Seminar – 22+29/04/2019

Ex. 1 Write the solution for the problem on $[0,\infty)\times\mathbb{R}^d$

$$\begin{aligned}&\partial_t u - \Delta u + cu = f \qquad t>0\\ &u = g \qquad t=0\end{aligned}$$

Ex. 2 Find all the harmonic functions $u(x,y)$ in $\mathbb{R}^2$ such that $(\partial_x u) = (\partial_y u)$.

Ex. 3 Let Ω be a bounded open set in $\mathbb{R}^d$, and let u_n be a sequence of harmonic functions in Ω. Suppose that u_n converges uniformly to some u. What can we say about u?

Ex. 4 Let u be a nonnegative harmonic function in the bi-dimensional unit disk. We want to prove that

$$\sup_{x\,:\,|x|\le 1/2} u(x) \le 9 \inf_{x\,:\,|x|\le 1/2} u(x)$$

(a) Recall that the Poisson kernel for a ball of radius r in $\mathbb{R}^d$ is given by $K(x,\xi) = \frac{r^2-|x|^2}{\omega_d r\,|\xi-x|^d}$. Prove that in our case

$$\tfrac{1-|x|}{2\pi\,(1+|x|)} \le K(x,\xi) \le \tfrac{1+|x|}{2\pi\,(1-|x|)}$$

(b) Deduce

$$\tfrac{1-|x|}{1+|x|}u(0) \le u(x) \le \tfrac{1+|x|}{1-|x|}u(0)$$

(c) From (b), obtain the thesis.

Ex. 5 A smooth nonnegative function u satisfies over a smooth bounded domain Ω

$$\begin{aligned}&\Delta u \ge u \qquad x\in\Omega\\ &\partial_{\hat n} u = 0 \qquad x\in\partial\Omega\end{aligned}$$

Find u.

Ex. 6 Suppose that $u_0\colon\mathbb{R}\to[0,\infty)$, $u_0\in L_2(\mathbb{R})\cap L_1(\mathbb{R})$, is such that $u_0(0) = u_0(1) = 0$. Consider the following problems

$$\begin{cases}\partial_t u = \Delta u & t\ge 0,\ x\in\mathbb{R}\\ u(t=0,x) = u_0(x)\end{cases}$$

$$\begin{cases}\partial_t v = \Delta v & t\ge 0,\ x\in[0,1]\\ v(t=0,x) = u_0(x)\\ v(t,x=0) = v(t,x=1) = 0\end{cases}$$

$$\begin{cases}\partial_t w = \Delta w & t\ge 0,\ x\in S^1\\ w(t=0,x) = u_0(x)\end{cases}$$

What inequalities hold between u, v and w? Find the limit of the solutions when $t\to\infty$. [*Beware:* Why v is *not* a solution to the third problem?]

Ex. 7 Find the solution to the following problems, where the functions $u_0(x)$ and $f(x,t)$ are smooth and compactly supported, $t \geq 0$ and $x \geq 0$

$$\begin{cases} \partial_t u = a^2 \Delta u + f(t,x) \\ u(t=0,x) = u_0(x) \\ u(t,x=0) = 0 \end{cases}$$

$$\begin{cases} \partial_t u = \Delta u + f(t,x) \\ u(t=0,x) = u_0(x) \\ (\partial_x u)(t=0,x) = 0 \end{cases}$$

$$\begin{cases} \partial_t u = \Delta u + f(t,x) \\ u(t=0,x) = u_0(x) \\ (\partial_x u)(t=0,x) = \alpha u(t=0,x) \end{cases}$$

What happens in the last problem when $\alpha \to 0$ and $\alpha \to +\infty$?

Ex. 8 Consider the heat equation on the real line

$$\begin{cases} \partial_t u = \Delta u + f(t,x) \\ u(t=0,x) = u_0(x) \end{cases}$$

We look for a solution $u(t,x)$ such that $u(t,x) = u_0(x)\, e^t$.

(a) Try to find a solution by taking the Fourier transform of the requirement $u(t,x) = u_0\, e^t$.
(b) Can u_0 be bounded?
(c) Find the solution in the case $u_0(x) = e^x$.

Ex. 9 Consider the Schroedinger problem

$$\begin{cases} i\hbar \partial_t u = -\hbar^2 \Delta u + V(x) u \end{cases}$$

where $x \in [-a,a]$, u satisfies the condition $(\partial_x u)(x) = 0$ for $x = \pm a$, and the potential V is given by (here $0 < \ell < a$)

$$V(x) = \begin{cases} 0 & \text{if } \ell < |x| \leq a \\ c/\ell & \text{if } \ell \geq |x| \end{cases}$$

(a) Solve the associated Sturm-Liouville problem.
(b) Give an example of a solution for which $\psi(0,x) = 0$ for $x \geq -\ell$, but ψ is not identically 0 on $[-a,a]$; and such that $|\psi(t,x)|^2 > 0$ for $t > 0$.
(c) What happens when $\ell \to \infty$?