Write your name and the version of your test (namely B). Feel free to answer in English or in Russian.

Ex. B1. Solve the problem in the unknown $u(t,x,y)$ on $(t,x,y) \in \mathbb{R} \times \mathbb{R}^2$

$$\begin{cases} \partial_t u = i\Delta u - \sin(t) \\ u(0,x,y) = (x+y)\, e^{-(x^2+y^2)/2} \end{cases}$$

Why does it admits a unique solution? Find it.

Ex. B2. An half-infinite bar radiates heat on its end proportionally to its temperature: solve the problem for $t > 0$ and $x \leq 0$

$$\begin{cases} \partial_t u = \partial_{xx} u \\ u(t=0,x) = e^{2x} \\ (\partial_x u)(t,x=0) = -u(t,x=0) \end{cases}$$

Ex. B3. Consider the problem in the unknown $u \colon \mathbb{R} \times [0,1] \to \mathbb{R}$, $u \equiv u(t,x)$

$$\begin{cases} \partial_{tt} u - 2\partial_t u = \partial_{xx} u + 2\partial_x u \\ u(0,x) = x\, e^{-x} \\ (\partial_t u)(0,x) = x\, e^{-x} \\ u(0,0) = u(0,1) = 0 \end{cases}$$

Find the solution explicitly.

Some useful integrals

$$\int_0^1 x\, \sin(k\,\pi\,x) dx = \tfrac{(-1)^{1+k}}{k\pi}, \qquad \int_0^1 x\, \cos(k\,\pi\,x) dx = \tfrac{(-1)^k - 1}{\pi^2 k^2}, \qquad k \in \mathbb{Z}$$

$$\int_{-\infty}^{+\infty} \exp\left(-\tfrac{x^2}{2}\right) \exp(i\,s\,x)\, dx = \sqrt{2\pi}\exp(-\tfrac{s^2}{2})$$

$$\int_{-\infty}^{+\infty} x \exp\left(-\tfrac{x^2}{2}\right) \exp(i\,s\,x)\, dx = i\sqrt{2\pi}\exp(-\tfrac{s^2}{2}) s$$

Sol. B1. It is better to consider the function $v(t,x) = u(t,x) + \cos(t)$. It satisfies

$$\begin{cases} \partial_t v = i\Delta v \\ v(0,x,y) = (x+y)\, e^{-(x^2+y^2)/2} \end{cases}$$

This has a unique solution as a well-posed Petrovsky problem with polynomial $P(d) = id_1^2 + id_2^2$, and thus with semigroup bound $e^{0t} = 1$. The solution is then

$$u(t,x,y) = -\sin(t) + \tfrac{1}{4\pi^2} \int_{\mathbb{R}^2} e^{-i(s_1\,x + s_2\,y)} e^{-i(s_1^2 + i s_2^2)t} \hat{v}_0(s_1,s_2) ds_1\, ds_2$$

where

$$\hat{v}_0(s_1,s_2) = \int_{\mathbb{R}^2} (x+y)\, e^{-(x^2+y^2)/2} e^{i(s_1 x + s_2 y)} = 2i\pi e^{-\frac{s_1^2+s_2^2}{2}} (s_1 + s_2)$$

so that $u(t,x,y) = -\cos(t) + \frac{(x+y)\exp(\frac{i(x^2+y^2)}{4t-2i})}{(1+2it)^2}$

Sol. B2. We want to write the solution as the solution to a heat equation on the whole line

$$\begin{cases} \partial_t v = \partial_{xx} v \\ u(t=0,x) = \varphi(x) \end{cases}$$

so that the condition $(\partial_x v)(t, x = 0) = -v(t, x = 0)$ is automatically ensured. To this aim, we consider for $x < 0$ a function φ solving for $x < 0$

$$\varphi'(x) + \varphi(x) = \varphi'(-x) + \varphi(-x)$$

or

$$\varphi(x) = \varphi(0)\, e^{-x} - e^{-x} \int_0^{-x} e^{-y}(\varphi'(y) + \varphi(y))\, dy = -e^{-x}\left(3e^{-x} - 4\right)$$

so that

$$u(t, x) = (p_t * \varphi)(x)$$

Sol. B3. We look for the separation of variables. If $u(t, x) = a(t)b(x)$, then $(a''(t)+2a'(t))b(x) = a(t)(b''-2b')$. This entails $b'' - 2b' = \lambda b$ with boundary conditions, $a''(+2a' = \lambda a$ for some constant λ. The first equation has solution for $\lambda_k = -1 - k^2\pi^2$, with k a non-negative integer. In such a case $b_k(x) = e^x \sin(k\,\pi\, x)$. The b_k's are orthogonal in $L_2(e^{-2x}dx)$. Thus $a_k(t) = e^{-t}(\alpha_k \cos(k\pi t) + \beta_k \sin(k\,\pi\, t))$. However, since $(\partial_t u)(0, x) = -u(0, x)$ it holds $\beta_k = 0$. Therefore

$$u(t, x) = \sum_{k\geq 0} \alpha_k e^{x-t} \cos(k\,\pi\, t) \sin(k\,\pi\, x)$$

where

$$\alpha_k = \frac{\int_0^1 u(0, x) b_k(x) e^{-2x}\, dx}{\int_0^1 b_k(x)^2 e^{-2x}\, dx} = 2\int_0^1 x\, \sin(k\,\pi\, x) dx = \tfrac{-2(-1)^k}{k\pi}$$