Ex. 1. Consider the problem

$$\begin{cases} \partial_{tt} u = (\partial_{xx} u)^2 \\ u(x,0) = x^2 \\ \partial_t u(x,0) = x \end{cases}$$

Do you expect it to feature an analytic solution? Find it, or prove that such a solution does not exist.

Ex. 2. Consider the problem

$$\begin{cases} \partial_{tt} u = (\partial_{xx} u)^2 \\ u(x,0) = x^2 \\ \partial_t u(x,0) = 1 + x^2 \end{cases}$$

Do you expect it to feature an analytic solution? Find it, or prove that such a solution does not exist.

Ex. 3. Consider the problem

$$\begin{cases} \partial_t u = \partial_{xx} u \\ u(0,x) = \frac{1}{1+x^2} \end{cases}$$

Do you expect it to feature an analytic solution? Find it, or prove that such a solution does not exist.

Ex. 4. Consider the problem, for $|x| < 1$

$$\begin{cases} \partial_t u = \partial_{xx} u \\ u(0,x) = \frac{1}{1-x} \end{cases}$$

Does it admit analytic solutions?

Ex. 5. The inviscid Burgers equation

$$\begin{cases} \partial_t u + u\,\partial_x u = 0 \\ u(0,x) = u_0(x) \end{cases}$$

appears in several models, for instance in hydrodynamics and in large-scale limits of electron propagation.

(a) Assuming that u_0 is analytic, do you expect it to have an analytic solution for all (t,x) (compare with the linear case $\partial_t u + c\partial_x u = 0$, where c is constant).

(b) Find an explicit solution for $u_0(x) = x$ and $u_0(x) = -x$, locally in t. What is the *blow up* time?

(c) Find an explicit solution for $u_0(x) = x^2$, locally in t. What is the *blow up* time?

(d) Give an example with u_0 bounded (on $\mathbb{R}$) for which there is no global analytic solution in the space $t \geq 0$. Give a non-trivial example for which there is such an analytic solution.

Sol 1. We are looking for solutions in the form

$$u(t,x) = \sum_{m,n} a_{m,n} \frac{t^m}{m!} \frac{x^n}{n!} \tag{1}$$

If we plug the power series into the equation, we get a recurrence equation for the coefficients $a_{m,n}$. However, this would a way to look for a generic solution. Here it is faster to reason as follows

$$u(0,x) = x^2, \quad (\partial_x u)(0,x) = 2x,, \quad (\partial_{xx} u)(0,x) = 2, \quad (\partial_x^n u)(0,x) = 0, \quad n \geq 3$$
$$(\partial_t u)(0,x) = x, \quad (\partial_{tx} u)(0,x) = 1, \quad (\partial_t \partial_x^n u)(0,x) = 0, \quad n \geq 2$$

(we did not use the equation up to here). However, using the equation

$$(\partial_{tt} u)(0,x) = \quad (\partial_{xx} u)^2(0,x) = 4, \quad (\partial_{tt} \partial_n^n u)(0,x) = 0, \quad n \geq 1$$
$$(\partial_t^n u)(0,x) = 0, \quad n \geq 3.$$

etc. We easily guess, calculating the previous functions at $x = 0$, $u(t,x) = x^2 + xt + 2t^2$, which is indeed a solution.

Sol 3. Let us the form (1) here. Then the equation and the initial conditions yield

$$\begin{cases} a_{m+1,n} = a_{m,n+2} \\ a_{0,2n+1} = 0 \\ a_{0,2n} = (-1)^n\,(2n)! \end{cases}$$

This implies $a_{m,2n+1} = 0$ and $a_{m,2n} = a_{0,2(m+n)}$. It is easily seen that radius of convergence in (1) is 0.

Sol 4. We. can reason as above, or more simply notice that, if a solution exists

$$(\partial_t^n u)(0,x) = \frac{(2n)!}{(1-x)^{2n+1}}$$

which would give $u(t,x) = \sum_n \frac{(2n)!}{n!} \frac{t^n}{(1-x)^{2n+1}}$, again with vanishing radius of convergence.

Sol 5. There is no reason for having a global analytic solution.

(a) The solution has $dx/dt = u$ as characteristic.

(b) It should be clear from (a) that $u(t,\cdot)$ is still linear in x. Looking thus for u in the form $u(t,x) = f(t)x$ we gather $f' + f^2 = 0$, with $f(0) = \pm 1$. Thus $u(t,x) = x/(t \pm 1)$. It blows up at time ∓ 1.

(c) Let us look more in general to the problem with analytic initial condition $u_0(x)$, and let us reason a bit informally to get an ansatz for the solution; we will check a posteriori the answer. We know that there is a smooth solution, at least for a short time $|t| < t_0$. The characteristic equation here is

$$\tfrac{dx}{dt} = u$$

However, as long as characteristics do not intersect (for a short enough time), u is constant along the characteristic. So the solution is $x = u\,t + c$, where c is nothing but the point where the characteristic curve is at $t = 0$, so that u in this formula is nothing but $u_0(c)$, so that $x = u_0(c)\,t + c$. In other words, using the substitution $\xi = x - u(t,x)\,t$ we get $u(t,x) = F(\xi)$, from which $F = u_0$; this is nothing but an implicit equation for u:

$$u = u_0(x - u\,t) \tag{2}$$

What we actually checked, is that a smooth solution $u(t,x)$ of the equation will satisfy (2) (as long as characteristics do not intersect). Conversely, given a solution to (2), we have by derivation

$$\partial_t u + u\partial_x u = u_0'(x - u\,t)\,(\partial_t u + u\partial_x u)$$

which yields that u is a solution (whenever $u_0'(x - ut) \neq 1$).

The point here is that it is not true that (2) admits a solution for all t. Certainly, if u_0 is Lipschitz, say with Lipschitz constant L, then by contraction we get a solution for $t < 1/L$. (On the other hand, it is easy to check that for such a t characteristics do not intersect.)

In the specific case $u_0(x) = x^2$, we get

$$u = (x - u\,t)^2 \quad \Rightarrow u(t,x) = \frac{1 + 2tx \pm \sqrt{1 + 4tx}}{2t^2} \tag{3}$$

Notice however that the solution with the $+$ sign explodes as $t \to 0$ (we just said that the solution with initial condition x^2 satisfies (3), not the opposite); so that $u(t,x) = \frac{1+2tx-\sqrt{1+4tx}}{2t^2}$. One can check that that $u(0,x) = x^2$ and

(d) Take $u_0(x) = \pm\arctan(x)$.