Ex 1. (Geodesic characteristics) In this exercise we will give a key example to motivate the method of characteristics.

(1a) Find all the C^1-solutions $u(t,x)$ for the equations in the unknown $u\colon \mathbb{R}\times\mathbb{R}$

$$\partial_t u = \partial_x u \tag{1}$$

(this means that we look for a continuously differentiable function $u(t,x)$ such that at every $(t,x)\in\mathbb{R}\times\mathbb{R}$ it holds $\left(\frac{\partial u}{\partial t}\right)(t,x) = \left(\frac{\partial u}{\partial x}\right)(t,x)$).

(1b) Check, without using the explicit form of the solutions, that C^2-solutions to (1) are also solutions to the wave equation

$$\partial_{tt} u = \partial_{xx} u$$

Let now $d \geq 1$ and let $V \in \mathbb{R}^d$ be a (tangent) vector in $\mathbb{R}^d$, and consider the equation in the unknown $u\colon \mathbb{R}\times\mathbb{R}^d \to \mathbb{R}$

$$\partial_t u = V\cdot\nabla u \tag{2}$$

where ∇ denotes the gradient operator in the variable $x=(x_1,\dots,x_d)$; thus (2) means $\left(\frac{\partial u}{\partial t}\right)(t,x) = \sum_{i=1}^d V_i \left(\frac{\partial u}{\partial x_i}\right)(t,x)$.

(1c) Find all C^1-solutions to (2). In particular, check that for every $u_0 \in C^1(\mathbb{R}^d)$ there exists a unique solution u to (2) such that $u(0,x)=u_0(x)$.

(1d) Let now u be a bounded measurable function $u\colon \mathbb{R}\times\mathbb{R}^d$. Consider the two statements

(A) $u\in C^1(\mathbb{R}\times\mathbb{R}^d)$ is a solution to (2).

(B) For each $\varphi \in C^1_c(\mathbb{R}\times\mathbb{R}^d)$ (the space of continuously differentiable and compactly supported functions), it holds

$$\int_{\mathbb{R}\times\mathbb{R}^d} u(t,x)\,(\partial_t\varphi)(t,x)\,dt\,dx = \int_{\mathbb{R}\times\mathbb{R}^d} u(t,x)\,V\cdot(\nabla\varphi)(t,x)\,dt\,dx$$

Are (A) and (B) equivalent? Does (A) imply (B)? Does (B) imply (A)? Give proofs or counterexamples. What about the relation of (A) and (B) with the (trivial) statement

(C) For each $\varphi \in C^1_c(\mathbb{R}\times\mathbb{R}^d)$ it holds for all $(t,x)\in\mathbb{R}\times\mathbb{R}^d$

$$u(t,x)\,(\partial_t\varphi)(t,x) = u(t,x)\,V\cdot(\nabla\varphi)(t,x)$$

Ex 2. (Solitons) In a famous episode dating 1844, the Scottish engineer John Scott Russell reported to the *British Association for the Advancement of Science* the following , York, September 1844[1]

> I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation.

The heigth $u(t,x)$ of the water in a shallow and narrow channel is, under suitable approximations (no viscosity, unidimensionality etc), well-described by the $(1+1)$-dimensional KdV equation

$$\partial_t u + \partial_{xxx} u = 6u\,\partial_x u \tag{3}$$

Can one guess, from this equation, the unusual phenomenon observed by Russel?

[*Hint:* Since we dropped viscosity from the model, we are looking for a stable travelling wave solution. Namely a function $u(t,x)$ solving (3) such that u converges to a constant (say 0) for $|x|\to\infty$, and such that $u(t,x)=f(x-a-vt)$ for some constants $a,v\in\mathbb{R}$. Also, notice that the function $g(x) := -\frac{1}{2\cosh(x)^2}$ satisfies $g'' = 3\,g^2 + g$.]

Sol 1a. Let u be a solution and consider the $C^1(\mathbb{R}\times\mathbb{R})$-function w defined as

$$w(s,y) = u(s-y,y), \qquad u(t,x) = w(t+x,x)$$

[1] Later Russel even built a narrow channel in his own garden to further study the behavior of waves.

Then[2] $(\partial_y w)(s,y) = -(\partial_t u)(s-y,y) + (\partial_x u)(s-y,y) = 0$. Thus $w(s,y) = f(s)$ for some f, and since $u(t,x) = w(t+x,x) = f(t+x)$. In other words, all solutions have the form $f(t+x)$ for some $C^1(\mathbb{R})$-function f.

Sol 1b. Since $\partial_t u = \partial_x u$, deriving w.r.t. t we gather $\partial_{tt} u = \partial_{tx} u = \partial_{xt} u = \partial_{xx} u$.

Sol 1c. Reasoning as in (1a) and setting $w(s,y) = u(s, y - Vs)$, we have $(\partial_s w)(s,y) = (\partial_t u)(s, y-Vs) - (V \cdot \nabla u)(s, y - Vs) = 0$. Thus $u(t,x) = f(x+Vt)$ for some $f \in C^1(\mathbb{R}^d)$.

Sol 1d. (A) implies (B). Indeed let's just multiply the equation by φ and integrate by parts. Since φ is compactly supported there are no boundary terms, and we get (B).

(B) does not imply (A). If u is smooth, we may just read backword the previous implication to obtain (A). However in the formulation (B) there is no need to assume u of class C^1 (indeed it's just bounded and measurable). Therefore take any bounded measurable (but not smooth) function $f\colon \mathbb{R}^d \to \mathbb{R}$, and let $u(t,x) = f(x+Vt)$. We claim that such a u satisfies (B). Indeed for each $\varphi \in C^1_c(\mathbb{R}\times\mathbb{R}^d)$, using the substitution $y = x + Vt$ and the auxiliary function $\psi(t,y) = \varphi(t, y - Vt)$ (check that it is also compactly supported!)

$$\begin{aligned}\int f(x+Vt)\,(\partial_t\varphi - V\cdot\nabla\varphi)(t,x))\;dt\,dx &= \int f(y)\,(\partial_t\varphi - V\cdot\nabla\varphi)(t, y-Vt))\,dt\,dy \\ &= \int_{\mathbb{R}^d} f(y)\left(\int_{\mathbb{R}} (\partial_t\psi)(t,y)dt\right)dy = 0\end{aligned}$$

Sol 2. If $u(t,x)$ solves (3) and vanishes at infinity, also $u(t, x-a)$ solves (3) and vanishes at infinity. So we can take $a = 0$ and look for solution $u(t,x) = f(x - vt)$. Then f satisfies the ODE

$$v\,f' + f''' = 3(f^2)'$$

and integrating we gather

$$v\,f + f'' - 3f^2 = C \tag{4}$$

for some integration constant C. To get an intuition about this ODE, we can look at it as a Newtonian particle in a potential. Namely setting $Z(t) = f(t)$ and $U(z) = -z^3 + \frac{1}{2}vz^2 - Cz$, (4) reads

$$\ddot{Z} = -U'(z)$$

This suggests that we need $C = 0$ and $v > 0$ if we want to find a non-zero solution vanishing at infinity. Then, using the hint, the wanted solution is simply

$$f(x) = -v\frac{1}{2\cosh(\sqrt{v}(x-vt))^2}$$

[2] Recall that $\partial_t u$ and $\partial_x u$ denote just the derivatives of u w.r.t. the first and second variable.