&(\partial_{t^k}u)(0, x) = 0, \quad k \geq 5
\end{aligned}$$

And indeed $u(t, x) = x + x\,t + t^2/2 + t^3/3 + t^4/12$ solves the problem.

Sol. B4. From d'Alambert formula

$$(\partial_x u)(t, x) = \frac{\varphi'(x - at) + \varphi'(x + at)}{2} + \frac{1}{2a}(\psi(x + at) - \psi(x - at))$$

Since φ is even, φ' is odd, so the previous formula vanishes at $x = 0$.