1К

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Замена переменных. Решение дифференциальных уравнений.

Задача 1. Введя новые независимые переменные ξ и η, решить следующее уравнение:

(1) $$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y}, \text{ если } \xi = x+y, \ \eta = x-y$$

Решение:

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial z}{\partial \eta}\cdot\frac{\partial \eta}{\partial x} = \frac{\partial z}{\partial \xi} + \frac{\partial z}{\partial \eta}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial \xi}\cdot\frac{\partial \xi}{\partial y} + \frac{\partial z}{\partial \eta}\cdot\frac{\partial \eta}{\partial y} = \frac{\partial z}{\partial \xi} - \frac{\partial z}{\partial \eta}$$

Следовательно, $\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = 2\frac{\partial z}{\partial \eta}$.

Получаем уравнение $\frac{\partial z}{\partial \eta} = 0$.

Следовательно, z – не зависит от η, т.е.

$$z = \varphi(\xi) \Rightarrow z = \varphi(x+y).$$

Получили общий вид решения уравн. (1).

Задача 2. Решить уравнение, введя новые переменные ξ и η:

(2) $$y\cdot\frac{\partial z}{\partial x} - x\cdot\frac{\partial z}{\partial y} = 0, \quad \xi = x, \ \eta = x^2+y^2$$

Решение:

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial \xi}+\frac{\partial z}{\partial \eta}\cdot 2x;\quad \frac{\partial z}{\partial y}=\frac{\partial z}{\partial \eta}\cdot 2y.$$

Подставляем в уравнение (2)

$$y\left(\frac{\partial z}{\partial \xi}-\frac{\partial z}{\partial \eta}\,2x\right)-x\cdot\frac{\partial z}{\partial \eta}\cdot 2\cdot y=y\cdot\frac{\partial z}{\partial \xi}=0$$

т.е. $y\cdot\frac{\partial z}{\partial \xi}=0 \Rightarrow z=\varphi(\eta)$, т.е.

$z=\varphi(x^2+y^2)$ общее решение (2)

Задача 3. Преобразовать ур-е, в виде новые переменн.

(3) $$x\cdot\frac{\partial z}{\partial x}+\sqrt{1+y^2}\cdot\frac{\partial z}{\partial y}=xy$$

$$u=\ln x,\quad v=\ln\left(y+\sqrt{1+y^2}\right).$$

Решение:

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{1}{x}$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}=\frac{\partial z}{\partial v}\cdot\frac{1+\frac{2y}{2\sqrt{1+y^2}}}{y+\sqrt{1+y^2}}=$$

$$=\frac{\partial z}{\partial v}\cdot\frac{1}{\sqrt{1+y^2}}$$

Подставляем в уравн.:

$$x\cdot\frac{\partial z}{\partial u}\cdot\frac{1}{x}+\sqrt{1+y^2}\cdot\frac{\partial z}{\partial v}\,\frac{1}{\sqrt{1+y^2}}=xy$$

$$\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}=x\cdot y;\quad x=e^u,\ y=\operatorname{sh} v$$

В новых перемен.: $$\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}=e^u\cdot\operatorname{sh} v.$$

Задача 4. Решить ДУ в частных производных:

(4) $$(x+y)\cdot\frac{\partial z}{\partial x} - (x-y)\frac{\partial z}{\partial y} = 0.$$

$$u = \ln\sqrt{x^2+y^2}, \qquad v = \operatorname{arctg}\frac{y}{x}$$

Решение:

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} = \frac{x}{x^2+y^2}\frac{\partial z}{\partial u} - \frac{y}{x^2+y^2}\cdot\frac{\partial z}{\partial v}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y} = \frac{y}{x^2+y^2}\cdot\frac{\partial z}{\partial u} + \frac{x}{x^2+y^2}\frac{\partial z}{\partial v}$$

Подставим в (4).

$$\left((x+y)\cdot\frac{x}{x^2+y^2} - (x-y)\cdot\frac{y}{x^2+y^2}\right)\frac{\partial z}{\partial u} - \left((x+y)\cdot\frac{y}{x^2+y^2} + (x-y)\frac{x}{x^2+y^2}\right)$$

$$\times\frac{\partial z}{\partial v} = 0$$

$$\frac{x^2+xy-xy+y^2}{x^2+y^2}\cdot\frac{\partial z}{\partial u} - \frac{xy+y^2+x^2-xy}{x^2+y^2}\frac{\partial z}{\partial v} = 0$$

$$\frac{\partial z}{\partial u} \quad \frac{\partial z}{\partial v} = 0$$

Получили уравнение, как в задаче 1.

Его общее решение: $z = \varphi(u+v)$,

Ответ: $z = \varphi\left(\ln\sqrt{x^2+y^2} + \operatorname{arctg}\frac{y}{x}\right).$

Задача 5. Преобразовать уравнение

(5) $(x-z)\cdot\frac{\partial z}{\partial x}+y\cdot\frac{\partial z}{\partial y}=0$, приняв x за функцию, а y и z за независимые перемен.

Решение: Напишем дифференциал dx:

$$dx=\frac{\partial x}{\partial z}\cdot dz+\frac{\partial x}{\partial y}\cdot dy=$$

$$=\frac{\partial x}{\partial z}\left(\frac{\partial z}{\partial x}\cdot dx+\frac{\partial z}{\partial y}dy\right)+\frac{\partial x}{\partial y}dy=$$

$$=\left(\frac{\partial x}{\partial z}\cdot\frac{\partial z}{\partial x}\right)dx+\left(\frac{\partial x}{\partial z}\cdot\frac{\partial z}{\partial y}+\frac{\partial x}{\partial y}\right)dy$$

Приравниваем коэффициенты при dx и dy:

$$1=\frac{\partial x}{\partial z}\cdot\frac{\partial z}{\partial x};\quad 0=\frac{\partial x}{\partial z}\cdot\frac{\partial z}{\partial y}+\frac{\partial x}{\partial y}$$

Отсюда: $\boxed{\frac{\partial z}{\partial x}=\frac{1}{\frac{\partial x}{\partial z}}}$; $\boxed{\frac{\partial z}{\partial y}=-\frac{\frac{\partial x}{\partial y}}{\frac{\partial x}{\partial z}}}$

Подставляем в ур (5):

$$\frac{x-z}{\frac{\partial x}{\partial z}}-y\cdot\frac{\frac{\partial x}{\partial y}}{\frac{\partial x}{\partial z}}=0 \Leftrightarrow \frac{\partial x}{\partial y}=\frac{x-z}{y}$$

Уравнение упростилось!

Задача 6. Преобразовать уравнение

(6) $(y-z)\cdot\frac{\partial z}{\partial x}+(y+z)\cdot\frac{\partial z}{\partial y}=0$, приняв x за функцию, $\boxed{u=y-z,}\ \boxed{v=y+z}$ - за новые независимые переменные

Решение: Напишем дифференциал ф-ии dx

$$dx=\frac{\partial x}{\partial u}\cdot du+\frac{\partial x}{\partial v}\cdot dv=$$

$$=\frac{\partial x}{\partial u}\left(dy-dz\right)+\frac{\partial x}{\partial v}\left(dy+dz\right)=$$

$$=\frac{\partial x}{\partial u}\left(dy-\frac{\partial z}{\partial x}dx-\frac{\partial z}{\partial y}dy\right)+\frac{\partial x}{\partial v}\left(dy+\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy\right)$$

$$=\left(-\frac{\partial x}{\partial u}\cdot\frac{\partial z}{\partial x}+\frac{\partial x}{\partial v}\cdot\frac{\partial z}{\partial x}\right)dx+\left(\frac{\partial x}{\partial u}-\frac{\partial x}{\partial u}\cdot\frac{\partial z}{\partial y}+\frac{\partial x}{\partial v}+\frac{\partial x}{\partial v}\frac{\partial z}{\partial y}\right)dy$$

Приравниваем коэффициенты при dx и dy:

$$1=-\frac{\partial x}{\partial u}\cdot\frac{\partial z}{\partial x}+\frac{\partial x}{\partial v}\cdot\frac{\partial z}{\partial x};\quad 0=\frac{\partial x}{\partial u}-\frac{\partial x}{\partial u}\frac{\partial z}{\partial y}+\frac{\partial x}{\partial v}+\frac{\partial x}{\partial v}\frac{\partial z}{\partial y}$$

Получаем:

$$\frac{\partial z}{\partial x}=\frac{1}{\frac{\partial x}{\partial v}-\frac{\partial x}{\partial u}};\quad \frac{\partial z}{\partial y}=-\frac{\frac{\partial x}{\partial u}+\frac{\partial x}{\partial v}}{\frac{\partial x}{\partial v}-\frac{\partial x}{\partial u}}$$

подставляем в уравнение (6):

$$(y-z)\frac{\partial z}{\partial x}+(y+z)\frac{\partial z}{\partial y}=\frac{u}{\frac{\partial x}{\partial v}-\frac{\partial x}{\partial u}}-v\cdot\frac{\frac{\partial x}{\partial u}+\frac{\partial x}{\partial v}}{\frac{\partial x}{\partial v}-\frac{\partial x}{\partial u}}=0$$

Общий знаменатель!

$$\frac{\partial x}{\partial u}+\frac{\partial x}{\partial v}=\frac{u}{v}\quad\left(v\neq 0,\ \frac{\partial x}{\partial v}\neq\frac{\partial x}{\partial u}\right)$$

Уравнение упростилось, стало линейным.

Задача 7. Решить уравнение, сделав замену переменных: $\boxed{\xi = x,}\ \boxed{\eta = y - x,}\ \boxed{\zeta = z - x}$

$$(7)\quad \frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$$

Решение: дифференцируем как сложную ф.

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial x}+\frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial x}+\frac{\partial u}{\partial \zeta}\cdot\frac{\partial \zeta}{\partial x}=$$

$$=\frac{\partial u}{\partial \xi}-\frac{\partial u}{\partial \eta}-\frac{\partial u}{\partial \zeta}$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial \eta};\quad \frac{\partial u}{\partial z}=\frac{\partial u}{\partial \zeta}$$

Подставляем в уравнение (7)

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{\partial u}{\partial \xi}=0 \Rightarrow$$ u – не зависит от ξ,

т.е. $u = f(\eta, \zeta)$

Ответ: $u = f(y - x,\ z - x)$.

Решим уравнение из задачи 6:

<u>Задача 8.</u> $\frac{\partial x}{\partial u}+\frac{\partial x}{\partial v}=\frac{u}{v}$

<u>Решение:</u> Сделаем замену переменных

$\xi = u, \quad \eta = v-u$

$$\frac{\partial x}{\partial u}=\frac{\partial x}{\partial \xi}\cdot\frac{\partial \xi}{\partial u}+\frac{\partial x}{\partial \eta}\cdot\frac{\partial \eta}{\partial u}=\frac{\partial x}{\partial \xi}-\frac{\partial x}{\partial \eta}$$

$$\frac{\partial x}{\partial v}=\frac{\partial x}{\partial \eta}\cdot\frac{\partial \eta}{\partial v}+\frac{\partial x}{\partial \xi}\cdot\frac{\partial \xi}{\partial v}=\frac{\partial x}{\partial \eta}$$

Подставляем в уравнение:

$$\frac{\partial x}{\partial u}+\frac{\partial x}{\partial v}=\frac{\partial x}{\partial \xi}-\frac{\partial x}{\partial \eta}+\frac{\partial x}{\partial \eta}=\frac{\partial x}{\partial \xi}=\frac{u}{v}$$

$$\frac{\partial x}{\partial \xi}=\frac{u}{v}=\frac{\xi}{\xi+\eta}=1-\frac{\eta}{\xi+\eta}$$

$\frac{\partial x}{\partial \xi}=1-\frac{\eta}{\xi+\eta}$ - интегрируем по ξ:

$x=\xi-\eta\cdot\ln(\xi+\eta)+\varphi(\eta)$ т.е.

$$\boxed{x=u-(v-u)\cdot\ln v+\varphi(v-u)}$$

Тогда решение задачи 6 из уравн.:
$u=y-z, \quad v=y+z, \quad v-u=2z$

$$x=(y-z)-(2z)\cdot\ln(y+z)+\varphi(2z)$$

Неявное уравнение!

Задача 9. Преобразовать к полярным координатам r и φ, полагая $x = r\cos\varphi$, $y = r\sin\varphi$ следующие выражения:

а) $w = x\cdot\dfrac{\partial u}{\partial y} - y\cdot\dfrac{\partial u}{\partial x}$; б) $w = x\cdot\dfrac{\partial u}{\partial x} + y\cdot\dfrac{\partial u}{\partial y}$

в) $w = \left(\dfrac{\partial u}{\partial x}\right)^2 + \left(\dfrac{\partial u}{\partial y}\right)^2$.

Решение: Имеем:

$$\boxed{\frac{\partial u}{\partial x}} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \varphi}\cdot\frac{\partial \varphi}{\partial x}; \quad \boxed{\frac{\partial u}{\partial y}} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \varphi}\cdot\frac{\partial \varphi}{\partial y}$$

Чтобы найти $\dfrac{\partial \varphi}{\partial x}$, $\dfrac{\partial \varphi}{\partial y}$, $\dfrac{\partial r}{\partial x}$, $\dfrac{\partial r}{\partial y}$ дифференцируем тождества

1. $\boxed{x = r\cos\varphi}$, 2. $\boxed{y = r\sin\varphi}$ по x и y:

$$1 = \cos\varphi\cdot\frac{\partial r}{\partial x} - r\sin\varphi\frac{\partial \varphi}{\partial x}; \quad 0 = \cos\varphi\cdot\frac{\partial r}{\partial y} - r\sin\varphi\cdot\frac{\partial \varphi}{\partial y}$$

$$0 = \sin\varphi\cdot\frac{\partial r}{\partial x} + r\cos\varphi\cdot\frac{\partial \varphi}{\partial x}; \quad 1 = \sin\varphi\cdot\frac{\partial r}{\partial y} + r\cos\varphi\cdot\frac{\partial \varphi}{\partial y}$$

Откуда находим:

$$\frac{\partial r}{\partial x} = \cos\varphi, \quad \frac{\partial \varphi}{\partial x} = -\frac{\sin\varphi}{r}$$

$$\frac{\partial r}{\partial y} = \sin\varphi, \quad \frac{\partial \varphi}{\partial y} = \frac{\cos\varphi}{r}$$

Тогда: $\boxed{\dfrac{\partial u}{\partial x}} = \dfrac{\partial u}{\partial r}\cdot\cos\varphi - \dfrac{\partial u}{\partial \varphi}\,\dfrac{\sin\varphi}{r}$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\sin\varphi + \frac{\partial u}{\partial \varphi}\cdot\frac{\cos\varphi}{r}$$

Подставляем в выражения для w:

a) $$w = r\cos\varphi\left(\cancel{\frac{\partial u}{\partial r}\sin\varphi} + \frac{\partial u}{\partial \varphi}\frac{\cos\varphi}{r}\right) - r\sin\varphi\left(\cancel{\frac{\partial u}{\partial r}\cos\varphi} - \frac{\partial u}{\partial \varphi}\frac{\sin\varphi}{r}\right) =$$

$$= \frac{\partial u}{\partial \varphi}$$

б) $$w = r\cos\varphi\left(\frac{\partial u}{\partial r}\cos\varphi - \cancel{\frac{\partial u}{\partial \varphi}\cdot\frac{\sin\varphi}{r}}\right) + r\sin\varphi\left(\frac{\partial u}{\partial r}\cdot\sin\varphi + \cancel{\frac{\partial u}{\partial \varphi}\frac{\cos\varphi}{r}}\right) =$$

$$= r\cdot\frac{\partial u}{\partial r}$$

в) $$w = \left(\frac{\partial u}{\partial r}\cos\varphi - \frac{\partial u}{\partial \varphi}\cdot\frac{\sin\varphi}{r}\right)^2 + \left(\frac{\partial u}{\partial r}\sin\varphi + \frac{\partial u}{\partial \varphi}\cdot\frac{\cos\varphi}{r}\right)^2 =$$

$$= \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\cdot\left(\frac{\partial u}{\partial \varphi}\right)^2$$

Задача 10 Решить уравнение

$$(xy+z)\frac{\partial z}{\partial x} + (1-y^2)\frac{\partial z}{\partial y} = x + yz,$$

сделав замену переменных и функции:

$$u = yz - x, \quad v = xz - y, \quad w = xy - z$$

Решение:

$$du = y\,dz + z\,dy - dx$$

$$dv = x\,dz + z\,dx - dy$$

$$dw = x\,dy + y\,dx - dz$$

$$dw = \frac{\partial w}{\partial u}\cdot du + \frac{\partial w}{\partial v}\cdot dv$$

Подставим:

$$x\,dy + y\,dx - dz = \frac{\partial w}{\partial u}\left(y\,dz + z\,dy - dx\right) + \frac{\partial w}{\partial v}\left(x\,dz + z\,dx - dy\right)$$

Собираем вместе коэфф при dz, dx, dy:

$$\left(\frac{\partial w}{\partial u}\cdot y + \frac{\partial w}{\partial v}\cdot x + 1\right)dz = \left(\frac{\partial w}{\partial u} - \frac{\partial w}{\partial v}\cdot z + y\right)dx + \left(x + \frac{\partial w}{\partial v} - \frac{\partial w}{\partial u}\cdot z\right)dy$$

т.е. $$dz = \frac{\frac{\partial w}{\partial u} - \frac{\partial w}{\partial v}\cdot z + y}{\frac{\partial w}{\partial u}\cdot y + \frac{\partial w}{\partial v}\cdot x + 1}\,dx + \frac{\frac{\partial w}{\partial v} - \frac{\partial w}{\partial u}z + x}{\frac{\partial w}{\partial u}y + \frac{\partial w}{\partial v}x + 1}\,dy$$

Отсюда:

$$\frac{\partial z}{\partial x} = \frac{\frac{\partial w}{\partial u} - \frac{\partial w}{\partial v}z + y}{\frac{\partial w}{\partial u}\cdot y + \frac{\partial w}{\partial v}x + 1}; \quad \frac{\partial z}{\partial y} = \frac{\frac{\partial w}{\partial v} - \frac{\partial w}{\partial u}\cdot z + x}{\frac{\partial w}{\partial u}\cdot y + \frac{\partial w}{\partial v} + 1}$$

Подставляем эти производные в уравнение:

$$\frac{(xy+z)\left(\frac{\partial w}{\partial u}\cdot-\frac{\partial w}{\partial v}z+y\right)}{\frac{\partial w}{\partial u}\cdot y+\frac{\partial w}{\partial v}\cdot x+1}+\frac{(1-y^2)\left(\frac{\partial w}{\partial v}-\frac{\partial w}{\partial u}z+x\right)}{\frac{\partial w}{\partial u}\cdot y+\frac{\partial w}{\partial v}x+1}=x+yz$$

После приведения подобных членов остается уравнение:

$$\frac{\partial w}{\partial v}=0.$$

т.е. $w=\varphi(u)$ – общее решение

<u>Ответ:</u> $\boxed{xy-z=\varphi(yz-x)}$

Получили неявную функцию.