 List 2 – deadline 11.12

1. Let $x := 0.x_1x_2\dots$ be the uniquely defined decimal expansion, i.e. $\sum_{k=1}^{n} x_k 10^{-k} \le x\ \forall n,\ x \in X := [0,1)$ and let $Tx := \frac{1}{20}\limsup_n x_n$. Check measurebility of $T : X \to X$ and find its Riemann/Lebesgue integrals and all invariant measures.
2. Let $X := [0,1] = \bigsqcup X_i$ and let $T : X \to X$ be a piecewise isometry, i.e. $\rho(Tx, Ty) = \rho(x,y)\ \forall x,y \in X_i$. Prove/disprove that $\mathcal{M}_T \ne \emptyset$.
3. Let $f(x) := x(1-x)$. Compute $\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} f(\{x+ky\})$ as a function of 2 variables $x,y \in [0,1],\ k \in \mathbb{N}$.
4. Let X be a finite set with a σ-algebra $\mathcal{B} = 2^X$ and let $T : X \to X$ be uniquely ergodic. The latter means that there exists the only one probabilistic invariant measure. Find ALL pairs of positive integers p,q, such that $T^{pn}x = T^{qn}y\ \ \forall x,y \in X$ and some $n \in \mathbb{N}$.
5. Let T be a continuous map from a compact metric space X into itself with an invariant measure μ. Consider ergodic averages $S^n\varphi(x) := \frac{1}{n}\sum_{k=0}^{n-1}\varphi(T^k x)$ with $\varphi \in C^0(X)$. Prove/disprove that:
   (a) $\exists f = f(\varphi) \in C^0(X) : \quad S^n\varphi(x) \stackrel{n\to\infty}{\longrightarrow} f(x)\ \ \forall x \in \mathrm{supp}\mu$
   (b) If $f(x) =$Const, then (T, X, μ) is ergodic.

Please do not wait for the deadline and try to prepare solutions in LaTex.