Remainder On a Riemannian manifold, the gradient operator is defined by the relation $\rangle \nabla f, V\rangle = df(V)$, for any smooth function f and tangent vector V. In other words, the metric tensor g maps the cotangent vector df in the tangent vector ∇f. On the Euclidean space $\mathbb{R}^n$, if we choose standard orthonormal coordinates $\mathbf{x}$ such that g coincides with the identity, we gather that ∇f is identified by the relation

$$f(y) = f(x) + (\nabla f)(x) \cdot (y - x) + o(|y - x|)$$

so that in these coordinates $\nabla f = (\partial_{x_1} f, \ldots, \partial_{x_n} f)$.

The divergence operator is defined as the adjoint of $-\nabla$ in $L_2(dx)$, where dx is the volume measure (Lebesgue on the Euclidian space $\mathbb{R}^n$). In other words, by the formula

$$\int dx\, (\nabla \cdot F)(x)\, f(x) = -\int dx\, F(x) \cdot (\nabla f)(x) \tag{1}$$

For instance in coordinates on $\mathbb{R}^n$, $\nabla \cdot F = \sum_{i=1}^n \partial_{x_i} F^i$, provided $F = (F^1, \ldots, F^n)$.

Finally, $\Delta f := \nabla \cdot (\nabla f)$, so on $\mathbb{R}^n$, $\Delta f = \sum_{i=1}^n \partial_{x_i x_i} f$.

Notice that the notation $\nabla f = \mathrm{grad} f$ and $\nabla \cdot F = \mathrm{div}(F)$ also exist.

Ex 1. Check that on $\mathbb{R}^n$ it holds, for a smooth tangent vector field F and a smooth function φ

$$\nabla \cdot (\varphi F) = (\nabla \varphi) \cdot F + \varphi \nabla \cdot F$$

These formulas admit wide generalizations, both in the context of differential geometry and geometric measure theory[1].

Ex 2. The polar coordinates on $\mathbb{R}^2$ are given by the change of variables

$$x = \varrho \cos(\theta)$$
$$y = \varrho \sin(\theta)$$

In other words, one identifies $\mathbb{R}^2 \simeq ([0, \infty) \times S^1)/\sim$, where $(\varrho, \theta) \sim (\varrho', \theta')$ iff $\varrho = \varrho' = 0$[2].

Calculate in polar coordinates: (a) the volume form (namely write $\int dxdy f(x, y)$ in polar coordinates); (b) the gradient of a function f; (c) the divergence of a vector field F; (d) the Laplacian of a function f.

Ex 3. On a bi-dimensional space, two (one-dimensional) circles of radius r and R, with $r < R$, are positioned in the vacuum. The smaller circle is attached to a generator at potential $1V$. The larger one, is grounded (that is, its potential is 0). Recall that the electric potential satisfies the equation $\Delta u = 0$ when no charges are on place. Calculate the electric potential in the space $r < |x| < R$.

Ex. 4 Recall how to calculate the signature of a quadratic form in $\mathbb{R}^n$. Then consider the equation

$$4\partial_{xx} u + 2\partial_{yy} u + 2\partial_{zz} u + 2\partial_{xy} u - 2\partial_{xz} u + 2\partial_{yz} u + (\partial_x u)^2 = 0$$

Is it parabolic, elliptic or hyperbolic?

Sol 1. It is enough to check the identity locally in coordinates. Thus it is the usual rule of derivation for products.

Sol 2. Let's be very precise here, even if the statements are trivial. We have a space $E = \mathbb{R}^2$, and we can identify it with $\mathbb{R} \times \mathbb{R}$ or with $([0, \infty) \times S^1)/\sim$. This means that there are bijections $\imath\colon E \to \mathbb{R} \times \mathbb{R}$ and $\jmath\colon ([0, \infty) \times S^1)/\sim$. Given measures, distances, functions etc on E, we can use these bijections to associate to them a corresponding object in $\mathbb{R} \times \mathbb{R}$ or in $([0, \infty) \times S^1)/\sim$. For instance if λ is the Lebesgue measure on E, then $\lambda \circ \imath^{-1}$ and $\lambda \circ \jmath^{-1}$ are the corresponding measures. We already know that $d\lambda \circ \imath^{-1}(x, y) = dx\, dy$. The exercise requires ask to calculate $\lambda \circ \jmath^{-1}$ in the (ϱ, θ) coordinates. We just need to use the change of variable formula with the determinant of the Jacobian of the change of variables. We easily get $d\lambda \circ \jmath^{-1}(\varrho, \theta) = \varrho\, d\varrho, d\theta$.

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[1]Actually an integral version of (1) can be used to define some useful concepts. For instance, one can say that $\varphi \in L_1(dx)$ has *bounded variation* if

$$\sup_{F \text{ measurable}\,:\, |F| \leq 1} \int_{\mathbb{R}^n} dx\, (\nabla \cdot F)(x)\, \varphi(x) < \infty$$

In such a case, $\nabla \varphi$ is identified with a (vectorial) measure. If $\Omega \subset \mathbb{R}^n$ is a measurable set, one says that it has *finite perimeter* if its characteristic function $\mathbf{1}_\Omega$ has bounded variation. In such a case, denote by $\sigma \hat{n}$ the measure $\nabla \mathbf{1}_\Omega$, which is clearly concentrated on $\partial \Omega$. Then (1) can be turned into

$$\int d\sigma \hat{n} \cdot F = -\int_\Omega dx\, (\nabla \cdot F)(x)$$

[2]However, in the coordinates (ϱ, θ) the metric tensor is not represented by the identity.

Using the Jacobian operator for the change of variables, we similarly calculate the differential operators, in particular $\Delta = \frac{1}{\varrho}\partial_\varrho(\varrho\partial_\varrho) + \frac{1}{\varrho^2}\partial_{\theta\theta}$.

Sol 3. We want to find a solution to the problem

$$\begin{cases} \Delta u = 0 & \text{in } r < |\mathbf{x}| < R \\ u = 1 & \text{in } |\mathbf{x}| = r \\ u = 0 & \text{in } |\mathbf{x}| = R \end{cases} \tag{2}$$

The whole point is that the problem has a rotational symmetry on $\mathbb{R}^2$. This means that if u is a solution to (2), and S_θ is a rotation of an angle θ in $\mathbb{R}^2$, then $v(\mathbf{x}) = u(S_\theta \mathbf{x})$ is also a solution. A priori, this *does not* imply that solutions are invariant under rotations (namely that $v = u$), but still it suggests that there can be rotationally symmetric solutions. Indeed, thanks to the linearity, we have that $\mathbf{x} \mapsto \int d\theta\, u(S_\theta \mathbf{x})$ is also a solution.

All this just suggests to make an ansatz for the solution in polar coordinates: u does not depend on the θ variable. Writing then $\delta u = 0$ in polar coordinates, we have

$$0 = \frac{1}{\varrho}\partial_\varrho(\varrho\partial_\varrho u) + \frac{1}{\varrho^2}\partial_{\theta\theta} u = \frac{1}{\varrho}\partial_\varrho(\varrho\partial_\varrho u) + 0$$

This yields $u(\mathbf{x}) = c_1 \log(|\mathbf{x}|/c_2)$, and imposing the boundary conditions, $c_2 = R$ and $c_1 = -1/\log(R/r)$.

Sol 3. Any method to determine the signature of a symmetric bilinear form (or any bilinear form) will work here. Here the fastest is probably to check that all the N-W determinants are positive.