Ex. 1. Solve the Sturm-Liouville problem in the unknown $y = y(x)$ for some fixed $k \in \mathbb{R}$

$$\begin{cases} y'' - \lambda y = 0 \\ y'(0) = y'(\ell) = 0 \end{cases}$$

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What happens for $k \to 0$ and $k \to \infty$?

Ex. 2. Solve the Sturm-Liouville problem in the unknown $u = u(x, y)$ in the domain $\Omega = [0, L_1] \times [0, L_2]$

$$\begin{cases} \Delta u - \lambda\, u = 0 & \text{in the interior of } \Omega. \\ \partial_{\hat{n}} u = 0 & \text{on } \partial\Omega. \end{cases}$$

where $\partial_{\hat{n}}$ denotes the derivative in the direction normal to the boundary.

Ex. 3. Let $\alpha \geq 0$, $r < R$ and $a, b \in \mathbb{R}$. Let $\Omega := \{x \in \mathbb{R}^d \,:\, r < |x| < R\}$. Solve the Sturm-Liouville problem

$$\begin{cases} -(\Delta u)(x) + \alpha\, |x|^2\, u(x) = E\, u(x) & \text{in } \Omega \\ u(x) = a & \text{if } |x| = r \\ u(x) = b & \text{if } |x| = R \end{cases}$$

[Hint: one can prove by induction, that the laplacian operator in polar coordinates reads

$$\Delta f = \tfrac{1}{\varrho^{d-1}} \partial_\varrho (\varrho^{d-1} f) + \Theta^d f$$

where Θ^d is a second-order differential operator in the angular variables. Suitably define $v = \phi(\varrho) u$ to reduce to the case $\alpha = 0$.]