Write your name and the version of your test (namely A). Feel free to answer in English or Russian.

Ex. A1. Consider the problem in the unknown $u\colon \mathbb{R}\times[0,1]\to\mathbb{R}$, $u\equiv u(t,x)$

$$\begin{cases} \partial_{tt}u + 2\partial_t u = \partial_{xx}u - 2\partial_x u \\ u(0,x) = e^x\, x \\ (\partial_t u)(0,x) = -e^x\, x \\ u(t,0) = u(t,1) = 0 \end{cases}$$

Find the solution explicitly.

Ex. A2. Solve the problem in the unknown $u(t,x,y)$ on $(t,x,y)\in\mathbb{R}\times\mathbb{R}^2$

$$\begin{cases} \partial_t u = i\Delta u + \cos(t) \\ u(0,x,y) = (x+y)\, e^{-(x^2+y^2)/2} \end{cases}$$

Why does it admits a unique solution? Find it.

Ex. A3. An half-infinite bar radiates heat on its end proportionally to its temperature: solve the problem for $t>0$ and $x\geq 0$

$$\begin{cases} \partial_t u = \partial_{xx}u \\ u(t=0,x) = e^{-2\,x} \\ (\partial_x u)(t,x=0) = -u(t,x=0) \end{cases}$$

Some useful integrals

$$\int_0^1 x\,\sin(k\,\pi\,x)dx = \tfrac{(-1)^{1+k}}{k\pi},$$

$$\int_0^1 x\,\cos(k\,\pi\,x)dx = \tfrac{(-1)^k-1}{\pi^2 k^2}, \qquad k\in\mathbb{Z}$$

$$\int_{-\infty}^{+\infty} \exp\left(-\tfrac{x^2}{2}\right)\exp(i\,s\,x)\,dx = \sqrt{2\pi}\exp(-\tfrac{s^2}{2})$$

Sol. A1. We look for the separation of variables. If $u(t,x)=a(t)b(x)$, then $(a''(t)+2a'(t))b(x) = a(t)(b''-2b')$. This entails $b''-2b'=\lambda b$ with $b(0)=b(1)=0$, $a''+2a'=\lambda a$ for some constant λ. The first equation has solution for $\lambda_k=-1-k^2\pi^2$, with k a non-negative integer. In such a case $b_k(x)=e^x\sin(k\,\pi\,x)$. The b_k's are orthogonal in $L_2(e^{-2x}dx)$. Thus $a_k(t)=e^{-t}(\alpha_k\cos(k\pi t)+\beta_k\sin(k\,\pi\,t))$. However, since $(\partial_t u)(0,x)=-u(0,x)$ it holds $\beta_k=0$. Therefore

$$u(t,x)=\sum_k \alpha_k e^{x-t}\cos(k\,\pi\,t)\sin(k\,\pi\,x)$$

where

$$\alpha_k = \frac{\int_0^1 u(0,x)b_k(x)e^{-2x}\,dx}{\int_0^1 b_k(x)^2 e^{-2x}\,dx} = 2\int_0^1 x\,\sin(k\,\pi\,x)dx = \tfrac{-2(-1)^k}{k\pi}$$

Sol. A2. It is better to consider the function $v(t,x)=u(t,x)+\sin(t)$. It satisfies

$$\begin{cases} \partial_t v = i\Delta v \\ v(0,x,y) = (x+y)\, e^{-(x^2+y^2)/2} \end{cases}$$

This has a unique solution as a well-posed Petrovsky problem with polynomial $P(d)=id_1^2+id_2^2$, and thus with semigroup bound $e^{0t}=1$. The solution is then

$$u(t,x,y) = -\sin(t) + \tfrac{1}{4\pi^2}\int_{\mathbb{R}^2} e^{-i(s_1\,x+s_2\,y)}e^{-i(s_1^2+is_2^2)t}\hat{v}_0(s_1,s_2)ds_1\,ds_2$$

where

$$\hat{v}_0(s_1, s_2) = \int_{\mathbb{R}^2} (x+y)\, e^{-(x^2+y^2)/2} e^{i(s_1 x + s_2 y)} = 2i\pi e^{-\frac{s_1^2+s_2^2}{2}} (s_1 + s_2)$$

so that $u(t,x,y) = -\sin(t) + \frac{(x+y)\exp(\frac{i(x^2+y^2)}{4t-2i})}{(1+2it)^2}$

Sol. A3. We want to write the solution as the solution to a heat equation on the whole line

$$\begin{cases} \partial_t v = \partial_{xx} v \\ u(t=0,x) = \varphi(x) \end{cases}$$

so that the condition $(\partial_x v)(t, x=0) = -v(t, x=0)$ is automatically ensured. To this aim, we consider for $x < 0$ a function φ solving for $x < 0$

$$\varphi'(x) + \varphi(x) = \varphi'(-x) + \varphi(-x)$$

or, for $x < 0$,

$$\varphi(x) = \varphi(0)\, e^{-x} - e^{-x} \int_0^{-x} e^{-y}(\varphi'(y) + \varphi(y))\, dy = \frac{1}{3} e^{-x} \left(4 - e^{3x}\right)$$

so that

$$u(t,x) = (p_t * \varphi)(x)$$