Ex. 1 Find all the solutions in the unknown $u = u(x, y)$ to the following problems

$$\begin{cases} \partial_{yy} u = 2 \\ u\big|_{y=0} = 1 \\ (\partial_y u)\big|_{y=0} = x \end{cases}$$

$$\begin{cases} \partial_{xx} u - \partial_x u = 0 \\ u\big|_{x=0} = 1 \\ (\partial_x u)\big|_{x=0} = 2 \end{cases}$$

$$\begin{cases} \partial_{xy} u + \partial_x u = 0 \\ u\big|_{y=x} = \sin(x) \\ (\partial_x u)\big|_{y=x} = 1 \end{cases}$$

Ex. 2 For what functions $a \in C^2(\mathbb{R})$ does the problem

$$\begin{cases} \partial_{xx} u + 5\partial_{xy} u + 6\partial_{yy} u = 0 \\ u\big|_{y=3x+2} = 4x^2 + 1 \\ (\partial_x u)\big|_{y=3x+2} = a(x) \end{cases}$$

admit solutions?

Ex. 3 Find all the solutions in the unknown $u = u(x, y)$ to the following problem

$$\begin{cases} \partial_{xx} u + 3\partial_{xy} u + 2\partial_{yy} u = 0 \\ u\big|_{y=3x+1} = 6x^2 + 1 \\ (\partial_y u)\big|_{y=3x+1} = 1 \end{cases}$$

Homework 1. Recall that the spherical coordinates in $\mathbb{R}^3$ are given by

$$\begin{aligned} x_1 &= \varrho \cos(\theta) \\ x_2 &= \varrho \sin(\theta) \cos(\varphi) \\ x_3 &= \varrho \sin(\theta) \sin(\varphi) \end{aligned}$$

For $R > r > 0$, find a solution to the equation

$$\begin{cases} \Delta u(x) = 0 & \text{in } r < |x| < R. \\ u(x) = a & \text{if } |x| = r. \\ u(x) = b & \text{in } |x| = R. \end{cases}$$

Homework 2. Does the problem

$$\begin{aligned} &\partial_{xx} u = 4\partial_{yy} u \\ &u\big|_{y=2x+1} = x^2 \\ &(\partial_y u)\big|_{y=2x+1} = 5x \end{aligned}$$

admit solutions?

Sol 1-2-3. All these problems are quite elementary. The following procedure solves them all:

- Find the canonical form of the equation using the characteristics, and the (ξ, η) change of variables (see the second seminar sheet).
- Explicitly find the general solution to the equation. Beware, the class of solutions may depend critically on the boundary where the conditions are given. In particular, on wether the boundary is characteristic or not[1].
- Impose the boundary conditions in the (ξ, η) variables.
- Get back to the original variables (the order of the last two points may be interchanged according to convenience).

[1] And yet, stating things like this is not correct. The boundary may overlap somewhere with characteristics, or they may share a tangent space somewhere etc. This is just an informal paradigm for thinking, and you should check case by case.

The only delicate point here, is that often the system to impose boundary conditions is differential. However, in the above examples, general solutions will depend on two real arbitrary functions, say f and g. Imposing the boundary conditions gives a system

$$\begin{cases} a\, f(\xi) + b\, g(\xi) = h_1(\xi) \\ c\, f'(\xi) + d\, g'(\xi) = h_2(\xi) \end{cases}$$

for some known constants a, b, c, d and functions h_1 and h_2 (which depend on the problem). At this point, we just derive the first equation in ξ and reduce the problem to a usual linear system for each ξ. (This trick would not work if a, b, c, d would depend on ξ).