Seminar – 15/04/2019

Ex. 1 For $\ell > 0$, solve the problem in the unknown $u(t,x)$

$$\begin{cases} \partial_t u = 4\partial_{xx} u \\ u(t,0) = u(t,\ell) = 0 \\ u(0,x) = x\mathbf{1}_{[0,\ell/2]}(x) + (\ell - x)\mathbf{1}_{[\ell/2,\ell]}(x) \end{cases}$$

Ex. 2 For $\ell > 0$, solve the problem

$$\begin{cases} \partial_t u + u = \partial_{xx} u \\ u(t,0) = u(t,\ell) = 0 \\ u(0,x) = \sin(32\pi x/\ell) + \sin(23\pi x/\ell) \end{cases}$$

Ex. 3 For $v(x)$ a continuous function on $[0,1]$, consider the problem

$$\begin{cases} \partial_{tt} u = \partial_{xx} u \\ u(t,0) = u(t,1) = 0 \\ u(0,x) = 0 \\ (\partial_t u)(0,x) = v(x) \end{cases}$$

Calculate the quantity

$$\int_0^1 (\partial_t u)^2(t,x) + (\partial_x u)^2(t,x)\, dx$$

Sol 1. We look for a solution $u(t,x) = \sum_n a_n(t)\, b_n(x)$, where actually each addend $a_n\, b_n$ is a solution of the PDE (but not a solution of the problem at the boundary $t = 0$). We get

$$a'\, b = 4\, a\, b'', \qquad b(0) = b(\ell) = 0$$

so that for some $\lambda \in \mathbb{R}$

$$\begin{aligned} a' &= 4\,\lambda\, a \\ b'' &= \lambda b, \qquad b(0) = b(\ell) = 0 \end{aligned}$$

This yields $b_n(x) = \alpha_n\, \sin(n\,\pi\, x/\ell)$, $\lambda_n = -\pi^2 n^2/\ell^2$, $a_n(t) = e^{-4\,\lambda_n t}$, so that

$$u(t,x) = \sum_n \alpha_n\, \sin(n\,\pi\, x/\ell)\, e^{-4\,\lambda_n t}$$

Now, since the $\sin(n\,\pi\, x/\ell)$ for an orthogonal eigenbase in L_2

$$\alpha_n \left(\int_0^\ell \sin(n\,\pi\, x/\ell)^2\, dx \right) = \int_0^\ell u(0,x)\, \sin(n\,\pi\, x/\ell)\, dx$$

Sol 2. Reasoning as above, we gather

$$\frac{a'}{a} = \frac{b'' - b}{b} = \lambda$$

for some constant $\lambda \in \mathbb{R}$. Thus we must solve the Sturm-Liouville problem $b'' = (\lambda + 1)b$ which is just the same as above, up to the change $\lambda \to \lambda + 1$.

Sol 3. In this case, reasoning as above (or by Fourier series) we get that

$$u(t,x) = \sum_n c_n \, \sin(n\,\pi\,t)\,\sin(n\,\pi\,x)$$

where the c_n are determined by

$$v(x) = (\partial_t u)(0,x) = \sum_n \pi\,n\,c_n\,\sin(n\,\pi\,x)$$

Moreover

$$(\partial_t u)(t,x) = \sum_n n\,\pi\,c_n\,\cos(n\,\pi\,t)\,\sin(n\,\pi\,x)$$

$$(\partial_x u)(t,x) = \sum_n n\,\pi\,c_n\,\sin(n\,\pi\,t)\,\cos(n\,\pi\,x)$$

and therefore $\int (\partial_t u)^2(t,x)\,dx = \frac{1}{2}\sum_n n^2\pi^2\,c_n^2\,\cos(n\,\pi\,t)^2$, $\int (\partial_x u)^2(t,x)\,dx = \frac{1}{2}\sum_n n^2\pi^2\,c_n^2\,\sin(n\,\pi\,t)^2$. Thus their sums coincides with $\|v\|_{L_2}^2$.