The constructible reals can be (almost) anything

Leo Harrington May 74

Theorem There are models of ZFC in which the set of constructible reals is, respectively, exactly the following set of reals

Δ^1_3, Δ^1_4, ..., Δ^1_ω = projective,

Δ^m_n, $1 \le n \le \omega$, $2 \le m \le \omega$.

We will give the proof for Δ^1_3 and then indicate a few of the modifications needed for the full result.

We will use the notation of [1]. Our model will be a substructure of a model constructed in that paper.

Working inside L, Jensen and Solovay in [1] construct a sequence of posets Q_i, $1 \le i < \omega$, and a Σ^1_3 predicate $\varphi(j)$ such that:

the Q_i's are homogeneous,

if Q_0 is the poset which collapses $\aleph_1^L$ to ω, and if $\bigoplus_{i\in\omega} G_i$ is $\bigoplus_{i\in\omega} Q_i$ generic over L, then for $j \geq 1$

$$L[G_0, G_j] \models \varphi(j), \text{ but}$$

$$L[\textstyle\bigoplus_{j\neq i} G_i] \models \neg\varphi(j).$$

Q_0 can be defined in various equivalent ways. We define it by: $\langle m, f\rangle \in Q_0$ iff $m \in \omega$, f a partial function from m to $\aleph_1$; $\langle m, f\rangle \leq \langle m', f'\rangle$ iff $m \leq m'$, $f'\restriction m = f$.

G_0 can be identified with a partial map from ω onto $\aleph_1^L$.

Let h_α be the αth constructible real. Code triples of integers as one positive integer and let $Z(n,\alpha) = \{\langle n, k, \ell\rangle : h_\alpha(k) = \ell\}$.

$$\text{Let } Z = \bigcup \{Z(n,\alpha) : n \in \text{dom of } G_0 \text{ and } G_0(n) = \alpha\}$$

$$\text{Let } N = L[G_0 \oplus (\textstyle\bigoplus_{i\in Z} G_i)].$$

$N \models \varphi(j)$ iff $j \in Z$, so each h_α is Δ^1_3 in N. For the converse:

Let $\mathcal{P}(n,\alpha) = \bigoplus_{i \in Z(n,\alpha)} Q_i$

Let $\mathcal{R}$ be the poset: $r \in \mathcal{R}$ iff $r = \langle m, f, g \rangle$, $\langle m, f \rangle \in Q_0$, domain of g = dom of f, for $n \in \text{dom } f$ $g(n) \in \mathcal{P}(n, f(n))$.

$G_0 \oplus (\bigoplus_{i \in Z} G_i)$ is $\mathcal{R}$-generic over L.

Since the Q_i's are homogeneous, for φ a parameterless sentence of set theory, $\langle m, f, g \rangle \Vdash \varphi \Rightarrow \langle m, f, 0 \rangle \Vdash \varphi$

For $r = \langle m, f, 0 \rangle$, $r' = \langle m', f', 0 \rangle$, let $r \leq r'$ mean: $f \subseteq f'$.

If $r \leq r'$ then $r' \Vdash$ "in N there is a generic filter on $\mathcal{R}$ which extends r".

So if θ is a Σ^1_3 sentence, and if $r \leq r'$, then $r \Vdash \theta \Rightarrow r' \Vdash \theta$.

Let $\theta_0(n)$, $\theta_1(n)$ be Σ^1_3 formulas. Let $r = \langle m, f, 0 \rangle \in \mathcal{R}$, and assume $r \Vdash$ "$\forall n (\theta_0(n) \vee \theta_1(n))$".

Suppose we have $n \ni r \Vdash\!\!\!/\ \theta_0(n)$ and $r \Vdash\!\!\!/\ \theta_1(n)$. (If there is no such n, then the Δ^1_3 real θ_0, θ_1 define - if they do define a real - must be constructible.)

Pick $r' = \langle m', f', 0\rangle \geq r \ni r' \Vdash \theta_0(n)$. Let $r^0 = \langle m', f, 0\rangle$. Since $r^0 \leq r$ we can find $\langle m'', f'', 0\rangle = r'' \geq r^0 \ni r'' \Vdash \theta_1(n)$. Let $r''' = \langle m'', f' \cup f'', 0\rangle$. Then $r''' \geq r$ and $r', r'' \leq r'''$, so $r''' \Vdash$ "$\theta_0(n) \wedge \theta_1(n)$". $\square$

The only real obstacle to generalizing this argument is that it used Shoenfield Absoluteness (Σ^1_3 goes up). This is not an insuperable obstacle since Shoen. Abs. was only used between N and N with a few of the G_i's deleted.

To get Δ^1_{n+3}: by examining [1] we see that $Q_i = Q(X_i)$ where $X_i \subseteq \aleph_2$ and Q is a canonical procedure for constructing posets from subsets of $\aleph_2$.

Let X_i', $i < \aleph_2$ be a sequence of subsets of $\aleph_2$ which is Δ_{n+1} definable and $\underset{\sim}{\Sigma}_n$ generic over $L_{\aleph_2}$. (By generic here we mean wrt the $\aleph_1$-closed poset which adds subsets of $\aleph_2$ — i.e. a Cohen subset of $\aleph_2$.)

Let $Q_i' = Q(X_i')$. Define R' as above and let N be generic over L via the poset $R' \oplus (\bigoplus_{\omega \leq i < \aleph_2} Q_i')$.

The Q_i''s, $i \geq \omega$, obscure things enough so that a submodel of N, gotten by deleting a few Q_i''s, $i < \omega$, from the above poset, will be a Σ^1_{n+3} correct substructure of N.

To get Δ^1_ω, just diagonalize the above arguments. This involves finding $X_{i,n}$, $i < \aleph_2$, $n < \omega$, $X_i \subseteq \aleph_2$, $\ni$

For each $m < \omega$. $\langle X_{i,m} \rangle_{i < \aleph_2}$ is

Δ_{m+1} definable over $L_{\aleph_2}$, and $\langle X_{i,n} \rangle_{i<\aleph_2, n \geq m}$ is $\underset{\sim}{\Sigma}_m$ generic over $L_{\aleph_2}[\langle X_{i,n} \rangle_{i<\aleph_2, n<m}]$. Such a sequence can be found.

The Δ^m_n's follow by a straightforward generalization, except for the Δ^m_1's. Here a different argument is needed – use the forcing which adds on a κ-closed unbounded subset to a stationary subset of κ^+ (where every ordinal in the stationary set has cofinality κ). This takes place over $L[G_0]$, where $\kappa = \aleph_{m-2}$.

[1] Jensen, Solovay, Some applications of almost disjoint sets, in: Math. Logic and Found. of Set Theory, Y. Bar-Hillel, ed.

Addendum: Models where Separation principles fail.

Kechris has observed that in the above model (for Δ^1_n), $PWO(\Sigma^1_n)$ and $PWO(\Pi^1_n)$ both fail (since $({}^{\omega}\omega)^L \in \Sigma^1_2$).

By deceding to §4 of [1] we can produce, for $n \geq 3$, models in which $Sep(\underset{\sim}{\Sigma}^1_n)$ and $Sep(\underset{\sim}{\Pi}^1_n)$ both fail. In fact we get models of ZFC in which:

there are two disjoint lightface Σ^1_n sets of reals which cannot be separated by a $\underset{\sim}{\Delta}^1_n$ set of reals, and also there are 2 disjoint lightface Π^1_n sets of reals which cannot be separated by a $\underset{\sim}{\Delta}^1_n$ set of reals.

This is done as follows. Start with L. Break $({}^{\omega}\omega)^L$ into 2 parts. Generically split (via the usual ω-closed posets) each part into 3 pieces. Then make (using §4 of [1] plus the modifications sketched above) the 1st 2 pieces of the 1st part Σ^1_n, and make the 1st 2 pieces of the 2nd part Π^1_n. It is possible to show that any $\underset{\sim}{\Delta}^1_n$ subset of $({}^\omega\omega)^L$ in this model is constructible from a real; and also that all 6 of the above pieces are generic over each real in this model.

By diagonalizing, as was done above for Δ^1_ω, one can show that there is a model in which the above failures of separation occur for all $n \geq 3$ simultaneously. In fact (we believe) there is a model of ZFC in which separation fails for all of the following at once:

$$\Sigma^1_n, \Pi^1_n, 3 \leq n < \omega, \quad \Sigma^m_n, \Pi^m_n, \quad 1 \leq n < \omega, 2 \leq m < \omega$$

Separation without Reduction

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Theorem There is a model of ZFC in which $\mathrm{Sep}(\utilde{\Pi}^1_3, \utilde{\Delta}^1_3)$ holds for sets of reals, but $\mathrm{Red}(\Sigma^1_3, \utilde{\Sigma}^1_3)$ fails.

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Let $\mathcal{C}$ be the cohen-real poset.

Let $f_{\xi,i}$, $\xi < \aleph_1$, $i \in \omega$ be the sequence of reals in L as constructed by Jensen-Solovay in [2].

Let Q be the poset which embodies the following process: 1st add onto L g, a $\mathcal{C}$-generic real, then add onto $L[g]$ (in the usual way) a real a s.t. $S(f_{\xi,i}) \cap a$ is finite for all ξ iff $g(i) = 0$.

Let $T(g, a)$ be the Π^1_2 predicate s.t. $T(g,a) \equiv \forall \xi < \aleph_1 \, \forall i \in \omega \, (S(f_{\xi,i}) \cap a$ is finite iff $g(i) = 0)$.

Let $\mathcal{P} = \overset{Q^{\omega}}{\sum_{\omega}} Q$. Notice that $\mathcal{P}$ is homogeneous.

<u>Claim 1</u> If g is $\mathcal{C}$-generic over L, then
$$L[g]^{\mathcal{D}} \models \neg \exists a\, T(g, a).$$
[The proof of this will be given later.]

Our model will be $\mathcal{B}$-generic over L where $\mathcal{B}$ is described below. [In the following, we will systematically identify a poset with its complete boolean algebra.]

<u>Def</u> For each $\alpha \leq \aleph_1$ we define a cba $\mathcal{B}_\alpha$, and at the same time we define the notion of an <u>autonomous</u> complete-subalg of $\mathcal{B}_\alpha$.

$\mathcal{B}_0 = 2$. $\mathcal{B}'$ is an autonomous c-subalg of $\mathcal{B}_0$ iff $\mathcal{B}' = 2$.

$\mathcal{B}_\lambda = \bigcup_{\alpha < \lambda} \mathcal{B}_\alpha$. $\mathcal{B}'$ is an aut c-subalg of $\mathcal{B}_\lambda$ iff $\mathcal{B}' \cap \mathcal{B}_\alpha$ is an aut c-subalg of $\mathcal{B}_\alpha$, for all $\alpha < \lambda$.

Given $\mathcal{B}_\alpha$: Hop into $L^{\mathcal{B}_\alpha}$. Let $\mathcal{U}_\alpha = \{\tau \mid \tau$ is a countably generated c-subalg of $\mathcal{D}\}$. So $\Sigma\, \mathcal{U}_\alpha$ is a poset in $L^{\mathcal{B}_\alpha}$. Let
$$\mathcal{B}_{\alpha+1} = \mathcal{B}_\alpha * (\Sigma\, \mathcal{U}_\alpha).$$

$\mathcal{B}'$ is an aut c-subalg of $\mathcal{B}_{\alpha+1}$ iff:

$\mathcal{B}'_\alpha =_{\text{def}} \mathcal{B}' \cap \mathcal{B}_\alpha$ is an aut c-subalg of $\mathcal{B}_\alpha$, and there is $\mathcal{U}'_\alpha$ in $L^{\mathcal{B}'_\alpha}$ s.t., with truth value $\mathbb{1}$, $L^{\mathcal{B}'_\alpha} \models \forall \mathcal{T} \in \mathcal{U}'_\alpha$ ($\mathcal{T}$ is a countably generated complete subalg of $\mathcal{L}\vartheta$), and $\mathcal{B}' = \mathcal{B}'_\alpha * (\Sigma\, \mathcal{U}'_\alpha)$, where this is viewed as a c-subalg of $\mathcal{B}_{\alpha+1}$ in the obvious way.

Let $\mathcal{B} = \mathcal{B}_{\omega_1}$.

<u>Claim 2</u> If $\mathcal{B}'' \subseteq \mathcal{B}$ is a countably generated c-subalg, then there is a countably gen aut c-subalg $\mathcal{B}'$ of $\mathcal{B}$ s.t. $\mathcal{B}'' \subseteq \mathcal{B}'$.

<u>prf</u>

If $\mathcal{B}'_i$, $i \in I$, is an aut c-subalg of $\mathcal{B}$, then $\bigcup_{i \in I} \mathcal{B}'_i$ generates an aut c-subalg of $\mathcal{B}$. Using this fact, the claim is relatively easy. ∇

<u>Claim 3</u> If $\mathcal{B}'$ is an aut c-subalg of $\mathcal{B}$, then $\mathcal{B}$ can be embedded, over $\mathcal{B}'$, into $\mathcal{B}' \oplus (\sum_{\omega_1} Q)$.

<u>proof</u>

Let $\mathcal{B}'_\alpha = \mathcal{B}' \cap \mathcal{B}_\alpha$. We will embed $\mathcal{B}_\alpha$, over $\mathcal{B}'_\alpha$, into $\mathcal{B}'_\alpha \oplus (\sum_{\omega_1 \cdot \alpha} Q)$, so that all of these embeddings commute with the inclusion maps.

There is no problem passing through limits.

So: Given $\mathcal{B}_\alpha \underset{\mathcal{B}'_\alpha}{\hookrightarrow} \mathcal{B}'_\alpha \oplus (\sum_{\geq_1 \cdot \alpha} Q)$:

We have
$$\mathcal{B}_{\alpha+1} = \mathcal{B}_\alpha * (\sum \mathcal{U}_\alpha)$$
$$\mathcal{B}'_{\alpha+1} = \mathcal{B}'_\alpha * (\sum \mathcal{U}'_\alpha)$$

Hop into $L^{\mathcal{B}_\alpha}$. We may view $\mathcal{U}'_\alpha$ as a subset of $\mathcal{U}_\alpha$. $T \in (\mathcal{U}_\alpha \sim \mathcal{U}'_\alpha)$ is a c-subalg of $\mathcal{D} = \sum_\omega Q$. So in $L^{\mathcal{B}_\alpha}$, $\sum(\mathcal{U}_\alpha \sim \mathcal{U}'_\alpha)$ is embeddable into $\sum_{\geq_1} Q$. So we have embeddings as follows:

$$\mathcal{B}_{\alpha+1} = \mathcal{B}_\alpha * (\sum(\mathcal{U}_\alpha \sim \mathcal{U}'_\alpha) \oplus \sum \mathcal{U}'_\alpha) \hookrightarrow$$

$$\mathcal{B}_\alpha * ((\sum_{\geq_1} Q) \oplus \sum \mathcal{U}'_\alpha) \underset{\mathcal{B}'_\alpha}{\hookrightarrow}$$

$$(\mathcal{B}'_\alpha \oplus \sum_{\geq_1 \cdot \alpha} Q) * ((\sum_{\geq_1} Q) \oplus \sum \mathcal{U}'_\alpha) \hookrightarrow$$

$$(\mathcal{B}'_\alpha * \sum \mathcal{U}'_\alpha) \oplus (\sum_{\geq_1 \cdot (\alpha+1)} Q) = \mathcal{B}'_{\alpha+1} \oplus (\sum_{\geq_1 \cdot (\alpha+1)} Q). \square$$

<u>Claim 4</u> If $\mathcal{B}'$ is a countably generated aut c-subalg of $\mathcal{B}$, and if (working in $L^{\mathcal{B}'}$) $\mathcal{B}''$ is a c-subalg of $\mathcal{D}$, then: $\mathcal{B}' * \mathcal{B}''$ can be embedded, over $\mathcal{B}'$, into $\mathcal{B}$ so that its image is an aut c-subalg of $\mathcal{B}$.

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Since $\mathcal{B}'$ is countably generated, $\mathcal{B}' \subseteq \mathcal{B}_\alpha$ for some α. Clearly $\mathcal{B}''$ can be viewed as a member of $\mathcal{U}_\alpha$. Thus $\mathcal{B}' * \mathcal{B}''$ is embedded as an aut c-subalg of $\mathcal{B}_{\alpha+1}$. $\square$

Now, let M be $\mathcal{B}$-generic over L. By claim 3, $\mathcal{B}$ is a c-subalg of $\sum_{\aleph_1} \mathbb{Q}$. Thus if $b \in M$ is a real, then b is generic over L via a c-subalg of $\sum_\omega \mathbb{Q} = \mathcal{D}$.

Def For $b \in M$, b is autonomous if the c-subalg of $\mathcal{B}$ determined by b (in the usual way) is an aut c-subalg of $\mathcal{B}$.

By claim 2, if $a \in M$ is a real, then there is an aut real $b \in M$ s.t. $a \in L[b]$.

Claim 5 If $b \in M$ is a real, then for all Σ^1_3 formulas φ

$$L[b]^{\mathcal{D}} \models \varphi(b) \implies M \models \varphi(b);$$

Furthermore, if b is an aut real, then

$$L[b]^{\mathcal{D}} \models \varphi(b) \iff M \models \varphi(b).$$

<u>prf</u>:

Assume $L[b]^{\mathcal{S}\ell} \models \varphi(b)$. Find an aut real c s.t. $b \in L[c]$. By claim 4, there is $G \in M$ s.t. G is $\mathcal{S}\ell$-generic over $L[c]$. Hence G is $\mathcal{S}\ell$-generic over $L[b]$. So $L[b,G] \models \varphi(b)$. Since Σ^1_3 goes up, $M \models \varphi(b)$.

Assume b is aut, and assume $M \models \varphi(b)$. By claim 3 we can find (somewhere) G s.t. G is $(\sum_{\tilde{<}_1} Q)$-generic over $L[b]$, and s.t. $M \subseteq L[b,G]$. So $L[b,G] \models \varphi(b)$. Thus $L[b]^{\mathcal{S}\ell} \models \varphi(b)$. $\square$

<u>Claim 6</u> There is a recursive correspondence $\varphi \longleftrightarrow \varphi^*$ between Σ^1_3 (& hence Π^1_3) formulas s.t. for all reals b in M,

$$L[b]^{\mathcal{S}\ell} \models \varphi(b) \quad \text{iff} \quad L[b] \models \varphi^*(b).$$

[The proof will be given later.]

<u>To show</u> $Red(\Sigma^1_3, \utilde{\Sigma}^1_3)$ <u>fails</u> <u>in</u> <u>M</u>:

In M, let $A = \{g \mid M \models \exists a\, T(g,a)\}$. So A is Σ^1_3. Let B = universal Σ^1_3 set of reals.

Consider $\omega^\omega \times A$, $B \times \omega^\omega$

[This trick is based on an idea of Sami's.]

Suppose we can find a $\underset{\sim}{\Sigma}^1_3$ reduction of this pair of Σ^1_3 sets, i.e., suppose we have A', B' both $\underset{\sim}{\Sigma}^1_3$ s.t.

$A' \subseteq \omega^\omega \times A$, $B' \subseteq B \times \omega^\omega$, $A' \cap B' = \emptyset$,

$A' \cup B' = (\omega^\omega \times A) \cup (B \times \omega^\omega)$.

Let p be an aut real s.t. A', B' are $\Sigma^1_3(p)$.

By claim 4, there ~~is~~ are reals g, a s.t. $\langle g, a \rangle$ is Q-generic over $L[p]$. Hence g is $\mathcal{C}$-generic over $L[p]$ and $g \in A$.

So we have, $d \notin B \Rightarrow$ $\{y \mid \langle d, y \rangle \in A'\} = A \Rightarrow \exists y (\langle d, y \rangle \in A'$ & y is $\mathcal{C}$-generic over $L[p])$. This last assertion is $\Sigma^1_3(p, d)$. Since $\{\langle p, e \rangle \mid \langle p, e \rangle \notin B$ & $e \in \omega\}$ is not $\Sigma^1_3(p)$, we have that there is an integer e s.t. $\exists y (\langle\langle p, e \rangle, y \rangle \in A'$ & y is $\mathcal{C}$-generic over $L[p])$, and s.t. $\langle p, e \rangle \in B$.

Let $A'' = \{y \mid \langle\langle p, e \rangle, y \rangle \in A'\}$. A'' is $\Delta^1_3(p)$ (since $y \notin A'' \Longleftrightarrow \langle\langle p, e \rangle, y \rangle \in B'$).

Let φ be a Π^1_3 formula s.t.

$y \in A''$ iff $M \models \varphi(p, y)$.

Notice that $A'' \subseteq A$.

By choice of e, there is $g \in A''$ s.t. g is $\mathcal{C}$-generic over $L[p]$.

So $M \models \varphi(p,g)$. Thus by claim 5, $L[p,g]^{\mathcal{D}} \models \varphi(p,g)$. So by claim 6 we have $L[p,g] \models \varphi^*(p,g)$. Find $\bar{g} \in \mathcal{C}$, $\bar{g} \leq g$ s.t. $\bar{g} \Vdash$ "$L[p,\dot{g}] \models \varphi^*(p,\dot{g})$".

By claim 4, there is g' in M s.t. g' is $\mathcal{C}$-generic over $L[p]$, $\bar{g} \leq g'$, and s.t. $\langle p, g' \rangle$ is an aut real.

Thus $L[p,g'] \models \varphi^*(p,g')$.

So by claim 6 and claim 5, $M \models \varphi(p,g')$. Thus $g' \in A$.

So, by claim 5, $L[p,g']^{\mathcal{D}} \models \exists a\, T(g',a)$.

But since p is generic over L via a subalg of $\mathcal{D}$, and since g' is $\mathcal{C}$-generic over $L[p]$, ~~we have~~ and since $\mathcal{D} \oplus \mathcal{D} = \mathcal{D}$, $\cancel{L[p,g']^{\mathcal{D}}}$ we have:

$$L[g']^{\mathcal{D}} \models \exists a\, T(g',a).$$

But g' is $\mathcal{C}$-generic over L. So this contradicts claim 1. $\square$

<u>To show $Sep(\underset{\sim}{\Pi}^1_3, \underset{\sim}{\Delta}^1_3)$ holds in M:</u>

Let φ_0, φ_1 be $\underset{\sim}{\Pi}^1_3$ formulas. We can find an aut real p s.t. φ_0, φ_1 are $\Pi^1_3(p)$.

Assume $M \models \forall x(\varphi_0(p,x) \to \neg\varphi_1(p,x))$.

Consider $A_i \underset{\text{def}}{=} \{x \mid L[p,x] \models \varphi_i^*(p,x)\}$.

By claims 5 and 6, $M \models \varphi_i(p,x) \implies x \in A_i$. If $A_0 \cap A_1 = \emptyset$ then it is easy to separate A_0 from A_1 by a $\Delta^1_3(p)$ set of reals. So it will be enough to show that $A_0 \cap A_1 = \emptyset$:

Assume $b \in A_0 \cap A_1$. b is generic over $L[p]$. By claim 3, b is generic over $L[p]$ via a c-subalgebra of $\mathcal{S}\mathcal{I}$. Say this c-subalg is $\mathcal{B}'$. Wma that "$b \in A_0 \cap A_1$" has truth value $\mathbb{1}$.

By claim 4, there is b' in M s.t. $\langle p, b' \rangle$ is aut, and s.t. b' is generic over $L[p]$ via $\mathcal{B}'$. Thus $b' \in A_0 \cap A_1$.

But then by claims 6 and 5,

$M \models \varphi_0(p,b') \wedge \varphi_1(p,b')$. Contradiction.

$\square$

Proof of Claim 1

Let $\langle g_i \rangle_{i\in\omega}$ be $(\sum_\omega \mathcal{C})$-generic over L. Let $\mathcal{R}_i$ = the usual poset which adjoins a real a s.t. $T(g_i, a)$. Let $\mathcal{R} = \sum_{i\geq 1} \mathcal{R}_i$.

We must show that:

$$L[\langle g_i \rangle_{i\in\omega}]^{\mathcal{R}} \models \neg \exists a\, T(g_0, a).$$

Assume we have $p \in \mathcal{R}$ s.t. $p \Vdash T(g_0, t)$ for some term t.

We will mimick the argument given in [2]

Let L_η be a countable elementary substructure of L_{ω_1} s.t. $L_\eta[\langle g_i \rangle_{i\in\omega}]$ contains everything needed to make the following argument work.

Pick ξ, k s.t. $f_{\xi,k} \notin L_\eta$ and s.t. $g_0(k) = 0$. Find $q \in \mathcal{R}$, $q \geq p$ s.t. $q \Vdash$ "$t \cap S(f_{\xi,k})$ is finite".

There is a finite seq $\vec{F}$ of $f_{\delta,j}$'s s.t. $f_{\xi,k}$ does not appear among $\vec{F}$, and s.t. q is in $L[\langle g_i \rangle_{i\in\omega}, \vec{F}][f_{\xi,k}]$.

Let $N = L[\langle g_i \rangle_{i\in\omega}, \vec{F}]$. So $f_{\xi,k}$ is $\mathcal{C}$-generic over N.

We may view q as $q(f_{\xi,k})$, i.e., $q(f_{\xi,k})$ is a term, from N, applied to $f_{\xi,k}$.

q is in R. Hence for some $n \geq 1$, q is in $\sum_{1 \leq i \leq n} R_i$. Hence for all $f_{\delta,j}$ not in N, $q(f_{\delta,j})$ will be in $\sum_{1 \leq i \leq n} R_i$ and will extend p, provided $g_i(j) = g_i(k)$, for $1 \leq i \leq n$.

~~But we can easily~~

Since $q \Vdash$ "$t \cap S(f_{\xi,k})$ is finite", and since $f_{\xi,k}$ is $\mathcal{C}$-generic over N, there is $F \in \mathcal{C}$, s.t. $F \subseteq f_{\xi,k}$, and s.t.

$$F \Vdash \text{“} q(\dot{f}) \Vdash (\text{“} t \cap S(\dot{f}) \text{ is finite”}) \text{”}.$$

Since $\langle g_i \rangle_{0 \leq i \leq n}$ is $\sum_{0 \leq i \leq n} \mathcal{C}$ generic over L, we can find $f_{\delta,j}$ s.t. $f_{\delta,j} \notin N$, and $F \subseteq f_{\delta,j}$, and for all i, $1 \leq i \leq n$, $g_i(j) = g_i(k)$, and $g_0(j) = 1$.

Thus $q(f_{\delta,j})$ is in R, and $q(f_{\delta,j})$ extends p, and

$$q(f_{\delta,j}) \Vdash \text{“} t \cap S(f_{\delta,j}) \text{ is finite”}.$$

Thus $q(f_{\delta,j}) \Vdash$ "$\neg T(g_0, t)$". Contradiction.

$\square$

Proof of Claim 6

Let $W = \{ D \mid D \in L$ and D is a max. incompatible subset of $\mathcal{D} \}$. Then W is Σ_1 over $L_{\aleph_1}$.

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$D \in W$ iff for some ordinal ξ, $D \in L_\xi$, and L_ξ is a model of a reasonable fragment of ZFC, and D is a max. incompatible subset of $\mathcal{D} \cap L_\xi$. $\square$

By adopting the proof of Lemma 2.3 of [1], we obtain from the above:

If φ is a Σ_n formula, possibly involving terms denoting members of $L^{\mathcal{D}}_{\aleph_1}$, and if $p \in \mathcal{D}$, then

$p \Vdash$ " $L^{\mathcal{D}}_{\aleph_1} \models \varphi$ " is a Σ_n over

$L_{\aleph_1}$ assertion about p, φ. $\square$

We have $b \in M$. Hence b is generic over L via a c-subalg of $\mathcal{D}$. So there is G, an L-generic filter on $\mathcal{D}$, s.t. $b \in L[G]$, i.e, $b = t(G)$ for some term t. [Note. G need not be in M.]

Claim For a term t which denotes a real in $L^{\mathcal{D}}$, and for $p \in \mathcal{D}$, let $\Theta(t, p, b)$ assert:

there is G, an L-generic filter on $\mathcal{L}\mathcal{O}$, s.t. $p \in G$ and $b = t(G)$.

Then $\Theta(t, p, b)$ is Π_1 over $L[b]$.

proof

Let $\Psi(t, p, b) \equiv \neg\Theta(t, p, b)$.

Using the recursion theorem, Ψ can be ~~given~~ put into a Σ_1 form as follows:

$\Psi(t, p, b)$ iff:

① there is $D \in W$ s.t. $\forall q \in D$, q compatable with $p \Rightarrow \exists n \in \omega\, (q \Vdash t(n) \neq b(n))$,

or ② $\exists D \in W$ s.t. $\forall q \in D$, q compatable with $p \Rightarrow$ there is an ordinal ρ s.t. $L_\rho[b] \models \Psi(t, q, b)$ but $L_\rho[b] \not\models \Psi(t, p, b)$.

The above just corresponds to the usual inductive definition of those p which are inconsistent with the interpretation of t as b. $\square$

Now, given φ a Σ^1_3 formula, define φ^* by, $\varphi^*(b)$ iff:

there is t a term denoting a real in $L^{\mathcal{L}\mathcal{O}}$, there is $p \in \mathcal{L}\mathcal{O}$ s.t. $\Theta(t, p, b)$, and s.t. (viewing p as a member of $\mathcal{L}\mathcal{O} \oplus \mathcal{L}\mathcal{O}$)

$p \Vdash$ "$L[t]^{\mathcal{L}\mathcal{O}} \models \varphi(t)$". $\square$

Remarks The above proof was directly inspired by a result of Sami, namely: there is a model of ZFC in which Sep$(\underset{\sim}{\Pi}^1_3, \underset{\sim}{\Delta}^1_3)$ holds, but Red(Σ^1_3, Σ^1_3) fails for sets of reals.

We believe that this result can be generalized by replacing 3 by any integer ≥ 3. We also believe that this result can be improved so as to obtain a model of ZFC in which both Sep$(\underset{\sim}{\Pi}^1_3, \underset{\sim}{\Delta}^1_3)$ and Sep$(\underset{\sim}{\Sigma}^1_3, \underset{\sim}{\Delta}^1_3)$ hold. At the moment though these beliefs are just expressions of faith (or is it hope?).

References

[1] L. Harrington, Long projective well orderings, to appear

[2] R. B. Jensen, R. M. Solovay, Some applications of almost disjoint sets, in: Math Logic and Found. of Set Theory, 84-104.

Addendum, Part II to "the constructible reals can be (almost) anything"

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Theorem For each $n \geq 3$, there is a model of ZFC in which Separation(Π^1_n, Δ^1_n) and Separation(Σ^1_n, Δ^1_n) both hold for sets of integers.

proof

We will do this for $n=3$. The modifications already sketched in this note suffice to generalize the following argument for the case $n>3$.

Let Q_i, $1 \leq i < \omega$ be as in §5 of [1]. Let Q_0 be defined by: $f \in Q_0$ iff: f is a total func from an integer to $(\aleph_1 + 1) \cup \{-1\}$. Q_0 is identical with the usual poset which collapses $\aleph_1$ to ω.

Let $\oplus_{i \in \omega} G_i$ be $\oplus_{i \in \omega} Q_i$ generic over L.

G_0 can be identified with a total map from ω onto $(\aleph_1 + 1) \cup \{-1\}$.

Let $Z(n, \alpha)$ be as before, for $0 \leq \alpha < \aleph_1$.

Let $Z(n, \aleph_1) = \{\langle n, k, \ell \rangle \mid k, \ell \in \omega\}$.

and let $Z(n, -1) = \emptyset$.

Let $Z = \bigcup \{Z(n, \alpha) \mid G_0(n) = \alpha\}$, and let $N = L[G_0 \oplus (\oplus_{i \in Z} G_i)]$.

Every constructible real is Δ^1_3 in N.

Let $P(n, \alpha) = \oplus_{i \in Z(n, \alpha)} Q_i$.

Let $\mathbb{R}$ be the poset: $r \in \mathbb{R}$ iff: $r = \langle f, g \rangle$, $f \in \mathbb{Q}_0$, $\operatorname{dom} f = \operatorname{dom} g$, & for all $n \in \operatorname{dom} f$ $g(n) \in \mathbb{P}(n, f(n))$. $\langle f, g\rangle \leq \langle f', g'\rangle$ iff: $f \leq f'$ & $\forall n \in \operatorname{dom} f$, $g(n) \leq g'(n)$.

N is $\mathbb{R}$-generic over L.

<u>def</u> Given $f, f' \in \mathbb{Q}_0$, $f \leq f'$ iff: $\forall n \in \operatorname{dom} f$, if $f(n) \neq -1$, then $n \in \operatorname{dom} f'$ & either $f(n) = f'(n)$ or $f'(n) = \aleph_1$.

<u>Fact</u> If φ is a Σ^1_3 (resp. Π^1_3) sentence and if $f \leq f'$ (resp. $f' \leq f$) and if $\langle f, 0\rangle \Vdash \varphi$, then $\langle f', 0\rangle \Vdash \varphi$. $\square$

Now, given Σ^1_3 (resp. Π^1_3) formulas $\theta_0(n)$, $\theta_1(n)$, and given $f \in \mathbb{Q}_0$ s.t. $\langle f, 0\rangle \Vdash$ "$\forall m \in \omega\, (\theta_0(m) \rightarrow \neg\theta_1(m))$", we <u>claim</u>: $\forall m \in \omega$ either $\langle f, 0\rangle \Vdash \neg\theta_0(m)$ or $\langle f, 0\rangle \Vdash \neg\theta_1(m)$.

[proof: If not pick $f_0 \geq f$, $f_1 \geq f$ s.t. $\langle f_0, 0\rangle \Vdash \theta_0(m)$ & $\langle f_1, 0\rangle \Vdash \theta_1(m)$. Define $\hat{f} \in \mathbb{Q}_0$ by: $\operatorname{dom} \hat{f} = \operatorname{dom} f_0 \cup \operatorname{dom} f_1$; for $n \in \operatorname{dom} f$, $\hat{f}(n) = f(n)$; for $n \in (\operatorname{dom} f_0 \cup \operatorname{dom} f_1) \sim \operatorname{dom} f$, $\hat{f}(n) = \aleph_1$ (resp. $\hat{f}(n) = -1$). Thus $\hat{f} \geq f$ and $f_0, f_1 \leq \hat{f}$ (resp. $f_0, f_1 \geq \hat{f}$). So $\langle \hat{f}, 0\rangle \Vdash$ "$\theta_0(m) \wedge \theta_1(m)$". $\ast$ $\square$]

But then $\{m \mid \langle f, 0\rangle \Vdash \neg\theta_1(m)\}$ is a constructible set of integers (and hence Δ^1_3 in N) which separates $\{m \mid N \models \theta_0(m)\}$ from $\{m \mid N \models \theta_1(m)\}$. $\square$