Statistical mechanics of close-packings in 2D lattices

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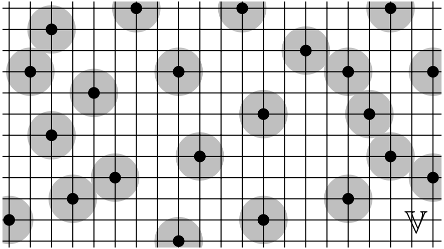
The hard-core model on a lattice



- A discrete set $\mathbb{L} \subset \mathbb{R}^2$
- A finite volume: $\mathbb{V} \subset \mathbb{L}$
- A configuration of particles: $\psi^{\mathbb{V}} \in \{0,1\}^{\mathbb{V}}$
- An admissible configuration:
  $\rho(x,y) \geq D \geq 1$ for any $\psi^{\mathbb{V}}_x = \psi^{\mathbb{V}}_y = 1$
- Activity/Fugacity: $u > 0$

- A boundary condition: an admissible configuration $\phi = \phi^{\mathbb{Z}^2}$
- An additional restriction: $\psi^{\mathbb{V}} \vee \phi^{\mathbb{Z}^2 \setminus \mathbb{V}}$ is admissible, $\psi^{\mathbb{V}} \in \mathcal{A}(\mathbb{V} \| \phi)$
- The statistical weight: $w(\psi^{\mathbb{V}} \| \phi) = u^{\sharp(\psi^{\mathbb{V}})}$
- The partition function: $\mathbf{Z}(\mathbb{V} \| \phi) = \sum_{\psi^{\mathbb{V}} \in \mathcal{A}(\mathbb{V} \| \phi)} w(\psi^{\mathbb{V}} \| \phi)$

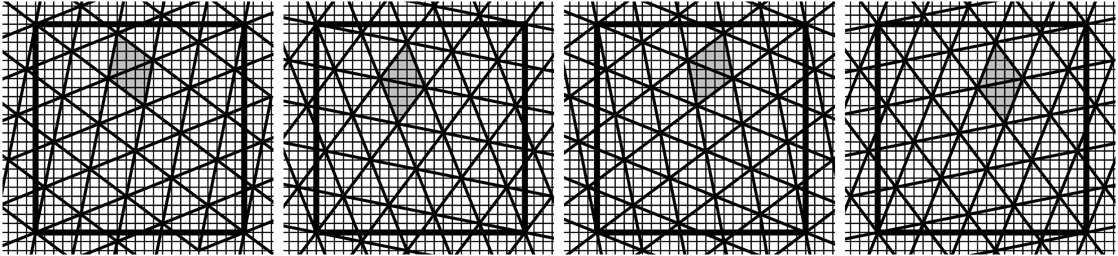
Gibbs measures on $\mathbb{L}$

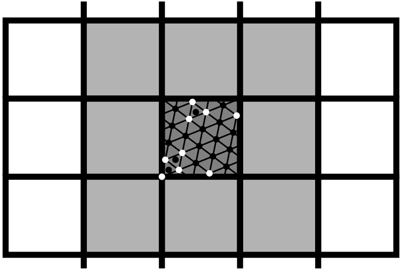
∘ The Gibbs measure: $\mu_{\mathbb{V}}(\psi^{\mathbb{V}}\|\phi) = w(\psi^{\mathbb{V}}\|\phi)/\mathbf{Z}(\mathbb{V}\|\phi)$
∘ A probability measure $\boldsymbol{\mu}$ on $\mathcal{X} = \{0,1\}^{\mathbb{L}}$ is called a D-HC *Gibbs/DLR measure* if (i) $\boldsymbol{\mu}(\mathcal{A}) = 1$, (ii) $\forall$ finite $\mathbb{V} \subset \mathbb{L}$ and a function $f : \phi \in \mathcal{X} \mapsto f(\phi) \in \mathbb{C}$ depending only on the restriction $\phi\restriction_{\mathbb{V}}$, the integral $\boldsymbol{\mu}(f) = \int_{\mathcal{X}} f(\phi)\mathrm{d}\boldsymbol{\mu}(\phi)$ has the form

$$\boldsymbol{\mu}(f) = \int\limits_{\mathcal{X}} \int\limits_{\{0,1\}^{\mathbb{V}}} f(\psi^{\mathbb{V}} \vee \phi\restriction_{\mathbb{L}\setminus\mathbb{V}})\mathrm{d}\mu_{\mathbb{V}}(\psi^{\mathbb{V}}\|\phi)\mathrm{d}\boldsymbol{\mu}(\phi).$$

One can say that under such measure $\boldsymbol{\mu}$, the probability of a configuration $\psi^{\mathbb{V}}$ in a finite volume $\mathbb{V} \subset \mathbb{L}$, conditional on a configuration $\phi\restriction_{\mathbb{L}\setminus\mathbb{V}}$, coincides with $\mu_{\mathbb{V}}(\psi^{\mathbb{V}}\|\phi)$, for $\boldsymbol{\mu}$-a.a. $\phi \in \{0,1\}^{\mathbb{L}}$.
∘ An extreme Gibbs measure (EGM) = a pure phase: cannot be written as a non-trivial convex combination of other Gibbs measures

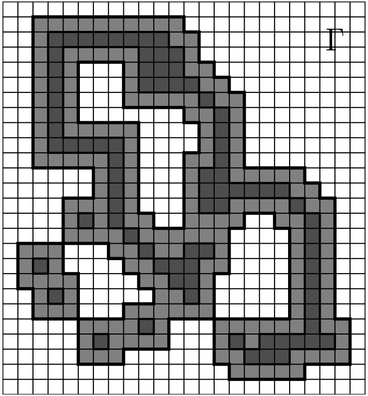
Pirogov-Sinai theory: ground states





◦ Ground states: no local excitation decreases the energy
◦ Periodic ground states (PGS): energy minimizers on a torus
◦ A common template
◦ A correct common template

Pirogov-Sinai theory: contours



∘ Contours, ensembles of compatible contours

$$\mathbf{Z}(\mathbb{V}\|\phi) = \sum_{\{\Gamma_i\}} \prod_i w(\Gamma_i)$$

∘ A Peierls condition: an excess in energy due to local a defect is proportional to the size of the defect:

$$w(\Gamma) \leq u^{-p(D)||\mathrm{Supp}\,(\Gamma)||}$$

∘ A Peierls constant: $p(D) > 0$

∘ A PGS as a boundary condition: generates a pure phase

Pirogov-Sinai theory: results

Theorem. *Finiteness of the PGS family and a strong enough Peierls condition imply that at least one PGS generates pure phase and any periodic pure phase is generated by some PGS.* (Pirogov-Sinai [1976], Zahradnik [1984])

◦ Non-periodic ground states

Theorem. *In dimension 2 the finite number of periodic ground states and strong enough Peierls condition imply that all pure phases are periodic.* (Dobrushin-Shlosman[1985])

◦ Stable and non-stable PGSs (dominance): ensemble of contours with truncated weight, polymer expansion, free energy.

◦ HC model admits a straightforward application of PS theory but the difficulty is in identifying PGSs and verifying Peierls condition

Local minimizing pattern

∘ An m-potential (Holztynski-Slawny [1978]):
- the energy of a configuration is a sum over all translations of a finite set B
- a minimum of the energy among all configurations in B is achieved at one or more minimizing configurations
- there exist at least one configuration in entire $\mathbb{L}$ which coincides with one of minimizing configurations on each translation of B in $\mathbb{L}$.

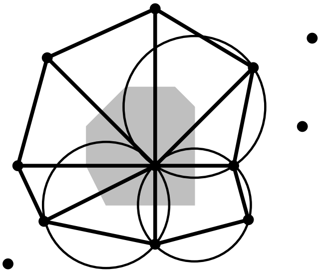
∘ A weaker version:
- a configuration is identified with a partition of $\mathbb{R}^2$ into bounded sets B_j
- the energy of a configuration $\{B_j\}$ is the sum of energies attributed to each B_j
- the variety of B's contains element(s) with minimal energy
- there exists a configuration in entire $\mathbb{L}$ formed solely by minimal elements

∘ A perfect configuration (PC): containins local minimizers only

∘ All PGSs are PCs

∘ The minimizing pattern is absent at boundaries between PGSs

Voronoi cells and disk close-packing in $\mathbb{R}^2$



- A Voronoi cell (VC), a convex polygon in $\mathbb{R}^2$:
  defined for each x with $\psi_x = 1$
- Vertices of a VC: centers of circumcircles
- A constituting polygon, C-triangle
- A Delaunay triangulation

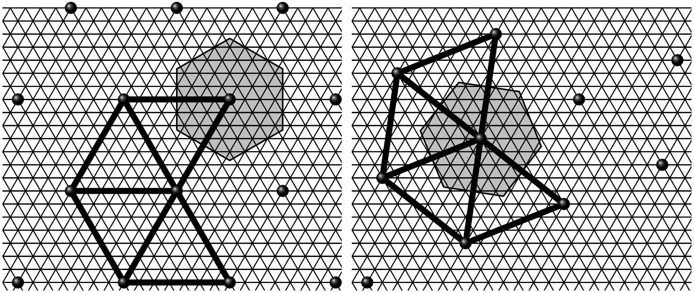
- The area of a VC as a local quantity to minimize:
  - find minimal VCs
  - find all configurations containing minimal VCs only

- The minimal Voronoi cell in $\mathbb{R}^2$ is a perfect hexagon

Voronoi cells and disk close-packing in $\mathbb{A}_2$

○ The minimal Voronoi cell in $\mathbb{A}_2$ is the same as in $\mathbb{R}^2$

○ For an admissible configuration containing a particle at the origin there are only finitely many possibilities to implement the minimal Voronoi cell centered at this particle



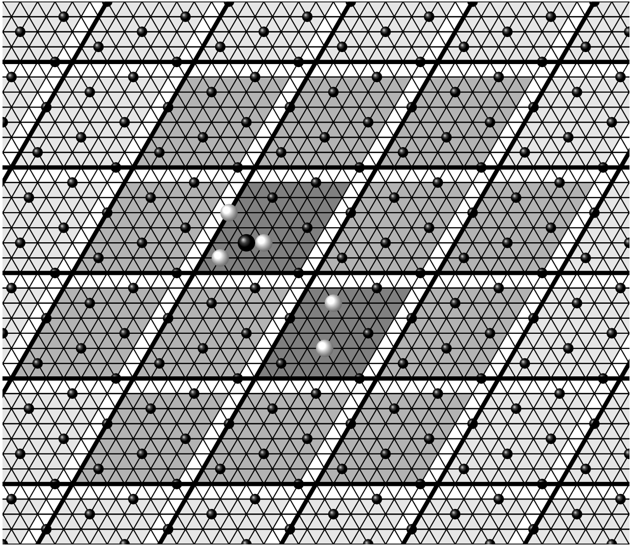
○ Each possibility corresponds to the configuration of particles forming a triangular D-sublattice of $\mathbb{A}_2$

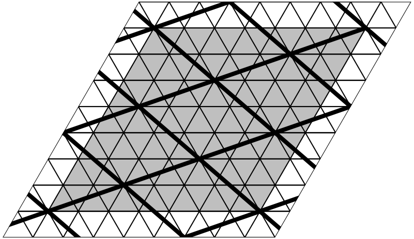
○ Each D-sublattice generates a class of PCs which contains its $\mathbb{A}_2$-symmetries.

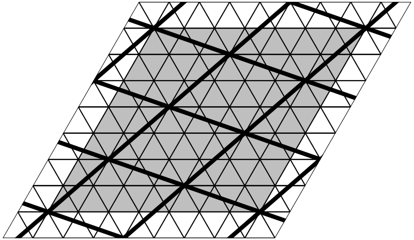
○ Depending on arithmetical structure of D^2, there may be one or more classes

○ All PGSs are from classes generated by D-sublattices

Templates and contours on $\mathbb{A}_2$







Peierls estimate on $\mathbb{A}_2$ via Voronoi cells

∘ **Theorem.** *Define the statistical weight* $w(\Gamma)$ *of a contour* Γ *by*

$$w(\Gamma) = u^{\sharp(\psi_\Gamma)-\sharp(\varphi_\Gamma)} = \prod_{x\in\psi_\Gamma} u^{-S^{-1}\left(\left|C_{\psi_\Gamma}(x)\right|-S\right)}.$$

Here and below, ψ_Γ *stands for the restriction* $\psi\restriction_{\mathrm{Supp}\,\Gamma}$*. Then*

$$w(\Gamma) \leq u^{-p(D)||\mathrm{Supp}\,\Gamma||}.$$

Next, $S = S(D) = D^2\sqrt{3}/2$ *is the area of the minimal Voronoi cell (a perfect hexagon of side length* $D\sqrt{3}/2$*),* $|C_{\psi_\Gamma}(x)|$ *is the area of the Voronoi cell* $C_{\psi_\Gamma}(x)$ *for* $x \in \psi_\Gamma$*,* $p(D) > 0$ *is a Peierls constant, and* $||\mathrm{Supp}\,\Gamma||$ *is the number of templates in* $\mathrm{Supp}\,\Gamma$*.*

∘ Here $p(D)$ is calculated via the difference $-s(D) < 0$ of areas between minimal and next-to-minimal Voronoi cells

Proof of Peierls estimate on $\mathbb{A}_2$

Consider x where $|C_{\psi_\Gamma}(x)| > S$; otherwise x does not contribute into $w(\Gamma)$. Observe that

$$\text{if} \quad |C_{\psi_\Gamma}(x)| - S \geq S \quad \text{then} \quad |C_{\psi_\Gamma}(x)| - S \geq \tfrac{1}{2} |C_{\psi_\Gamma}(x)| \,.$$

On the other hand,

$$\text{if} \quad |C_{\psi_\Gamma}(x)| - S < S \quad \text{then} \quad |C_{\psi_\Gamma}(x)| - S \geq s(D) \geq \tfrac{s(D)}{2S} |C_{\psi_\Gamma}(x)| \,.$$

According to the definition of a φ-correct common parallelogram, we have an inequality

$$\sum_{x \in \psi_\Gamma} |C_{\psi_\Gamma}(x)| \, \mathbf{1}\Big(|C_{\psi_\Gamma}(x)| > S \Big) \geq \frac{1}{9D^2} |\mathrm{Supp}\,\Gamma| \,.$$

Also, $||\mathrm{Supp}\,\Gamma|| = \dfrac{2}{D^4\sqrt{3}} |\mathrm{Supp}\,\Gamma|$. Thus, we can take

$$p(D) = \frac{1}{9} \min\left(\frac{1}{2}, \frac{s(D)}{\sqrt{3}} D^{-2} \right). \quad \blacksquare$$

Peierls estimates on $\mathbb{H}_2$ and $\mathbb{Z}^2$

∘ $\mathbb{H}_2$ can be considered as $\mathbb{A}_2$ with removed 1/3 of sites (a $\sqrt{3}$-sublatice)

∘ For $D^2 \bmod 3 = 0$ the above $\mathbb{A}_2$ theory is applied to $\mathbb{H}_2$ verbatim. The PGSs are still triangular D-sublattices and their $\mathbb{H}_2$ symmetries

∘ Let $D^* = D_*(D) \geq D$ be the nearest value such that $D_*^2 \bmod 3 = 0$ and $D_*^2 = m^2 + n^2 + mn$.

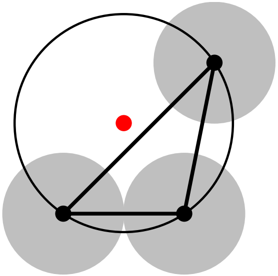
∘ For $D^2 \bmod 3 \neq 0$, $D^2 \neq 1, 4, 7, 13, 16, 28, 49, 64, 67, 97, 133, 157, 256$ the PGSs are D_*-sublattices from $\mathbb{A}_2$.

∘ However for $D^2 \bmod 3 \neq 0$ the Voronoi cells do not work.

∘ The Voronoi cells do not work for $\mathbb{Z}^2$ either, and we need a different approach

Saturated configurations and minimal triangles

○ Saturation of an admissible configuration
○ In $\mathbb{R}^2$ saturation implies (Chang-Wang [2010]):



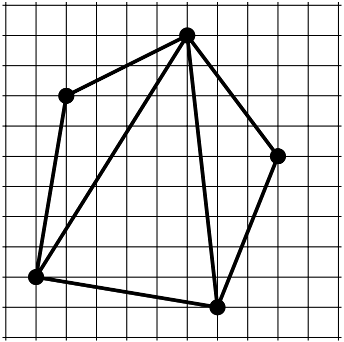
- all circumradii are $\leq D$ as otherwise a particle can be placed into the circumcenter of the corresponding constituting polygon

- all Delaunay triangles have angles $\leq 2\pi/3$

- amount of triangles is twice the amount of particles

○ A minimal triangle (M-triangle): an admissible acute $\mathbb{L}$-triangle (sides not shorter than D) with a minimal possible area
○ M-triangles: better local minimizers than VCs

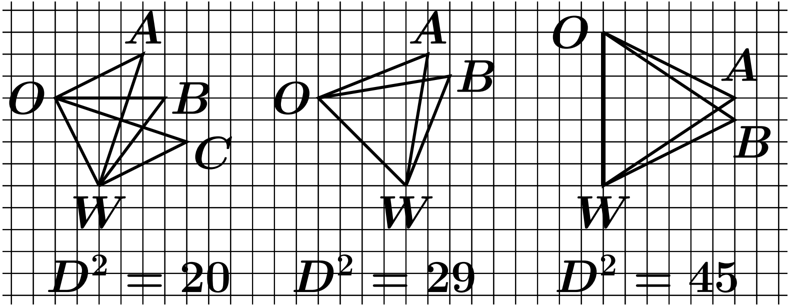
Area-minimization of Delaunay triangles in $\mathbb{Z}^2$



- All circumradii are $\leq D + \sqrt{2}$
- Obtuse triangles with less than minimal area
- Triangle groups and redistributed area
- Area redistribution is also useful in dealing with other issues, especially on $\mathbb{H}_2$

M-triangles and sliding in $\mathbb{Z}^2$

- Sliding in terms of M-triangles



- In absence of sliding:
  - an M-triangle generates a unique perfect configuration (PC)
  - all PGSs are sublattices or their $\mathbb{Z}^2$-symmetries (shifts, rotations, reflections)
  - $\mathbb{Z}^2$-symmetries define equivalence classes of PCs/PGSs
  - the PGS sublatices are non-square for $D^2 > 20$

M-triangles and PGSs on $\mathbb{Z}^2$

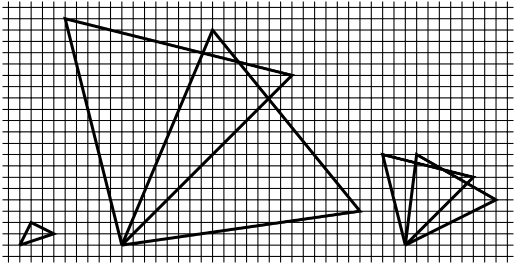
Theorem. (i) *For any attainable* D, *every* PGS *is obtained as a tessellation by* M-*triangles and their* $\mathbb{Z}^2$-*shifts.*

(ii) *Furthermore, if* D *is non-sliding then every* PGS *is obtained from a max-dense sub-lattice by means of* $\mathbb{Z}^2$-*congruences. Consequently, for any non-sliding* D *the* PGS *set* $\mathcal{P}(D)$ *is finite.*

∘ A similar theorem is true for $\mathbb{A}_2$ and $\mathbb{H}_2$ except for the exceptional values of D

Classes of M-triangles on $\mathbb{Z}^2$

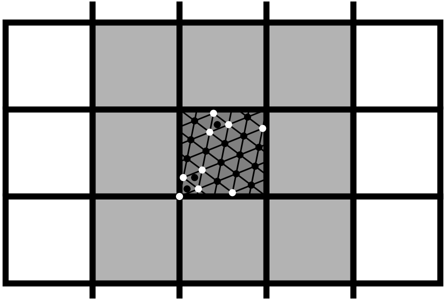
- **Class S** (sliding): finite



- **Class A** (a unique M-triangle, a unique implementation), $d = 5$, $[5|5|10]$
- **Class B0** (a unique M-triangle, a non-unique implementation), $d = 425$, $[425|425|450]$
- **Class B1** (a non-unique M-triangle, a unique implementation for each), $d = 65$, $[65|65|80]$, $[68|68|72]$
- **Class B2** (a non-unique M-triangle, a non-unique implementation), $d = 180610$, $[d|d+60|d+145]$, $[d|d+60|d+145]$, $[d|d+115|d+135]$ and $[d|d+115|d+135]$
- Classes A, Bx are infinite
- A general strategy: M-triangles $\to$ PCs $\to$ PGSs $\to$ EGMs

C-triangles, M-triangles and defects in $\mathbb{Z}^2$

∘ Let ψ^* be a saturation of a given D-AC ψ.



∘ If an added occupied site $x \in \psi^* \setminus \psi$ lies in a template then, clearly, this template is incorrect (more precisely, non-φ-correct in φ for each $\varphi \in \mathcal{P}$). We say that such a template is an *s-defect* (in ψ).

∘ Another possibility for a defect is where, in the saturation ψ^*, a template has a non-empty intersection with one of C-triangles that is not an M-triangle. We call it a *t-defect* (again in ψ). C-triangles with obtuse angles $> 2\pi/3$ lead to t-defects by definition.

∘ Finally, an incorrect (but actually perfect) template can be simply a neighbor of an s- or a t-defect. We call it an *n-defect* (still in ψ).

A Peierls estimate via M-triangles in $\mathbb{Z}^2$

∘ **Theorem.** *Consider a contour* Γ *in* $\psi \in \mathcal{A}(D)$ *containing* $||\mathrm{Supp}\Gamma|| = m$ *(incorrect) templates . Set* $m = i+j+k$ *where* i, j, k *give the amount of s-, t- and n-defects in* ψ_Γ*. Then the amounts* $\sharp(\psi_\Gamma), \sharp(\varphi_\Gamma)$ *of occupied sites in* ψ_Γ *and* φ_Γ *satisfy*

$$\sharp(\psi_\Gamma) \leq mS(D) - i - \max\left(1, \frac{j}{8S(D)}\right), \ \sharp(\varphi_\Gamma) = mS(D)$$

∘ Let Δ be a connected component of φ'-correct templates enclosed by a connected component of φ''-correct templates, $\varphi', \varphi'' \in \mathcal{P}(D)$. Then, in absence of sliding, any extension of this restricted configuration to a $\psi \in \mathcal{A}(D)$ contains a closed chain of adjacent non-minimal C-triangles enclosing Δ. This chain constitutes t-defects in the corresponding contour Γ.

∘ **Theorem.** *Assume the value* D *is non-sliding. For a contour* Γ *containing* $||\mathrm{Supp}\Gamma||$ *(incorrect) templates*

$$w(\Gamma) = u^{\sharp(\psi_\Gamma)-\sharp(\varphi_\Gamma)} \leq u^{-p(D)||\mathrm{Supp}\Gamma||}, \text{ where } p(D) = 1/(72S(D))$$

and $\frac{\sqrt{3}D^2}{2} < S(D) < \frac{\sqrt{3}D^2}{2} + \sqrt{2}D$

A Peierls estimate via M-triangles in $\mathbb{A}_2$ and $\mathbb{H}_2$

◦ **Theorem.** *For a contour* Γ *containing* $||\mathrm{Supp}\Gamma||$ *(incorrect) templates*

$$w(\Gamma) \leq u^{-p(D)||\mathrm{Supp}\Gamma||}, \quad p(D) = 1/(72S(D)),$$

where $S(D) = \frac{\sqrt{3}D^2}{2}$ *for* $\mathbb{A}_2$*, and* $\frac{\sqrt{3}D^2}{2} \leq S(D) \leq \frac{\sqrt{3}D_*^2(D)}{2}$ *for* $\mathbb{H}_2$*.*

◦ M-triangle approach is more technically involved but leads to better estimates

◦ For the case $D^2 = 64$ in $\mathbb{H}_2$ the coresponding unique M-triangle generates PGSs satisfying above Peierls estimate while the minimal Voronoi cell does not generate an admissible configuration in $\mathbb{H}_2$. Consequently, the PGSs consist of next-to-minimal rather than minimal Voronoi cells.

Results for Class A (a unique M-triangle)

- A complete phase diagram for large u
- **Theorem.** *Assume an attainable value D is of Class* A. *Then:*

(i) *The cardinality* $\sharp\mathcal{P}(D) = mS(D)$ *where* $m = 1, 2$ *or* 4 *for* $\mathbb{Z}^2$, $m = 1, 2$ *for* $\mathbb{A}_2$, $m = 2/3, 4/3$ *for* $\mathbb{H}_2$. *The* PGSs *are obtained from each other by* $\mathbb{L}$-*congruences.*

(ii) *There exists a value* $u_* = u_*(D, \mathbb{L}) \in (0, \infty)$ *such that for* $u \geq u_*$ *the following assertions hold true. Every* EGM $\boldsymbol{\mu} \in \mathcal{E}(D)$ *is generated by a* PGS $\varphi \in \mathcal{P}(D)$: $\boldsymbol{\mu} = \lim\limits_{\mathbb{V} \nearrow \mathbb{Z}^2} \mu_{\mathbb{V}}(\,\cdot\,||\varphi)(= \boldsymbol{\mu}_\varphi)$. *The measures* $\boldsymbol{\mu}_\varphi$ *are mutually disjoint* ($\boldsymbol{\mu}_{\varphi'} \perp \boldsymbol{\mu}_{\varphi''}$ *for* $\varphi' \neq \varphi''$) *and inherit the symmetry properties of their respective* PGSs. *Moreover,* $\sharp\mathcal{E}(D) = \sharp\mathcal{P}(D)$.

Results for Class B (a non-unique M-triangle)

∘ A phase diagram modulo dominance for large u

∘ **Theorem.** *Assume an attainable value D is of Class* B *with $J = J(D) > 1$ equivalence classes of* PGS. *Then:*

(i) *The cardinality $\sharp\mathcal{P}_j(D) = m_j S(D)$ where $m_j = 1, 2$ or 4 for $\mathbb{Z}^2$, $m_j = 1, 2$ for $\mathbb{A}_2$, $m_j = 2/3, 4/3$ for $\mathbb{H}_2$, $j = 1, 2, \dots, J$. The equivalent* PGSs *are obtained from each other by $\mathbb{L}$-congruences.*

(ii) *There exists a value $u_* = u_*(D, \mathbb{L}) \in (0, \infty)$ such that for $u \geq u_*$ the following assertions hold true. Every* EGM *$\boldsymbol{\mu} \in \mathcal{E}(D)$ is generated by a* PGS *φ from a dominant class $\mathcal{P}_j(D)$: $\boldsymbol{\mu} = \lim\limits_{\mathbb{V}\nearrow\mathbb{Z}^2} \mu_{\mathbb{V}}(\,\cdot\,||\varphi)(= \boldsymbol{\mu}_\varphi)$. The measures $\boldsymbol{\mu}_\varphi$ are mutually disjoint ($\boldsymbol{\mu}_{\varphi'} \perp \boldsymbol{\mu}_{\varphi''}$ for $\varphi' \neq \varphi''$) and inherit the symmetry properties of their respective* PGSs. *Furthermore, $1 \leq \sharp\mathcal{E}(D) = \sum\limits_{j:\, \mathcal{P}_j(D) \text{ is dominant}} \sharp\mathcal{P}_j(D)$.*

Polymer series and Dominant PGSs

◦ The twofold entropy of contours:

- the number of ways to draw $\mathrm{Supp}\Gamma$ with $||\mathrm{Supp}\Gamma|| = m$ is $\leq 4 \cdot 3^{m-1}$
- the number of ways to select ψ_Γ is $\leq 2^{D^2 S(D)} \leq 2^{D^4}$

◦ The statistical weight $w(\pi)$ of a polymer $\pi = \{\Gamma_i\}$ satisfies

$$|w(\pi)| \leq u^{(\log_u c(D) - p(D)) \sum_i ||\mathrm{Supp}\Gamma_i||}$$

◦ The polymer series for the free energy has the following form

$$\sum_{k=1}^{\infty} a_k u^{-k}, \text{ where } |a_k| \leq c(D)^k$$

◦ Dominance of a PGS: the truncated free energy is minimal (and consequently equal to free energy)

◦ We expect that for our models the dominance can always be seen at the second order u^{-2} in the polymer series

Dominant PGSs: orders of perturbations

∘ The constant $p(D)$ is too small to be practical. Need an alternative approach to contour enumeration

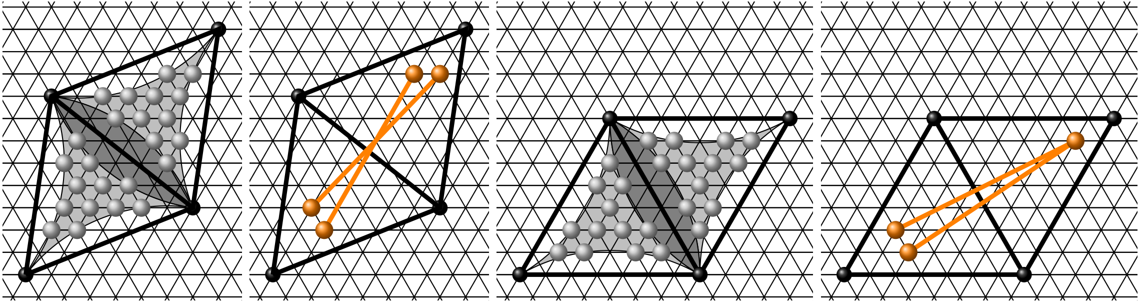
∘ By construction $w(\Gamma) = u^{-k},\ k \in \mathbb{N}$

∘ It is possible to show that the only way to create Γ with $w(\Gamma) = u^{-1}$ is to remove a particle. It does not discriminate between the PGSs

∘ There are multiple ways to create Γ with $w(\Gamma) = u^{-2}$ by removing $n \geq 3$ particles and inserting $n-2$ particles, without violating admissibility.

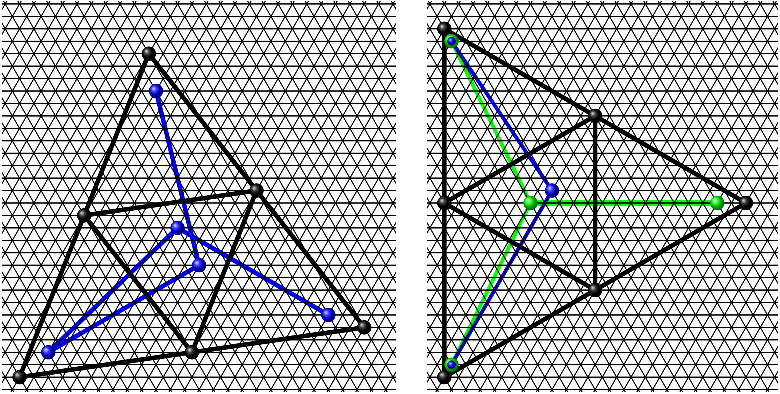
∘ In analyzed examples for $D^2 = 49, 147, 169$ on $\mathbb{A}_2$ we verified that $n \leq 6$

Perturbation examples: $D^2 = 49$



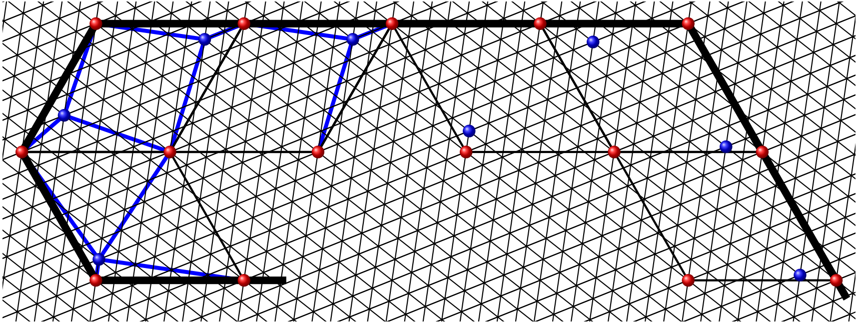
- Gray single-particle insertions ($n = 3$) do not discriminate between PGSs
- Orange pair insertions ($n = 4$) discriminate: the PGS class on the RHS dominates (non-inclined)

Perturbation examples: $D^2 = 147$



- Blue triples represent 3-particle insertions ($n = 5$). A green quadruple represents a 4-particle insertion ($n = 6$)
- In total, for $D^2 = 147$ the PGS class on the RHS dominates

The bound $n \le 6$ for $D^2 \le 169$: the proof



- For $\varphi \in \mathcal{P}(D)$, we assign, to a pair $\big($an insertion at site y, a particle removed by this insertion at site $x \in \varphi\big)$, a repelling force $f_r \ge 0$, $r = \rho(x,y)^2$, so that $\forall\, y \in \mathbb{A}_2$,

$$F(y) := \sum_{x\in\varphi} f_r \mathbf{1}(\rho(x,y)^2 = r,\ x \text{ removed by } y) = 1.$$

- Can define f_r so that

$$G(x) := \sum_y f_r \mathbf{1}(\rho(x,y)^2 = r,\ x \text{ removed by } y) \le 1$$

- The proof is computer-assisted. The bound $n \le 6$ not expected to hold for larger D

- $\forall$ admissible n-particle insertion Δ and $x \in \varphi$, the deficit $\delta(x) := 1 - G(x) \ge 0$
  $\sum_{x\in\varphi} \delta(x) =$ (the difference of the amounts of removed and inserted particles).

- The selection of f_r is not unique but it is possible to choose f_r such that $\delta(x) \ge 1/3$ for $x \in \varphi \cap \partial\Delta$.

- An implication is that on $\mathbb{A}_2$, an admissible insertion of order u^{-2} with $n \ge 7$ for $D^2 \le 169$ is impossible

- We expect that on $\mathbb{Z}^2$ and $\mathbb{H}_2$ the future theory will go along similar lines.

PGSs and EGMs for exceptional values of D^2 on $\mathbb{H}_2$

∘ Recall: the exceptional values on $\mathbb{H}_2$ are D^2 =1, 4, 7, 13, 16, 28, 49, 64, 67, 97, 133, 157, 256.

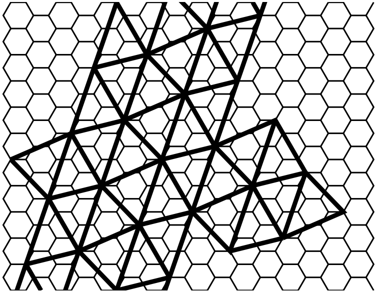
∘ For $D^2 = 4, 7, 133$: have sliding, offer no results, expect uniqueness of an EGM.

∘ The case $D^2 = 1$ is trivial: here the only PGS is the whole of $\mathbb{H}_2$. The EGM is unique for all $u > 0$: a Bernoulli measure with probability of occupied/vacant site $u/(1+u)$ and $1/(1+u)$.

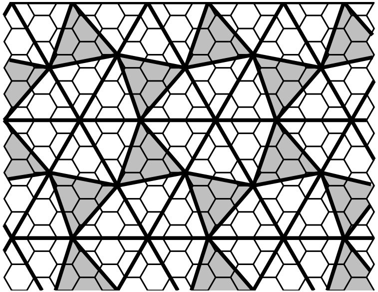
∘ The remaining values are divided into 3 classes E1: D^2 =13, 16, 28, 49, 64, 97, 157; E2: D^2 =16, 256; E3: D^2 =67.

∘ For values D^2 from E1: the PGSs are constructed from quadrilaterals formed by two adjacent D-admissible $\mathbb{H}_2$-triangles with the minimal squared side-length D^2, one of which is an M-triangle. (For $D^2 = 64$ the two triangles are congruent.)

PGSs and EGMs for Classes E1, E2



E1



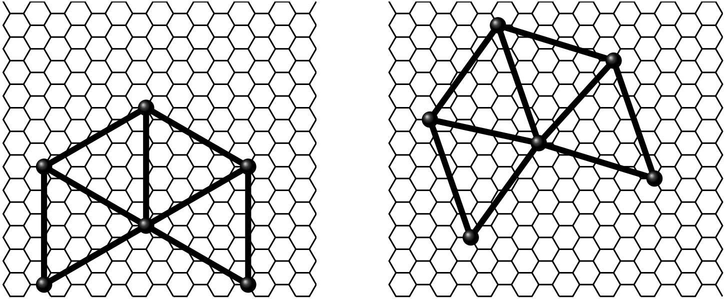
E2

∘ For values $D^2 = 16,\ 256$ from E2: the PGSs are constructed from equilateral triangles of squared side-lengths $(2D+1)^2$ (outer) and $D^2 + D + 1$ (inner)

∘ **Theorem.** *Let D be from Class* E1 *or* E1*. Then each EGM is generated by a PGS and each PGS generates an EGM. All assertions stated for Class* A *hold true.*

PGSs and EGMs for Class E3

○ For the remaining value $D^2 = 67$ the situation is as in Class B: we have a competition between two groups of PGSs. They are build (a) from equilateral D_*-triangles where $D_*^2 = 75$, (b) from quadrilaterals, similarly to Class E1. It turns out that type (a) is dominant and type (b) not.



(a) (b)