Ex 0. Briefly recall the algorithm to put a semilinear second-order PDE in canonical form, in the 2-dimensional case.

Ex 1. Consider the PDE in the unknown $u \equiv u(x, y)$

$$\partial_{xx}u - 2\,\sin(x)\partial_{xy}u - \cos(x)^2\partial_{yy}u = \cos(x)\partial_y u \tag{1}$$

Classify the PDE, calculate the characteristics and write it in canonical form. Then find explicitely all smooth solutions to (1).

Ex 2. Consider the PDE in the unknown $u \equiv u(x, y)$

$$x\,\partial_{xx}u - 2\,x^2\partial_{xy}u + 2x^3\,\partial_{yy}u = \partial_x u$$

Classify the PDE, calculate the characteristics and write it in canonical form. Then find explicitely those solutions such that $\sup_{x,y} \frac{|u(x,y)|}{(|x|+|y|+1)} < \infty$. What about solutions that are bounded at infinity but may diverge at $(x, y) = (0, 0)$?

Ex 3. Find the canonical form of the PDE

$$\partial_{xx}u - 2x\,\partial_{xy}u + x^2\partial_{yy}u = 2\partial_y u$$

Sol 0. Suppose we are given the equation in the unknown $u \equiv u(x, y)$

$$a(x, y)\partial_{xx}u + 2\,b(x, y)\partial_{xy}u + c(x, y)\partial_{yy}u = F(x, y, u, \partial_x u, \partial_y u) \tag{2}$$

We want to write the equation in a canonical form, with $b = 0$ and $a, c = \pm 1$, via change of variables (even if sometimes also the case $a = c = 0$ and $b = 1$ is considered canonical and can simplify calculations). This is just a general method, so we will assume that all the coefficients are smooth, denominators non-vanishing etc. In concrete cases, one has to check singular points or maybe adjust the strategy (for instance, if $a(x, y)$ vanishes at some point, and $c(x, y)$ never vanishes, just replace the role of x and y in the strategy below etc).

Classification and characteristics. First, we look at the determinant $D(x, y) := b^2(x, y) - a(x, y)c(xy)$. A point (x, y) is called hyperbolic if $D(x, y) > 0$, elliptic if $D(x, y) < 0$ and parabolic if $D(x, y) = 0$. In any case, we look at the ODE in the unknown $y = y(x)$ (beware the signs)

$$a(x, y)\left(\frac{dy}{dx}\right)^2 - 2\,b(x, y)\left(\frac{dy}{dx}\right) + c(x, y) = 0$$

Or equivalently the two first order ODEs

$$\frac{dy}{dx} = \frac{b(x, y) \pm \sqrt{b^2(x, y) - a(x, y)c(x, y)}}{a(x, y)} \tag{3}$$

Notice that this is a non-linear and non-autonomous ODE, so that there is no reason for it to have an explicit solution. Nevertheless, generically each of the equations in (3) will feature a solution depending on an arbitrary constant. Thus, we here *assume* that we can write the solutions in (4), at least in a parametric form, namely that there exist function ϕ and ψ such that the solutions with sign $\pm$ satisfy respectively

$$\phi(x, y) = c, \qquad \psi(x, y) = c \tag{4}$$

For instance, if $\frac{b(x,y)+\sqrt{b^2(x,y)-a(x,y)c(x,y)}}{a(x,y)} = f(x)$ is independent of y, then $y(x) = c + \int_0^x f(z)dz$. In this case we can just take $\phi(x, y) = y - \int_0^x f(z)dz$.

If the coefficients are smooth, in general the space will be partitioned in (open) regions of hyperbolic and elliptic points, and a closed region of parabolic points. We should study these cases separately.

Hyperbolic case $D > 0$. In this case, the solutions are real. We consider the change of variables

$$\xi(x, y) = \tfrac{1}{2}(\phi(x, y) + \psi(x, y)) \qquad \eta(x, y) = \tfrac{1}{2}(\phi(x, y) - \psi(x, y)) \tag{5}$$

provided the determinant $\partial_x\phi\partial_y\psi - \partial_y\phi\partial_x\psi$ does not vanish (equivalently $\partial_x\xi\partial_y\eta - \partial_y\xi\partial_x\eta \neq 0$). In this case, we can define a function v by

$$u(x, y) = v(\xi(x, y), \eta(x, y))$$

since, by the inverse function theorem, (5) is locally invertible. At this point, we can replace all the derivatives of u with derivatives of v. E.g. $\partial_x u = (\partial_\xi v)\partial_x \xi + (\partial_\eta v)\partial_x \eta$. When in doubt, always write down things explicitly. E.g. the latter relations truly means

$$(\partial_x u(x,y) = (\partial_\xi v)(\xi(x,y),\eta(x,y))(\partial_x \xi)(x,y) + (\partial_\eta v)(x,y)(\partial_x \eta)(x,y)$$

Replacing all the derivatives in (2), and -if needed- replacing $x = X(\xi,\eta)$, $y = Y(\xi,\eta$, where (X,Y) is the inverse of the function $(\xi(x,y),\eta(x,y))$, one gets an equation of the form

$$\partial_{\xi\xi} v - \partial_{\eta\eta} v = G(\xi,\eta,v,\partial_\xi v,\partial_\eta v)$$

Only in the hyperbolic case, one may also choose the coordinate ϕ and ψ, define w via $u(x,y) = w(\phi(x,y),\psi(x,y))$ and similarly obtain the equation

$$-\partial_{\phi\psi} w = H(\phi,\psi,w,\partial_\phi w,\partial_\psi w)$$

for a suitable H.

Elliptic case $D < 0$. The procedure is the same in this case, but the solutions to the ODEs are complex conjugates. Then $\phi = \bar{\psi}$. One should set

$$\xi(x,y) = \frac{1}{2}(\phi(x,y) + \psi(x,y)) \qquad \eta(x,y) = \frac{1}{2i}(\phi(x,y) - \psi(x,y))$$

Then continue as above. One gets an equation of the form

$$\partial_{\xi\xi} v + \partial_{\eta\eta} v = G(\xi,\eta,v,\partial_\xi v,\partial_\eta v)$$

Parabolic case $D = 0$. In this case, $\phi = \psi$. So we set $\xi(x,y) = \phi(x,y)$, while $\eta(x,y)$ can be chosen arbitrarily, provided $\partial_x \xi \partial_y \eta - \partial_y \xi \partial_x \eta \neq 0$. The simplest choices are usually $\eta = x$ or $\eta = y$. The canonical form of the equation will be

$$\partial_{\eta\eta} v = G(\xi,\eta,v,\partial_\xi v,\partial_\eta v)$$

Sol 2. In this case we get $\xi = y + \frac{1}{2}x^2$, $\eta = \frac{1}{2}x^2$ and

$$\partial_{\xi\xi} v + \partial_{\eta\eta} v = 0$$

In other words, the general solution is given by $u(x,y) = v(y + \frac{1}{2}x^2, \frac{1}{2}x^2)$, for an arbitrary harmonic function v. Recall that an harmonic function has at most linear growth iff it is affine (since an harmonic function on the plane is -say- the real part of an analytic function). Since the dependence of ξ and η is quadratic in x, we have the only possibility $v(\xi,\eta) = a(\xi - \eta) + b$, and $u(x,y) = ay + b$.