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Borel Equivalence Relations
Structure and Classification

Vladimir Kanovei

American Mathematical Society

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Preface

Classification problems for different types of mathematical structures have been in the center of interests in descriptive set theory during the last 15–20 years. Assume that X is a class of mathematical structures identified modulo an equivalence relation E. This can be, for example, countable groups modulo the isomorphism relation, unitary operators over a fixed space $\mathbb{C}^n$ modulo conjugacy, probability measures over a fixed Polish space modulo identification of measures having the same null sets, or, for instance, reals modulo Turing reducibility.[1] Suppose that Y is another class of mathematical structures identified modulo an equivalence relation F. The classification problem is then to find out whether there is a *definable*, or *effective* injection $\Theta : X/\mathsf{E} \to Y/\mathsf{F}$. Such a map Θ is naturally considered as a classification of objects in X in terms of objects in Y, in a way that respects the quotient structure over E and F, respectively. The existence of such a map can be a result of high importance, especially when objects in Y are of simpler mathematical nature than those in X.

In many cases, it turns out that the classes of structures X and Y can be considered as Borel sets in Polish (that is, separable, completely metrizable) spaces, so that E, F become Borel (as sets of pairs) or, more generally, analytic relations, while reduction maps are usually required to be Borel.[2] Then the problem can be studied by methods of descriptive set theory, where it takes the following form: if E, F are Borel (or more complicated) equivalence relations on Polish sets, resp., X, Y, does there exist a Borel *reduction* of E to F (that is, a Borel map $\vartheta : X \to Y$) satisfying

$$x \mathrel{\mathsf{E}} x' \iff \vartheta(x) \mathrel{\mathsf{F}} \vartheta(x') : \quad \text{for all } x, x' \in X?$$

If such a map ϑ exists, then E is said to be *Borel reducible* to F, symbolically

$$\mathsf{E} \le_{\mathrm{B}} \mathsf{F}.$$

Then an injection $\Theta : X/\mathsf{E} \to Y/\mathsf{F}$ can be defined by simply putting $\Theta([x]_{\mathsf{E}}) = [\vartheta(x)]_{\mathsf{F}}$, where $[x]_{\mathsf{E}}$ is the E-equivalence class of x. The *Borel equivalence* or *bi-reducibility* $\sim_{\mathrm{B}}$ and *strict reducibility* $<_{\mathrm{B}}$ are naturally introduced so that

$$\mathsf{E} \sim_{\mathrm{B}} \mathsf{F} \quad \text{iff} \quad \text{both } \mathsf{E} \le_{\mathrm{B}} \mathsf{F} \text{ and } \mathsf{F} \le_{\mathrm{B}} \mathsf{E}, \quad \text{and}$$
$$\mathsf{E} <_{\mathrm{B}} \mathsf{F} \quad \text{iff} \quad \mathsf{E} \le_{\mathrm{B}} \mathsf{F} \text{ but } \neg\, \mathsf{F} \le_{\mathrm{B}} \mathsf{E}.$$

The study of Borel and other effective equivalence relations under Borel reducibility by methods of descriptive set theory revealed a remarkable structure of

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[1] The examples are taken from HJORTH's book [**Hjo00b**] and KECHRIS' survey paper [**Kec99**], where many more examples of this type are given.

[2] That is, maps with Borel graphs. Baire measurable maps and reductions satisfying certain algebraic requirements are also applied [**Far00**] as well as $\mathbf{\Delta}^1_2$ and more complicated reductions [**Hjo00b, Kan98**], however they are not systematically considered in this book.

mutual $\leq_B$-reducibility and $\leq_B$-irreducibility of Borel equivalence relations of different types. This book presents a selection of basic Borel reducibility/irreducibility results in this area.

Originally, these were informal notes for a short course on Borel reducibility and dichotomy theorems given at Universität Bonn in Winter 2000/2001 for graduate and undergraduate students specializing in set theory. The first purpose of the notes was to give a self-contained treatment of several dichotomy theorems in the theory of reducibility of Borel equivalence relations, mainly those obtained in 1990s, in a form as unified as generally would be possible. This original rather short version was deposited at **arXiv** under the title *Varia, ideals and equivalence relations*. But pursuing the goal of self-containedness, the text has been gradually increased in size about three times with respect to the very first version. The last addition is a brief technical introduction into classical and effective descriptive theory.

Still the book does *not* contain much on such topics as Polish and Borel groups and their actions, ergodic theory, model theory, the structure of countable and hyperfinite equivalence relations, in relation to which one may be advised to study BECKER and KECHRIS [**BK96**] or the recent monographs of HJORTH [**Hjo00b**] and KECHRIS and MILLER [**KM04**].

The prospective reader should have some degree of experience with modern descriptive set theory, including Borel sets and methods of *effective* descriptive set theory. Some knowledge of *forcing* is expected as well, because the Cohen forcing for Polish spaces and the Gandy–Harrington forcing is the *sine qua non* for several of the most important arguments in this book.

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Introduction

For the convenience of the reader, we give an informal resumé of the content, without going into technical details.

Chapters 1 and 2 contain a brief introduction into descriptive set theory. Here we do not aim to give a really systematic, broad introduction, but rather a technical introduction with basic definitions, some comments, and proofs of several principal theorems of both classical and effective descriptive set theory—exactly those theorems used in the remainder of the book.

We continue in **Chapters 3 and 4** with introductory material on ideals and equivalence relations. We discuss several types of ideals on $\mathbb{N}$, and we sketch SOLECKI's proof that P-ideals, Borel polishable ideals, and the tailmeasure ideals of LSC submeasures on $\mathbb{N}$ are one and the same. Each Borel ideal $\mathscr{I}$ on a set A generates a Borel equivalence relation $\mathsf{E}_{\mathscr{I}}$ on $\mathscr{P}(A)$ such that $x \mathrel{\mathsf{E}_{\mathscr{I}}} y$ iff the symmetric difference $x \mathbin{\triangle} y$ belongs to $\mathscr{I}$, for all $x, y \subseteq A$. Some other Borel and analytic equivalence relations and important families of them are defined, for instance:

- the equalities Δ_X on Borel sets X,
- the equivalence relation E_0 of equality of infinite binary sequences (elements of $2^{\mathbb{N}}$) modulo a finite number of terms,
- the equivalence relation E_1 of equality of infinite sequences in $(2^{\mathbb{N}})^{\mathbb{N}}$ modulo a finite number of terms,
- the equivalence relation E_3 defined on the set $(2^{\mathbb{N}})^{\mathbb{N}}$ so that $x \mathrel{\mathsf{E}_3} y$ iff $x(k) \mathrel{\mathsf{E}_0} y(k)$ for all k,
- the "summable" equivalence relation E_2 defined on the set $2^{\mathbb{N}}$ so that $a \mathrel{\mathsf{E}_2} b$ iff $\sum_{k \in a \triangle b} \frac{1}{k} < +\infty$, where $a \mathbin{\triangle} b$ is the set of all $k \geq 1$ such that $a(k) \neq b(k)$,
- the density-0 equivalence relation Z_0 defined on $2^{\mathbb{N}}$ so that $a \mathrel{\mathsf{Z}_0} b$ iff $\lim_{n \to \infty} \frac{\#([0,n) \cap (a \triangle b))}{n} = 0$,
- the equivalence relations ℓ^p induced on $\mathbb{R}^{\mathbb{N}}$ by the natural (componentwise) action of additive groups of corresponding Banach spaces,
- the equivalence relation T_2 defined on the set $\mathbb{R}^{\mathbb{N}}$ so that $x \mathrel{\mathsf{T}_2} y$ iff
$$\{x(k) : k \in \mathbb{N}\} = \{y(k) : k \in \mathbb{N}\}$$
and hence called *the equality of the countable sets of reals*,
- *countable* Borel equivalence relations (those in which each equivalence class is at most countable), and in particular E_∞, the $\leq_{\mathrm{B}}$-largest one in this family,

and some others. Some of these equivalence relations are especially interesting because of their connection with certain large classes of Borel equivalence relations. For instance E_0 is $\le_{\text{B}}$-largest in the family of *hyperfinite* equivalence relations, E_∞ is $\le_{\text{B}}$-largest in the family of Borel countable equivalence relations, T_2 is connected with various isomorphism relations of countable structures, and so on.

Chapter 5 introduces the notion of Borel reducibility and presents a diagram of Borel reducibility of some key equivalence relations (in particular those defined above). The diagram, Figure 1 on page 68, begins with equalities on finite, countable, and continuum size Borel sets. This linearly ordered part ends with E_0, above which the linearity breaks. There exist at least two $\le_{\text{B}}$-incomparable distinguished Borel equivalence relations, $\le_{\text{B}}$-minimal above E_0, namely E_1 and E_3, and perhaps also E_2. Less studied higher levels are most likely even more complicated. The main content of this book consists of the proofs of different reducibility/irreducibility results related to the diagram. These main results are formulated and briefly commented upon in Section 5.6.

Another group of theorems consists of *dichotomy theorems* presented in Section 5.7. One of them, the 1st dichotomy theorem of SILVER (Theorem 5.7.1), asserts that any Borel equivalence relation E satisfies one of two (obviously incompatible) requirements $\mathsf{E} \le_{\text{B}} \Delta_{\mathbb{N}}$ or $\Delta_{2^{\mathbb{N}}} \le_{\text{B}} \mathsf{E}$. The 2nd dichotomy (Theorem 5.7.2) of HARRINGTON, KECHRIS, and LOUVEAU asserts that any Borel equivalence relation E satisfies one of two (incompatible) requirements $\mathsf{E} \le_{\text{B}} \Delta_{2^{\mathbb{N}}}$ or $\mathsf{E}_0 \le_{\text{B}} \mathsf{E}$. Three more dichotomy theorems clarify the structure of $\le_{\text{B}}$-intervals between E_0 and one of the relations E_1, E_2, E_3.

Chapter 6 contains proofs of several assorted reducibility/irreducibility theorems whose only common property is the rather elementary character of their proofs in the sense that only quite standard methods of real analysis and topology are involved. But some of the results are really tricky, in particular, the $\le_{\text{B}}$-incomparability of E_2 and the density-0 equivalence relation Z_0, or the HJORTH–DOUGHERTY theorem that shows that ℓ^p is Borel reducible to ℓ^q iff $p \le q$.

The following **Chapter 7** is devoted to the class of countable Borel equivalence relations. Studies of recent years demonstrated that this is an extremely rich family of equivalence relations. Among others, it includes hyperfinite equivalence relations, a comparably elementary type among countable Borel ones. It turns out that all countable (Borel) equivalence relations are induced by Borel actions of countable groups (Theorem 7.4.1) Another theorem (Theorem 7.5.1) shows that not all countable Borel equivalence relations are hyperfinite, in particular, E_∞ is not hyperfinite. We also prove a useful result (Theorem 7.3.1) on σ-additivity of the notions of smoothness and hyperfiniteness as functions of Borel domains.

In the next **Chapter 8** we consider the class of *hyperfinite* equivalence relations. It admits several different but equivalent characterizations, for instance, being induced by a Borel action of the additive group of the integers $\mathbb{Z}$ or being induced by a Borel action of the group $\langle \mathscr{P}_{\mathtt{fin}}(\mathbb{N})\,; \triangle \rangle$ of finite subsets of $\mathbb{N}$ with the symmetric difference as the group operation. Theorem 8.1.1 in **Chapter 8** proves the equivalence of several known characterizations. Some other theorems on hyperfinite equivalence relations (like the hyperfiniteness of the tail equivalence relations or the classification modulo Borel isomorphism between the domains) are discussed at the end of the chapter.

We come back to non-hyperfinite Borel countable equivalence relations in **Chapter 9**, where some modern results in this area are presented, mainly without proofs, in particular those related to amenable and treeable equivalence relations, as well as those induced by free actions of certain groups. It must be said that all known proofs of modern results on countable equivalence relations are based on advanced techniques of egrodic theory to the extent that makes it inappropriate to present any such proofs in this book.

Chapter 10 contains proofs of the 1st and 2nd dichotomy theorems. The key ingredient of the proofs is the *Gandy–Harrington forcing*, a technique based on the topology generated by non-empty Σ^1_1 sets. Here methods of effective descriptive set theory play an essential role. We also consider a forcing associated to the equivalence relation E_0: it consists of all E_0-large Borel sets $X \subseteq 2^{\mathbb{N}}$, that is, such that $\mathsf{E}_0 \restriction X$ is not smooth.

Chapter 11 is devoted to the equivalence relation E_1 and the corresponding ideal $\mathscr{I}_1$ of all sets $x \subseteq 2^{\mathbb{N}} \times 2^{\mathbb{N}}$ such that $x \subseteq \{0, 1, \ldots, n\} \times \mathbb{N}$ for some n. The most important property of E_1 is that it is not "polishable"; that is, it does not belong to the family of equivalence relations induced by Borel actions of Polish groups (Theorem 11.8.1). It has been conjectured and verified in some important particular cases that E_1 is a $\le_{\mathrm{B}}$-least equivalence relation among non-"polishable" equivalence relations. For instance, the conjecture is true for equivalence relations of the form $\mathsf{E}_{\mathscr{I}}$. This result is based on SOLECKI's theorem on characterization of polishable ideals (Theorem 3.5.1).

Here we prove the 3rd dichotomy (Theorem 5.7.3, in the form of Theorem 11.3.1) of KECHRIS and LOUVEAU: it asserts that any Borel equivalence relation E such that $\mathsf{E}_0 \le_{\mathrm{B}} \mathsf{E} \le_{\mathrm{B}} \mathsf{E}_1$ satisfies either $\mathsf{E}_0 \sim_{\mathrm{B}} \mathsf{E}$ or $\mathsf{E} \sim_{\mathrm{B}} \mathsf{E}_1$, and hence the strict $<_{\mathrm{B}}$-interval between E_0 and E_1 is empty. Borel ideals admit an even stronger result (Theorem 11.1.1 of KECHRIS): if $\mathsf{E}_{\mathscr{I}} \le_{\mathrm{B}} \mathsf{E}_1$ then the ideal $\mathscr{I}$ is Borel isomorphic to exactly one of the ideals $\mathscr{I}_1$, $\mathtt{Fin}$ (finite subsets of $\mathbb{N}$), or the product of $\mathtt{Fin}$ and the (trivial) ideal $\mathscr{P}(\mathbb{N})$ of all subsets of $\mathbb{N}$.

Chapter 12 considers equivalence relations induced by Borel actions of the group S_∞ of all permutations of $\mathbb{N}$. This group includes, for instance, various isomorphism relations of countable structures. In particular we prove (Theorem 12.5.2) that any Borel equivalence relation E, Borel reducible to a Polish action of S_∞, satisfies $\mathsf{E} \le_{\mathrm{B}} \mathsf{T}_\xi$ for some countable ordinal ξ, where $\{\mathsf{T}_\xi\}_{\xi<\omega_1}$ is H. FRIEDMAN's $<_{\mathrm{B}}$-increasing transfinite sequence of Borel equivalence relations. The next **Chapter 13** on *turbulence* makes use of this result.

Turbulent, or, more exactly, *generically* turbulent group actions are characterized by the property that almost all, in the sense of the Baire category, orbits, and even local orbits of the action are somewhere dense. This property separates a class of equivalence relations very different from those induced by actions of S_∞. Extending, in a certain direction, HJORTH's results on turbulence, we prove (Theorem 13.5.3) that generically turbulent Borel equivalence relations are not Borel reducible to equivalence relations in a large family $\mathscr{F}_0$ of all equivalence relations that can be obtained from $\Delta_{\mathbb{N}}$ (the equality on $\mathbb{N}$) by countable transfinite iterations of the operations of countable power, Fubini product, and some other. Note that not all equivalence relations in $\mathscr{F}_0$ are "polishable"; for instance, E_1, a non-"polishable" one, belongs to this family.

Chapter 14 contains one principal result: the 6th dichotomy (Theorem 5.7.6, in the form of Theorem 14.2.1) of HJORTH and KECHRIS, saying that the strict $<_{\mathrm{B}}$-interval between E_0 and E_3 is empty, similar to the interval between E_0 and E_1 by the 3rd dichotomy theorem. The proof still involves the Gandy–Harrington forcing, but the splitting construction is different and slightly more complicated than the one applied in the proof of the 3rd dichotomy.

The $<_{\mathrm{B}}$-interval between E_0 and the "summable" equivalence relation E_2 is not known yet to be empty, athough it is expected to be such. However, the 4th dichotomy (Theorem 5.7.4, in the form of Theorem 15.2.1) of HJORTH in **Chapter 15** significantly restricts the domain $<_{\mathrm{B}}$-below E_2 to countable Borel equivalence relations. The proof makes use of another splitting construction based on the Gandy–Harrington forcing.

The next **Chapter 16** presents $\mathbf{c}_0$-equalities—a family of Borel equivalence relations similar to the density-0 equivalence relation Z_0. This family was extensively studied by FARAH, LOUVEAU, and VELICKOVIC. In particular, it was found that it contains a continuum size subfamily of pairwise $\leq_{\mathrm{B}}$-incompatible equivalence relations (Theorem 16.6.3).

The problem considered in **Chapter 17** has the opposite character with respect to the results in Chapter 13. We introduce a family of *pinned* (Borel or analytic) equivalence relations E—those satisfying the property that in any generic extension of the universe every stable E-class contains an element of the ground universe. This family contains, for instance, all orbit equivalence relations of Polish actions of complete left-invariant groups, all Borel equivalence relations with $\boldsymbol{\Sigma}^0_3$ equivalence classes, some turbulent equivalence relations, and many more. The most notable example of a non-pinned equivalence relation is T_2, the equality of countable sets of reals, for which the non-pinned stable class consists of all $x \in (2^{\mathbb{N}})^{\mathbb{N}}$ in the extension such that the set $\{x(n) : n \in \mathbb{N}\}$ is equal to the set of all $a \in 2^{\mathbb{N}}$ in the ground model. Theorem 17.1.3 proves that non-pinned Borel equivalence relations, in particular, T_2, are not Borel reducible to pinned ones.

The final **Chapter 18** presents a recent theorem due to ROSENDAL on the cofinality of Borel ideals in the $\leq_{\mathrm{B}}$-structure of Borel equivalence relations of general form. In other words, for any Borel equivalence relation E there exists a Borel ideal $\mathscr{I}$ such that $\mathsf{E} \leq_{\mathrm{B}} \mathsf{E}_{\mathscr{I}}$. A $\leq_{\mathrm{B}}$-cofinal ω_1-sequence of Borel equivalence relations of the form $\mathsf{E}_{\mathscr{I}}$ is defined.

For the convenience of the reader, an appendix (**Appendix A**) is added on some issues related to forcing. It explains the setup and basic terminology of forcing in this book, and it discusses important details related to Cohen and Gandy–Harrington forcing.

General set-theoretic notation used in this book.

- $\mathbb{N} = \{0, 1, 2, \ldots\}$: natural numbers; $\mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$.
- $X \subseteq Y$ iff $\forall x\,(x \in X \Longrightarrow x \in Y)$: *the inclusion*, and this holds also in the case when $X = Y$.
- $X \subsetneqq Y$ or sometimes $X \subset Y$ means that $X \subseteq Y$ but $Y \not\subseteq X$: *strict inclusion*.
- $X \subseteq^* Y$ means that the difference $X \smallsetminus Y$ is finite.
- $\{x : \Phi(x)\}$ is the set (or class) of all sets x such that $\Phi(x)$.
- If $X \subseteq A \times B$ and $a \in A$, then $(X)_a = \{b : \langle a, b\rangle \in X\}$, a *cross-section*.
- $\mathtt{card}\, X$ is the cardinality of a set X, equal to the number of elements of X whenever X is finite.
- $\mathtt{dom}\, P = \{x : \exists y\, (\langle x, y\rangle \in P)\}$ and $\mathtt{ran}\, P = \{y : \exists x\, (\langle x, y\rangle \in P)\}$ are the *domain* and *range* of any set P that consists of pairs.
- In particular, if $P = f$ is a function, then $\mathtt{dom}\, f$ and $\mathtt{ran}\, f$ are the domain and the range of f.
- Functions are routinely identified with their graphs; that is, if $P = f$ is a function, then $f = \{\langle x, f(x)\rangle : x \in \mathtt{dom}\, f\}$, so that $y = f(x)$ is equivalent to $\langle x, y\rangle \in f$.
- Geometrically, if $P \subseteq \mathbb{X} \times \mathbb{Y}$, where $\mathbb{X}$ and $\mathbb{Y}$ are Polish spaces, then $\mathrm{pr}\, P = \{x : \exists y\, (\langle x, y\rangle \in P)\}$, *the projection*, will sometimes be used instead of $\mathtt{dom}\, P$, but in fact $\mathtt{dom}\, P = \mathrm{pr}\, P$.
- $f[X] = \{f(x) : x \in X \cap \mathtt{dom}\, f\}$, the *$f$-image* of X.
- $f^{-1}[Y] = \{x \in \mathtt{dom}\, f : f(x) \in Y\}$, the *$f$-preimage* of Y.
- $\triangle$ is the symmetric difference.
- $\exists^\infty x \ldots$ means "there exist infinitely many x such that $\ldots$",
  $\forall^\infty x \ldots$ means "for all but finitely many $x \cdots$ holds".
- $\{x_a\}_{a \in A}$ is the map f defined on A by $f(a) = x_a$, $\forall a$.
- $\mathscr{P}(X) = \{x : x \subseteq X\}$ and $\mathscr{P}_{\mathtt{fin}}(X) = \{x : x \subseteq X \text{ is finite}\}$.
- $\varnothing$ is the empty set, Λ is the empty sequence; basically, $\varnothing = \Lambda$.
- $X^{<\omega}$ is the set of all finite sequences of elements of a given set X.
- In particular $2^{<\omega} \subsetneqq \mathbb{N}^{<\omega}$ denote, respectively, the set of all finite sequences of numbers $0, 1$ and the set of all finite sequences of natural numbers.
- $\mathtt{lh}\, s$ is the *length* of a finite sequence s.
- If x is any set, then $s^\frown x$ is the sequence obtained by adjoining x as the right-most term to a given finite sequence s.
- $s \subset t$ means that the sequence t is a proper extension of s.

CHAPTER 1

Descriptive set theoretic background

Generally speaking, we assume that the reader of this book has some knowledge of both classical and effective descriptive set theory in Polish spaces, including Borel and projective hierarchy, Borel sets and functions, analytic ($\boldsymbol{\Sigma}^1_1$) and coanalytic ($\boldsymbol{\Pi}^1_1$) sets, basic notions of effective descriptive set theory, and the like. Such a reader can skip this chapter or give it a surface scan. Here we introduce the projective hierarchy of pointsets of the Baire space $\mathbb{N}^{\mathbb{N}}$ and product spaces of the form $\mathbb{N}^k \times (\mathbb{N}^{\mathbb{N}})^\ell$, and a finer effective hierarchy based on the definability of pointsets by analytic formulas.

1.1. Polish spaces

A *Polish space* is a topological space that admits a compatible complete separable metric. In other words, it is required that there exists a complete separable metric that induces the given topology. As a matter of fact, such a metric is usually presented explicitly for typical Polish spaces considered, so one can say that a Polish space is simply a complete separable metric space. Nevertheless, most important structures associated with Polish spaces are built upon their topologies without a direct reference to any particular complete separable metric space.

EXAMPLE 1.1.1. The set $\mathbb{N} = \{0, 1, 2, \dots\}$ of all natural numbers is a Polish space with discrete metrics. □

EXAMPLE 1.1.2. The *Baire space* $\mathbb{N}^{\mathbb{N}}$ consists of all infinite sequences of natural numbers. The distance

$$\rho(x, y) = \frac{1}{n} \text{ for } x \neq y \in \mathbb{N}^{\mathbb{N}}, \quad \text{where} \quad n = \mathtt{min}\{n : x(n) \neq y(n)\},$$

converts $\mathbb{N}^{\mathbb{N}}$ into a Polish space. The induced topology is the same as the product topology on $\mathbb{N}^{\mathbb{N}}$ with the discrete topology on each copy of $\mathbb{N}$. The sets of the form

$$\mathscr{O}_s(\mathbb{N}^{\mathbb{N}}) = \{x \in \mathbb{N}^{\mathbb{N}} : s \subset x\}, \quad \text{where} \quad s \in \mathbb{N}^{<\omega},$$

that is, s a finite sequence of natural numbers, are basic clopen nbhds in $\mathbb{N}^{\mathbb{N}}$. □

EXAMPLE 1.1.3. The *Cantor discontinuum* $2^{\mathbb{N}}$ consists of all infinite dyadic sequences with the same distance as $\mathbb{N}^{\mathbb{N}}$. This is a Polish space, actually a closed set in $\mathbb{N}^{\mathbb{N}}$. The sets of the form

$$\mathscr{O}_s(2^{\mathbb{N}}) = \{x \in 2^{\mathbb{N}} : s \subset x\}, \quad \text{where} \quad s \in 2^{<\omega},$$

that is, s a finite sequence of numbers $0, 1$, are basic clopen nbhds in $2^{\mathbb{N}}$.

The power set $\mathscr{P}(\mathbb{N}) = \{x : x \subseteq \mathbb{N}\}$ is commonly identified with $2^{\mathbb{N}}$ by means of identification of every set $x \subseteq \mathbb{N}$ with its *characteristic function* $\chi_x \in 2^{\mathbb{N}}$. The same Polish topology on $\mathscr{P}(\mathbb{N})$ can be generated by sets of the form $\{x \subseteq \mathscr{P}(\mathbb{N}) : x \cap [0, n) = u\}$, where $n \in \mathbb{N}$ and $u \subseteq [0, n)$. □

Blanket Agreement 1.1.4. Elements of $\mathbb{N}^{\mathbb{N}}$ are routinely called *reals* in modern descriptive set theoretic publications, and we will follow this practice in some cases. In those few cases in this book where the true reals (members of the real line $\mathbb{R}$) are considered, we will make a clear distinction. □

Example 1.1.5. All spaces of the form $\mathbb{N}^k \times (\mathbb{N}^{\mathbb{N}})^\ell$ are Polish. They will be called *product spaces*. If $\ell = 0$, then the space $\mathbb{N}^k \times (\mathbb{N}^{\mathbb{N}})^\ell = (\mathbb{N})^k$ is discrete; otherwise, it is homeomorphic to $\mathbb{N}^{\mathbb{N}}$. □

Let us introduce a useful system of bijections between product spaces.

Definition 1.1.6 (Bijections). (i) Put $\pi(i,j) = 2^i(2j+1)-1$; thus π is a bijection of $\mathbb{N}^2$ onto $\mathbb{N}$. Then define $\pi(i_1,\dots,i_k,i_{k+1}) = \pi(\pi(i_1,\dots,i_k), i_{k+1})$ by induction on k, so that in each arity $k \geq 2$, the map π is a bijection of $\mathbb{N}^k$ onto $\mathbb{N}$. This allows us to define a system of inverse maps $(m)^k_i$, where $i < k \geq 2$, so that $\pi((m)^k_0, (m)^k_1, \dots, (m)^k_{k-1}) = m$ for all $k \geq 2$ and m. In particular $(m)^2_0 = i$ and $(m)^2_1 = j$ iff $m = \pi(i,j) = 2^i(2j+1)-1$.

In addition define $(m)^1_0 = m$ and $(m)^k_i = 0$ in the "wrong" case $k \leq i$.

(ii) We define an enumeration $\mathbb{N}^{<\omega} = \{\mathsf{s}_n : n \in \mathbb{N}\}$ of the set $\mathbb{N}^{<\omega}$ of all finite sequences of natural numbers as follows. If $n > 0$, then let

$$n' = (n)^2_0, \quad n'' = (n)^2_1 + 1, \quad \mathsf{s}_n = \langle (n')^{n''}_0, (n')^{n''}_1, \dots, (n')^{n''}_{n''-1} \rangle .$$

Separately, put $\mathsf{s}_0 = \Lambda$, the empty sequence. The enumeration just defined satisfies the following requirements: $\mathtt{lh}\, \mathsf{s}_n \leq n$ and $\mathsf{s}_n \subset \mathsf{s}_m \Longrightarrow n < m$.

(iii) For $x \in \mathbb{N}^{\mathbb{N}}$ and $j < \ell$ define $(x)^\ell_j \in \mathbb{N}^{\mathbb{N}}$ so that $(x)^\ell_j(n) = x(n\ell + j)$, $\forall n$. Clearly, $x \mapsto \langle (x)^\ell_0, (x)^\ell_1, \dots, (x)^\ell_{\ell-1} \rangle$ is a bijection and a homeomorphism of the Baire space $\mathbb{N}^{\mathbb{N}}$ onto $(\mathbb{N}^{\mathbb{N}})^\ell$. In addition put $(x)^\ell_i = x$ in the "wrong" case $\ell \leq i$.

(iv) Moreover, even the infinite product $(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$ is homeomorphic to $\mathbb{N}^{\mathbb{N}}$ by means of the map $x \mapsto \{(x)_n\}_{n \in \mathbb{N}}$, where, for $x \in \mathbb{N}^{\mathbb{N}}$ and $n \in \mathbb{N}$, a point $(x)_n \in \mathbb{N}^{\mathbb{N}}$ is defined by $(x)_n(k) = x(2^n(2k+1)-1)$, $\forall k$. □

There are many other Polish spaces, like the real line $\mathbb{R}$, the space $\mathbb{R}^{\mathbb{N}}$ (of all infinite real sequences with the product topology) and many others, some of which will be considered below. On the other hand, the Baire space $\mathbb{N}^{\mathbb{N}}$ is an adequate representative of this class of spaces because basic descriptive set theoretic phenomena look similar in *all* uncountable Polish spaces, simply because all of them are *Borel isomorphic* by Theorem 1.2.2 below.

1.2. Pointsets. Borel sets

To distinguish sets in Polish spaces, they sometimes call them *pointsets*. Thus a pointset is a subset of a Polish space. Descriptive set theory studies mainly those pointsets which can be defined or constructed, beginning with open sets, by means of certain operations. There are several different hierarchies of pointsets, classifying them in accordance with the length and complexity of such a construction. The most important of them are the Borel and projective hierarchies.

Recall that *Borel sets* in a given space $\mathbb{X}$ are those that belong to the smallest σ-algebra $\mathbf{Bor}(\mathbb{X})$ of sets $Y \subseteq \mathbb{X}$ which contains all open sets. The *Borel hierarchy* of Borel sets in $\mathbb{X}$ consists of *Borel classes* $\mathbf{\Sigma}^0_\xi$, $\mathbf{\Pi}^0_\xi$, $\mathbf{\Delta}^0_\xi$, where $1 \leq \xi < \omega_1$. The classes are defined by induction on ξ as follows:

$\mathbf{\Sigma}^0_1$ consists of all open sets in $\mathbb{X}$;

$\mathbf{\Sigma}^0_\xi$ (for $\xi > 1$) contains all countable unions of sets that belong to classes $\mathbf{\Pi}^0_\eta$, $1 \le \eta < \xi$;

$\mathbf{\Pi}^0_\xi$ contains all complements of sets in $\mathbf{\Sigma}^0_\xi$ to $\mathbb{X}$; that is, a set $X \subseteq \mathbb{X}$ belongs to $\mathbf{\Pi}^0_\xi$ iff its complement $\mathbb{X} \setminus X$ belongs to $\mathbf{\Sigma}^0_\xi$;

$\mathbf{\Delta}^0_\xi$ contains all sets that belong simultaneously to $\mathbf{\Sigma}^0_\xi$ and to $\mathbf{\Pi}^0_\xi$.

LEBESGUE proved long ago in [**Leb05**] that the classes with bigger indices ξ strictly include those with smaller ones, and

$$\mathbf{Bor}(\mathbb{X}) = \bigcup_{1 \le \xi < \omega_1} \mathbf{\Sigma}^0_\xi = \bigcup_{1 \le \xi < \omega_1} \mathbf{\Pi}^0_\xi = \bigcup_{1 \le \xi < \omega_1} \mathbf{\Delta}^0_\xi .$$

DEFINITION 1.2.1. A *standard Borel space* is a Polish space with the associated Borel structure $\mathbf{Bor}(\mathbb{X})$. A *Borel isomorphism* is any bijection $f : X \xrightarrow{\text{onto}} Y$, where X, Y are Borel sets in Polish spaces, such that f-images $f[X']$ of Borel sets $X' \subseteq X$ and f-preimages $f[Y']$ of Borel sets $Y' \subseteq Y$ are Borel sets. □

Note that a Borel isomorphism $f : X \xrightarrow{\text{onto}} Y$ induces a $\subseteq$-isomorphism $X' \mapsto f[X']$ of the Borel algebra $\mathbf{Bor}(X)$ of all Borel sets $X' \subseteq X$ onto $\mathbf{Bor}(Y)$.

The following theorem (see § 37 in [**Kur66**], or 15.2, 15.6, 13.7 in [**Kec95**]) shows that different uncountable Polish spaces, and even uncountable Borel sets in Polish spaces, have essentially the same standard Borel spaces.

THEOREM 1.2.2. *Suppose that X, Y are Borel sets in Polish spaces. Then*

(i) *If X, Y are uncountable, then they are Borel isomorphic; that is, there is a Borel isomorphism $f : X \xrightarrow{\text{onto}} Y$.*

(ii) *If $f \subseteq X \times Y$ is a 1-to-1 function, Borel in the sense that its graph is Borel as a subset of $X \times Y$, then f is a Borel isomorphism.*

(iii) *There exists a closed set $P \subseteq \mathbb{N}^\mathbb{N}$ and a continuous 1-to-1 map $f : P \xrightarrow{\text{onto}} X$, and every such a map f is a Borel isomorphism.* □

We are not going to prove this theorem in full generality (see references above), but we give the proof for the case of product spaces in Section 2.12. Let us point out an important corollary.

COROLLARY 1.2.3. *If X is a set in a Polish space $\mathbb{X}$, then there is a Polish topology τ on X that produces exactly the same Borel subsets of X as the original topology and contains all relatively open subsets of X in the sense of the original Polish topology.*

PROOF. By Theorem 1.2.2(iii) there is a closed set $P \subseteq \mathbb{N}^\mathbb{N}$ and a Borel isomorphism $f : P \xrightarrow{\text{onto}} X$ that is simultaneously a contunuous map. Let τ consist of all sets $Y \subseteq X$ such that the preimage $f^{-1}[Y]$ is open in P. Then τ is just a copy of the Polish topology of P and, hence, it is Polish too. Moreover, τ contains all sets $Y \subseteq X$ relatively open in the original Polish topology since f is continuous. And τ produces the same Borel sets as the original Polish topology since f is a Borel isomorphism. □

1.3. Projective sets

The most common definition of *projective* pointsets includes interplay between different Polish spaces. The *projective hierarchy* of sets in Polish spaces consists of *projective classes* $\mathbf{\Sigma}^1_n$, $\mathbf{\Pi}^1_n$, $\mathbf{\Delta}^1_n$ of pointsets. The classes are defined by induction on n. As long as only *product* spaces (those of the form $\mathbb{X} = \mathbb{N}^k \times (\mathbb{N}^\mathbb{N})^\ell$) are considered, the definition is as follows:

$\mathbf{\Sigma}^1_0$: consists of all open sets in Polish spaces of the form $\mathbb{X} = \mathbb{N}^k \times (\mathbb{N}^\mathbb{N})^\ell$;

$\mathbf{\Pi}^1_n$: consists of all complements of sets in $\mathbf{\Sigma}^1_n$;

$\mathbf{\Delta}^1_n$: consists of all sets that belong simultaneously to $\mathbf{\Sigma}^1_n$ and to $\mathbf{\Pi}^1_n$;

$\mathbf{\Sigma}^1_{n+1}$: consists of all projections of sets in $\mathbf{\Pi}^1_n$;

where the *projection* of a set $P \subseteq \mathbb{X} \times \mathbb{N}^\mathbb{N}$ is the set

$$\operatorname{pr} P = \{x \in \mathbb{X} : \exists y \in \mathbb{N}^\mathbb{N} \, (\langle x, y\rangle \in P)\}.$$ [1]

And finally $\bigcup_n \mathbf{\Sigma}^1_n = \bigcup_n \mathbf{\Pi}^1_n$ is the class of all projective sets.

Sets in $\mathbf{\Sigma}^1_1$ are also called *Souslin*, or *analytic*, sets. Accordingly, sets in $\mathbf{\Pi}^1_1$ are called *co-Souslin* or *coanalytic*.

One can equivalently define $\mathbf{\Sigma}^1_{n+1}$ as the class of all continuous images of $\mathbf{\Pi}^1_n$ sets that situate in *the same* space $\mathbb{X}$. Such a definition of projective classes extends to every Polish space $\mathbb{X}$, with the following modification of the initial step: $\mathbf{\Sigma}^1_1$ is the class of all continuous images of Borel sets in this space.[2] And in such a modified form, the projective hierarchy obeys the following theorem of classical descriptive set theory, easily provable by induction on n.

THEOREM 1.3.1. *Suppose that X, Y are Borel sets in Polish spaces, and $f : X \xrightarrow{\text{onto}} Y$ is a Borel isomorphism. If $n \geq 1$ and $Z \subseteq X$, then Z is $\mathbf{\Sigma}^1_n$ iff $f[Z]$ is $\mathbf{\Sigma}^1_n$. The same holds for $\mathbf{\Pi}^1_n$ and $\mathbf{\Delta}^1_n$.* □

The Souslin theorem (Corollary 2.3.4 below) asserts that the Borel algebra $\mathbf{Bor}(\mathbb{X})$ of an arbitrary Polish space coincides with the class of all $\mathbf{\Delta}^1_1$ subsets of $\mathbb{X}$. It easily follows that any Borel isomorphism between Polish spaces induces an isomorphism between the projective hierarchies, and, hence by Theorem 1.2.2, the projective structure is essentially one and the same in all uncountable Polish spaces. We give KECHRIS' book [**Kec95**] as a general reference in matters of Borel and projective sets in Polish spaces.

1.4. Analytic formulas

Surprisingly, it turns out that a number of difficult descriptive set theoretic theorems involve tools and methods originally designed in recursion theory for entirely different goals. This leads us to a branch of modern descriptive set theory called *effective*, and we have to give credit to ADDISON, who demonstrated in

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[1] If $\mathbb{X} = \mathbb{N}^k \times (\mathbb{N}^\mathbb{N})^\ell$, then the space $\mathbb{X} \times \mathbb{N}^\mathbb{N}$, formally identified with $\mathbb{N}^k \times (\mathbb{N}^\mathbb{N})^{\ell+1}$, still belongs to the category of product spaces.

[2] Here Borel sets can be replaced by $\mathbf{G}_\delta$ sets *w.l.o.g.*, but the class $\mathbf{\Pi}^1_0$ of all closed sets, as in the definition for the spaces $\mathbb{N}^m \times (\mathbb{N}^\mathbb{N})^k$, is not sufficient any more. Indeed and for instance, for σ-compact spaces like $\mathbb{R}$, all projections, even all continuous images of closed sets, are sets $\mathbf{F}_\sigma$, which form a proper subclass of $\mathbf{\Sigma}^1_1$.

[**Add59b, Add59a**] the technical advantages of effective methods and even notational conventions in descriptive set theory.

This direction follows classical descriptive set theory in that it considers hierarchies of pointsets based on essentially the same operations as classical hierarchies. However, there is an important additional aspect that does not belong to the initial circle of basic ideas of descriptive set theory. Namely, Borel and projective sets are classified not only on the base of the number of iterations of basic operations necessary to produce a given set from open sets, but also in matters of definability of those original open sets.

BLANKET AGREEMENT 1.4.1. From now on we will develop effective descriptive set theory only for sets in product spaces (those of the form $\mathbb{N}^k \times (\mathbb{N}^{\mathbb{N}})^\ell$). This is by far not the maximal known generality, but still it covers all applications in this book and allows us to avoid technicalities of a more general treatment. □

The structure of these spaces allows us to employ a simple language to describe pointsets. This language, the language of second-order Peano arithmetic, contains two types of variables:

type 0: with the domain $\mathbb{N}$ (letters k, l, m, n, etc., are used), and

type 1: with the domain $\mathbb{N}^{\mathbb{N}}$ (letters x, y, z, a, b, c, etc., are used).

Terms can be obtained from variables by means of the following rules:

1) a variable of type $i = 0, 1$ is a term of type i;
2) every natural number is a term of type 0;
3) if t, s are terms of type 0, then so are $t + s$, ts, and t^s;
4) if t, s are terms of types $0, 1$, respectively, then $s(t)$ is a term of type 0.
5) if t, u, r are terms of type 0, then so is $(t)^u_r$, with the intended meaning of $(t)^u_r$ in accordance with Definition 1.1.6(i);
6) if t, r are terms of type 0, then so are $\mathtt{lh}\, \mathtt{s}_t$ and $\mathtt{s}_t(r)$, with the intended meaning of $\mathtt{s}_t$ in accordance with Definition 1.1.6(ii), and we understand that $\mathtt{s}_n(k) = 0$ in the "wrong" case $k \geq \mathtt{lh}\, \mathtt{s}_n$;
7) if t is a term of type 1 and r, u of type 0, then $(t)_r$ and $(t)^u_r$ are terms of type 1, with the intended meaning in accordance with Definition 1.1.6(iii,iv).

For instance, $x(2^k + (y)_j(3n))$ is a term of type 0.

The following classes of formulas of this language are distinguished as follows:

elementary: those formulas of the form $t = t'$, $t < t'$, $t \leq t'$, where t, t' are terms (for example, variables) of type 0;

analytic: all formulas obtained from elementary formulas by means of propositional connectives and quantifiers of either type; that is, all (well-defined) formulas of the language of the second-order Peano arithmetic;

arithmetic: those analytic formulas that do not include quantifiers of type 1 (over $\mathbb{N}^{\mathbb{N}}$);

bounded: those arithmetic formulas that include quantifiers only of the form $\exists\, k < t$ and $\forall\, k < t$, where k is a variable of type 0 and t is a term; in particular, all quantifier-free formulas are bounded;

Σ^0_n and Π^0_n: arithmetic formulas of the form

(0) $$\exists\, k_1\, \forall\, k_2\, \exists\, k_3 \ldots \exists\, (\forall\,) k_n\, \varphi \quad \text{and} \quad \forall\, k_1\, \exists\, k_2\, \forall\, k_3 \ldots \forall\, (\exists) k_n\, \varphi\,,$$

respectively, where φ is a bounded formula;

Σ^1_n and Π^1_n: analytic formulas of the form

(1) $$\exists\, x_1\, \forall\, x_2\, \exists\, x_3 \ldots \exists\, (\forall\,) x_n\, \forall\, (\exists\,) m\, \varphi \quad \text{and} \quad \forall\, x_1\, \exists\, x_2\, \forall\, x_3 \ldots \forall\, (\exists\,) x_n\, \exists\, (\forall\,) m\, \varphi\,,$$

respectively, where φ is a bounded formula.

A *quantifier prefix* is a left-most part of an analytic formula which consists of a string of quantifiers. The quantifier prefices of formulas in (0) are called Σ^0_n*-prefix* and Π^0_n*-prefix*, and those of formulas in (1) are called Σ^1_n*-prefix* and Π^1_n*-prefix*.

1.5. Transformation of analytic formulas

The equivalences included in Table 1 allow us to convert complicated analytic formulas by means of simplification of the quantifier prefix to a form that makes it possible to immediately evaluate the type of the formula (and the set defined by the formula). Note that the prelast equivalence $(\forall^0\, \exists^1)$ expresses the countable axiom of choice, while $(\exists^0\, \forall^1)$ expresses the dual statement.

EXAMPLE 1.5.1. If $\varphi(x, y, k, m)$ is a Σ^1_n formula, and $n \geq 1$, then the formulas

$$\exists\, x\; \varphi(x, y, k, m)\,, \quad \exists\, k\; \varphi(x, y, k, m)\,, \quad \forall\, k\; \varphi(x, y, k, m)$$

TABLE 1. Transformation rules (see Definition 1.1.6 on $(n)^k_i$, etc.)

$(\forall^< \exists^0 \to \exists^0 \forall^<)$	$\forall\, i < j\; \exists\, k\; \varphi(i, j, k)$	$\iff$	$\exists\, k\; \forall\, i < j\; \varphi(i, j, (k)^j_i)$
$(\exists^< \forall^0 \to \forall^0 \exists^<)$	$\exists\, i < j\; \forall\, k\; \varphi(i, j, k)$	$\iff$	$\forall\, k\; \exists\, i < j\; \varphi(i, j, (k)^j_i)$
$(\exists^0 \exists^0 \to \exists^0)$	$\exists\, i\; \exists\, j\; \varphi(i, j)$	$\iff$	$\exists\, n\; \varphi((n)^2_0, (n)^2_1)$
$(\forall^0 \forall^0 \to \forall^0)$	$\forall\, i\; \forall\, j\; \varphi(i, j)$	$\iff$	$\forall\, n\; \varphi((n)^2_0, (n)^2_1)$
$(\exists^1 \exists^1 \to \exists^1)$	$\exists\, x\; \exists\, y\; \varphi(x, y)$	$\iff$	$\exists\, z\; \varphi((z)^2_0, (z)^2_1)$
$(\forall^1 \forall^1 \to \forall^1)$	$\forall\, x\; \forall\, y\; \varphi(x, y)$	$\iff$	$\forall\, z\; \varphi((z)^2_0, (z)^2_1)$
$(\forall^0 \exists^0 \to \exists^1 \forall^0)$	$\forall\, i\; \exists\, j\; \varphi(i, j)$	$\iff$	$\exists\, x\; \forall\, i\; \varphi(i, x(i))$
$(\exists^0 \forall^0 \to \forall^1 \exists^0)$	$\exists\, i\; \forall\, j\; \varphi(i, j)$	$\iff$	$\forall\, x\; \exists\, i\; \varphi(i, x(i))$
$(\exists^1 \exists^0 \to \exists^1)$	$\exists\, x\; \exists\, j\; \varphi(x, j)$	$\iff$	$\exists\, y\; \varphi((y)_0, (y)_1(0))$
$(\forall^1 \forall^0 \to \forall^1)$	$\forall\, x\; \forall\, j\; \varphi(x, j)$	$\iff$	$\forall\, y\; \varphi((y)_0, (y)_1(0))$
$(\forall^0 \exists^1 \to \exists^1 \forall^0)$	$\forall\, i\; \exists\, x\; \varphi(i, x)$	$\iff$	$\exists\, x\; \forall\, i\; \varphi(i, (x)_i)$
$(\exists^0 \forall^1 \to \forall^1 \exists^0)$	$\exists\, i\; \forall\, x\; \varphi(i, x)$	$\iff$	$\forall\, x\; \exists\, i\; \varphi(i, (x)_i)$

belong to the same type, in the sense that they can be converted to equivalent $\varSigma^1_n$ formulas by means of the transformation rules, $(\exists^1\exists^1 \to \exists^1)$, $(\exists^1\exists^0 \to \exists^1)$, respectively, and the combination of $(\forall^0\exists^1 \to \exists^1\forall^0)$ and $(\forall^1\forall^0 \to \forall^1)$. □

EXAMPLE 1.5.2. Suppose that $n \ge 1$. An analytic formula that has a $\varSigma^1_n$ prefix followed by an arithmetic formula can be converted to the equivalent $\varSigma^1_n$ form, and the same for $\varPi^1_n$. To prove this, let φ be, e.g.,

$$\exists x\, \forall y\, \exists k\, \forall m\, \psi(x,y,k,m),$$

where ψ is bounded and $n = 2$. Then the quantifier prefix has the form $\exists^1\forall^1\exists^0\forall^0$. We convert it to $\exists^1\forall^1\forall^1\exists^0$ and then to $\exists^1\forall^1\exists^0$ by means of the rules $(\exists^0\forall^0 \to \forall^1\exists^0)$ and $(\forall^1\forall^1 \to \forall^1)$, respectively.

In particular it follows that every arithmetic formula is convertible to both $\varSigma^1_1$ and $\varPi^1_1$ form, equivalent to the original formula. □

EXAMPLE 1.5.3. Suppose that $n \ge 1$. The conjunction and the disjunction of the two $\varSigma^1_n$ formulas can be converted to equivalent $\varSigma^1_n$ forms, and the same for $\varPi^1_n$. To prove this, let φ and φ' be the formulas

$$\exists x\, \forall y\, \exists k\, \psi(x,y,k) \quad \text{and} \quad \exists x\, \forall y\, \exists k\, \psi'(x,y,k)\,,$$

where ψ, ψ' are bounded and $n = 2$. Then $\varphi \wedge \varphi'$ is equivalent to

$$\exists x\, \exists x'\, \forall y\, \forall y'\, \exists k\, \exists k'\, (\psi(x,y,k) \wedge \psi'(x',y',k'))\,,$$

and appropriate transformation rules convert this to $\varSigma^1_2$. □

1.6. Effective hierarchies of pointsets

Free variables of analytic formulas can be substituted by particular elements of $\mathbb{N}$ (type 0) or $\mathbb{N}^{\mathbb{N}}$ (type 1). These elements are then called *parameters*.[3] This leads us to a classification of pointsets in product spaces $\mathbb{N}^k \times (\mathbb{N}^{\mathbb{N}})^\ell$, which takes into account both the position of the formula φ in the definition of the form

$$X = \{\langle \vec{n}, \vec{x}\rangle \in \mathbb{N}^k \times (\mathbb{N}^{\mathbb{N}})^\ell : \varphi(\vec{n}, \vec{x})\}$$

in the hierarchy of formulas defined in Section 1.4 and the list of parameters that occur in the formula φ.

DEFINITION 1.6.1. Suppose that $A \subseteq \mathbb{N}^{\mathbb{N}}$.

Then $\varSigma^i_n(A)$ is the class of all pointsets in spaces of the form $\mathbb{N}^k \times (\mathbb{N}^{\mathbb{N}})^\ell$, definable by $\varSigma^i_n$-formulas with parameters that belong to A. The class $\varPi^i_n(A)$ is defined similarly, while $\varDelta^i_n(A) = \varSigma^i_n(A) \cap \varPi^i_n(A)$.

We write $\varSigma^i_n$, resp., $\varSigma^i_n(a)$, instead of $\varSigma^i_n(A)$, in cases $A = \varnothing$ and $A = \{a\}$, respectively, and similarly for $\varPi$ and $\varDelta$. □

The classes whose notation includes the characters $\varSigma$, $\varPi$, $\varDelta$ will be called *effective*, opposite to projective classes $\mathbf{\Sigma}^1_n$, $\mathbf{\Pi}^1_n$, $\mathbf{\Delta}^1_n$. Note that if $A \subseteq 2^{\mathbb{N}}$ is at most countable, then every class $\varSigma^i_n(A)$ is countable as well. In particular $\varSigma^i_n$ and all classes $\varSigma^i_n(a)$, $a \in 2^{\mathbb{N}}$, are countable. The same for $\varPi$ and $\varDelta$.

Alternatively, classes $\mathbf{\Sigma}^i_n$, $\mathbf{\Pi}^i_n$, $\mathbf{\Delta}^i_n$ are sometimes called *boldface* classes, while $\varSigma^i_n(A)$, $\varPi^i_n(A)$, $\varDelta^i_n(A)$ are called *lightface*.

[3] In fact, such a substitution is inessential for type 0 since every natural number is straightforwardly definable by a quantifier-free formula.

Blanket Agreement 1.6.2. The character Γ routinely denotes one of Σ, Π, Δ, and accordingly $\mathbf{\Gamma}$ denotes one of $\mathbf{\Sigma}$, $\mathbf{\Pi}$, $\mathbf{\Delta}$. □

Example 1.6.3. Every set defined by a bounded formula is Δ^0_1. For instance

$$P = \{m : \forall k < m \, \forall n < m \, (m \neq nk)\},$$

the set of all primes, is Δ^0_1. In fact, it is not so easy to find a Δ^0_1 set *not* definable by a bounded formula. □

Note that the projective hierarchy is a special case of the effective one:

Proposition 1.6.4. *For sets X in product spaces, we have $\mathbf{\Gamma}^i_n = \Gamma^i_n(\mathbb{N}^\mathbb{N})$. More precisely, $X \in \mathbf{\Gamma}^i_n$ if and only if $X \in \Gamma^i_n(a)$ for some $a \in \mathbb{N}^\mathbb{N}$.*

Recall that by Blanket Agreement 1.6.2, Γ here is one of Σ, Π, Δ, and $\mathbf{\Gamma}$ is one of $\mathbf{\Sigma}$, $\mathbf{\Pi}$, $\mathbf{\Delta}$.

Proof (sketch). We leave it as an *exercise* for the reader to show that bounded analytic formulas define clopen subsets of the spaces considered. It immediately follows that all sets definable by Σ^0_1 formulas (with arbitrary parameters) are open; therefore, $\Sigma^0_1(\mathbb{N}^\mathbb{N}) \subseteq \mathbf{\Sigma}^0_1$.

Conversely, suppose that X is an open set, say, in $\mathbb{N}^\mathbb{N}$. If $X = \varnothing$, then the result is obvious, so we can assume that $X \neq \varnothing$. Then there exists a set $\varnothing \neq S \subseteq \mathbb{N}^{<\omega}$ satisfying $X = \bigcup_{s \in S} \mathscr{O}_s(\mathbb{N}^\mathbb{N})$. (Recall that $\mathscr{O}_s(\mathbb{N}^\mathbb{N}) = \{x \in \mathbb{N}^\mathbb{N} : s \subset x\}$ for $s \in \mathbb{N}^{<\omega}$. Sets of this form are basic clopen nbhds in $\mathbb{N}^\mathbb{N}$; see Example 1.1.2.) Put $N = \{n : \mathtt{s}_n \in S\}$. Obviously $\varnothing \neq N \subseteq \mathbb{N}$ and $S = \{\mathtt{s}_n : n \in N\}$. There exists $a \in \mathbb{N}^\mathbb{N}$ such that $N = \{a(n) : n \in \mathbb{N}\}$. Then

$$x \in X \iff \exists n \, (\mathtt{s}_{a(n)} \subset x) \iff \exists n \, \forall k < \mathtt{lh}\, \mathtt{s}_{a(n)} \, (x(k) = \mathtt{s}_{a(n)}(k)).$$

The formula in the right-hand side is Σ^0_1 with $a \in \mathbb{N}^\mathbb{N}$ as the only parameter. Therefore, X belongs to $\Sigma^0_1(a)$.

Thus, $\Sigma^0_1(\mathbb{N}^\mathbb{N}) = \mathbf{\Sigma}^0_1$. This implies the result required for all arithmetic classes Σ^0_n, Π^0_n, Δ^0_n. As for projective classes, the quantifier $\exists x$ (in our assumptions, $\exists x \in \mathbb{N}^\mathbb{N}$) corresponds to the projection, while the quantifier $\forall x$ corresponds to the combination "complement–projection–complement". □

Exercise 1.6.5. Prove, using the last claim of 1.5.2, that if $p \in \mathbb{N}^\mathbb{N}$ and $n \in \mathbb{N}$, then $\Sigma^0_n(p) \cup \Pi^0_n(p) \subseteq \Delta^1_1(p)$. □

1.7. Characterization of Σ^0_1 sets

This section presents two results related to Σ^0_1 sets.

Lemma 1.7.1 (Σ^0_1= recursively enumerable). *A set $X \subseteq \mathbb{N}^m$ is Σ^0_1 iff it is recursively enumerable* (in the sense of recursion theory). *In particular, a map $f : \mathbb{N}^m \to \mathbb{N}$ is recursive iff it is Δ^0_1.*

Proof (Sketch). Suppose that $X \subseteq \mathbb{N}$ is recursively enumerable; that is, there exists a computer program C that computes $C(n) = 1$ iff $n \in x$ and, otherwise, computes some $C(n) \neq 1$ or just does not compute anything. The formula $\varphi(n, k) :=$ "$C(n) = 1$ after not more than k steps of C with input n" can be shown to be bounded. Now $n \in x \iff \exists k \, \varphi(n, k)$, and hence X is Σ^0_1.

To prove the converse it suffices to note that sets defined by any bounded formulas are recursive (even primitive recursive). □

LEMMA 1.7.2 (Σ_1^0= effective open). *A set* $X \subseteq \mathbb{N}^\mathbb{N}$ *is* Σ_1^0 *iff there exists a* Σ_1^0 *set* $N \subseteq \mathbb{N}$ *such that* $X = \bigcup_{n \in N} \mathcal{O}_{s_n}(\mathbb{N}^\mathbb{N})$.

PROOF (Sketch). If such a set N exists, then X is Σ_1^0 since

$$x \in X \iff \exists n\,(n \in N \wedge \mathbf{s}_n \subset x).$$

The converse needs a bit more work. Suppose that $X = \{x : \exists m\, \varphi(m,x)\}$, where φ is bounded. Because of the boundedness, the truth definition of $\varphi(m,x)$ for any given $x \in \mathbb{N}^\mathbb{N}$ and $m \in \mathbb{N}$ can be represented as the computation by a computer program (depending on the structure of φ but not on the values of x, m), where x is an input in the form of an infinite string of numeric values $x(k)$, $k \in \mathbb{N}$, and m as a single number, and the result is obtained after a finite number of steps. In particular the computation appeals to only finitely many values $x(k)$.

If $s \in \mathbb{N}^{<\omega}$, then let $s^\frown \mathbf{0} \in \mathbb{N}^\mathbb{N}$ be the extension of s by zeros. We put

$W =$ the set of all pairs $\langle m, n\rangle$ such that there exists a computation of the truth value of $\varphi(m, \mathbf{s}_n{}^\frown \mathbf{0})$ that yields "true" and does not refer to the values $(s^\frown \mathbf{0})(j)$, $j \geq \mathtt{lh}\, \mathbf{s}_n$.

Then W is Σ_1^0; therefore, the set $N = \{n : \exists m\,(\langle m,n\rangle \in W)\}$ is Σ_1^0 as well, and easily $X = \{x \in \mathbb{N}^\mathbb{N} : \exists n \in N\,(\mathbf{s}_n \subset x)\}$. □

1.8. Classifying functions

Functions are in the scope of this classification as well. To avoid repetition, we suppose below that $\mathbb{X}$ and $\mathbb{Y}$ are arbitrary product spaces.

DEFINITION 1.8.1. Given any class of sets K, a map $F : \mathbb{X} \to \mathbb{Y}$ *belongs to* K, or is a *K-function*, if its graph $\Gamma_F = \{\langle x, y\rangle : x \in X \wedge f(x) = y\}$ is a set in K. □

LEMMA 1.8.2. *If* $n \geq 1$, *then every* Σ_n^1 *function* $F : \mathbb{X} \to \mathbb{Y}$ *is* Δ_n^1. *If* $\mathbb{Y} = \mathbb{N}^k$, $k \in \mathbb{N}$, *then every* Σ_n^0 *function* $F : \mathbb{X} \to \mathbb{Y}$ *is* Δ_n^0.

PROOF. Indeed $F(x) = y \iff \forall y'\,(y' \neq y \implies F(x) \neq y')$. To transform the right-hand side to Π_n^1, apply the rules of Table 1 on page 12. □

In some cases, the next definition is more suitable than Definition 1.8.1. We give it in the case when the receiving space is $\mathbb{N}$ or $\mathbb{N}^\mathbb{N}$.

DEFINITION 1.8.3. A function $F : \mathbb{X} \to \mathbb{N}^\mathbb{N}$ is *K-measurable* if the set

$$\Gamma_F^* = \{\langle x, n, k\rangle : x \in X \wedge F(x)(n) = k\} \subseteq \mathbb{X} \times \mathbb{N}^2$$

(not to be mixed with the graph Γ_F) belongs to K.[4] A function $F : \mathbb{X} \to \mathbb{N}$ is *K-measurable* if its true graph $\Gamma_F = \{\langle x, n\rangle : x \in X \wedge n = f(x)\}$ belongs to K.

Δ_1^0-measurable functions are called *recursive*. □

Thus, for a function $F : \mathbb{X} \to \mathbb{N}$, K-measurability is the same as class K in the sense of Definition 1.8.1. The following fact is less trivial.

LEMMA 1.8.4. *Suppose that* $p \in \mathbb{N}^\mathbb{N}$. *A function* $F : \mathbb{X} \to \mathbb{N}^\mathbb{N}$ *is of class* $\Delta_1^1(p)$ *iff it is* $\Delta_1^1(p)$*-measurable.*

[4] See the end of Section 1.9 regarding the measurability in the case of boldface classes K.

PROOF. Obviously, $F(x) = y$ iff $\forall n\,\forall k\,(\langle x,n,k\rangle \in \Gamma^*_F \Longrightarrow y(n) = k)$, and if Γ^*_F is $\Delta^1_1(p)$, then the latter formula can be transformed to both Σ^1_1 and Π^1_1 using Table 1 on page 12. Conversely,

$$\langle x,n,k\rangle \in \Gamma^*_F \iff \forall y\,(F(x) = y \Longrightarrow y(n) = k) \iff \exists y\,(F(x) = y \wedge y(n) = k),$$

and this gives Δ^1_1 for Γ^*_F as well. □

Another important case is Σ^0_1-measurability. For functions into $\mathbb{N}^\mathbb{N}$, the notions of a Σ^0_1 function and a Σ^0_1-measurable function do not coincide; moreover, Σ^0_1 functions into $\mathbb{N}^\mathbb{N}$ simply do not exist because functions $F : \mathbb{X} \to \mathbb{N}^\mathbb{N}$ cannot have Σ^0_1 (hence, they are open by Proposition 1.6.4) graphs. Still we have the following:

LEMMA 1.8.5. *If* $p \in \mathbb{N}^\mathbb{N}$ *and a function* $F : \mathbb{X} \to \mathbb{N}^\mathbb{N}$ *is* $\Sigma^0_1(p)$*-measurable, then it is* $\Delta^0_1(p)$*-measurable and (the graph of) F is* $\Pi^0_1(p)$.

PROOF. Clearly, $\langle x,n,k\rangle \in \Gamma^*_F \iff \forall k' \neq k\,(\langle x,n,k'\rangle \notin \Gamma^*_F)$, and this proves the first claim. To prove the second one, note that

$$F(x) = y \iff \forall n\,\forall k\,(\langle x,n,k\rangle \in \Gamma^*_F \Longrightarrow k = y(n))\,. \qquad \square$$

Note that every $a \in \mathbb{N}^\mathbb{N}$ is a map $\mathbb{N} \to \mathbb{N}$; hence, it is a subset of the product space $\mathbb{N}^2$, and under this angle a subject of classification. For instance, we say that a is a Δ^1_1 element of $\mathbb{N}^\mathbb{N}$ if so is its graph $\{\langle n,k\rangle : a(n) = k\}$. The next lemma connects the class of $a \in \mathbb{N}^\mathbb{N}$ with the class of its singleton.

LEMMA 1.8.6. *Suppose that* $a \in \mathbb{N}^\mathbb{N}$. *Then the following conditions are equivalent:* a *is* Δ^1_1, a *is* Σ^1_1, $\{a\}$ *is* Δ^1_1, $\{a\}$ *is* Σ^1_1.

PROOF. Suppose that a is Σ^1_1. Then

$$b \in \{a\} \iff \forall n\,\forall k\,(b(n) = k \Longrightarrow a(n) = k) \iff \forall n\,\forall k\,(a(n) = k \Longrightarrow b(n) = k).$$

Changing $a(n) = k$ to a suitable Σ^1_1 formula and using the table of transformation rules (Table 1), we obtain $\{a\} \in \Sigma^1_1$ from the middle formula and $\{a\} \in \Pi^1_1$ from the right-hand formula. To prove the converse, suppose that $X = \{a\}$ is Σ^1_1. Then

$$a(n) = k \iff \exists b\,(b \in X \wedge b(n) = k) \iff \forall b\,(b \in X \Longrightarrow b(n) = k).$$

Once again $a \in \Sigma^1_1$ follows from the middle formula and $a \in \Pi^1_1$ from the right-hand formula. □

1.9. Closure properties

Proposition 1.6.4, together with the transformation rules presented in Table 1 allows us to prove standard closure properties for projective and effective classes known from books like [**Kur66, Sho01, Mos80, MW85, Kec95**].

PROPOSITION 1.9.1. *Suppose that* $p \in \mathbb{N}^\mathbb{N}$. *Then*

(i) *Every class of the form* $\Gamma^i_n(p)$ *or* $\mathbf{\Gamma}^i_n$ *is closed under finite unions and intersections of sets in the same space and under bounded type-*0 *quantifiers* $\exists k < n$ *and* $\forall k < n$.

(ii) *Classes of the form* $\Delta^i_n(p)$ *are also closed under complements.*

(iii) *Classes* $\Sigma^i_n(p)$ *are closed under the type-*0 $\exists$ *quantifier (that is, under the type-*0 *projection): if* $\mathbb{X}$ *is a product space and* $P \subseteq \mathbb{X} \times \mathbb{N}$ *is* $\Sigma^i_n(p)$*, then* $\operatorname{pr} P = \{x \in \mathbb{X} : \exists k\,(\langle x, k\rangle \in P)\}$ *is a set in* $\Sigma^i_n(p)$*. Accordingly, classes* $\Pi^i_n(p)$ *are closed under the type-*0 $\forall$ *quantifier.*

(iv) *Classes* $\Sigma^1_n(p)$, $n \geq 1$, *are closed under the type-*1 $\exists$ *quantifier (that is, under the type-*1 *projection): if* $\mathbb{X}$ *is a product space and* $P \subseteq \mathbb{X} \times \mathbb{N}^{\mathbb{N}}$ *is* $\Sigma^1_n(p)$*, then* $\operatorname{pr} P = \{x \in \mathbb{X} : \exists z\,(\langle x, z\rangle \in P)\}$ *is* $\Sigma^1_n(p)$*. Accordingly, classes* $\Pi^1_n(p)$ *are closed under the type-*1 $\forall$ *quantifier.*

(v) *Boldface classes* $\mathbf{\Sigma}^0_\xi$ *are closed under the type-*0 $\exists$ *quantifier and under countable unions (of sets in the same space), while classes* $\mathbf{\Pi}^0_\xi$ *are closed under the type-*0 $\forall$ *quantifier and under countable intersections.*

(vi) *Boldface classes* $\mathbf{\Gamma}^1_n$, $n \geq 1$, *are closed under type-*0 *quantifiers* $\exists$ *and* $\forall$ *and under countable unions and countable intersections. It follows that all Borel sets are* $\mathbf{\Delta}^1_1$.[5]

(vii) *Boldface classes* $\mathbf{\Sigma}^1_n$, $n \geq 1$, *are closed under the type-*1 $\exists$ *quantifier (that is, under the type-*1 *projection). Accordingly, classes* $\mathbf{\Pi}^1_n$, $n \geq 1$, *are closed under the type-*1 $\forall$ *quantifier.*

(viii) *Classes of the form* $\Gamma^i_n(p)$ *are closed under* $\Delta^0_1(p)$*-measurable substitutions and* $\Delta^0_1(p)$*-measurable preimages, and classes of the form* $\Gamma^1_n(p)$ *are also closed under* $\Delta^1_1(p)$ *substitutions and* $\Delta^1_1(p)$*-preimages.*

(ix) *Classes* $\mathbf{\Gamma}^i_n$ *are closed under continuous substitution and continuous preimages, and classes* $\mathbf{\Gamma}^1_n$ *are also closed under* $\mathbf{\Delta}^1_1$ *substitution and* $\mathbf{\Delta}^1_1$ *preimages.*

Claims (v), (vi), (vii), (ix) *are true for all Polish (not only product) spaces.*

Proof (sketch). We prove a couple of less trivial items.

(vi) Let us show that $\mathbf{\Sigma}^1_1$ is closed under countable operations. Suppose that X_n, $n \in \mathbb{N}$, are $\mathbf{\Sigma}^1_1$ sets in $\mathbb{N}^{\mathbb{N}}$. Then by definition there exist closed sets $W_n \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ such that $X_n = \operatorname{pr} W_n = \{x : \exists y\,(\langle x, y\rangle \in W_n)\}$. Clearly, the set

$$W = \{\langle x, n^\frown y\rangle : n \in \mathbb{N} \wedge \langle x, y\rangle \in W_n\}$$

is closed in $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$, and hence $\bigcup_n X_n = \operatorname{pr} W$ is a $\mathbf{\Sigma}^1_1$ set. Moreover, the set $W' = \{\langle x, y\rangle : \forall n\,(\langle x, (y)_n\rangle \in W_n)\}$ is closed too; therefore, $\bigcap_n X_n = \operatorname{pr} W'$ is $\mathbf{\Sigma}^1_1$.

(viii) Suppose that $F : \mathbb{X} \to \mathbb{N}^{\mathbb{N}}$ is a Δ^0_1-measurable function and $Y \subseteq \mathbb{N}^{\mathbb{N}}$ is a Σ^0_1 set. Prove that the preimage $X = F^{-1}[Y] = \{x \in \mathbb{X} : F(x) \in Y\}$ is Σ^0_1. By Lemma 1.7.2 there exists a Σ^0_1 set $M \subseteq \mathbb{N}$ such that $Y = \bigcup_{m \in M} \mathscr{O}_m(\mathbb{N}^{\mathbb{N}})$. On the other hand, the set $\Gamma^*_F = \{\langle x, n, k\rangle : F(x)(n) = k\}$ is Δ^0_1. Yet obviously

$$F(x) \in Y \iff \exists m\,(m \in M \wedge \forall n < \mathtt{lh}\,\mathtt{s}_m\,(\langle x, n, \mathtt{s}_m(n)\rangle \in \Gamma^*_F)).$$

Now suppose that $F : \mathbb{X} \to \mathbb{N}^{\mathbb{N}}$ is a Δ^1_1 function and $Y \subseteq \mathbb{N}^{\mathbb{N}}$ is a Σ^1_1 set. To prove that the preimage $X = F^{-1}[Y] = \{x \in \mathbb{X} : F(x) \in Y\}$ is Σ^1_1, note that

$$F(x) \in Y \iff \exists y\,(y \in Y \wedge \langle x, y\rangle \in F). \qquad \square$$

We end with a result often used implicitly in many arguments.

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[5] And conversely, all $\mathbf{\Delta}^1_1$ sets are Borel; see Corollary 2.3.4.

COROLLARY 1.9.2. *Let* $\mathbb{X}$ *be a Polish space and* $\mathbf{K} = \mathbf{\Gamma}^i_n$ *one of the Borel or projective classes. A set* $X \subseteq \mathbb{N} \times \mathbb{X}$ *belongs to* $\mathbf{K}$ *iff all cross-sections* $X_j = \{x \in \mathbb{X} : \langle j, x\rangle \in X\}$, $j \in \mathbb{N}$, *belong to* $\mathbf{K}$.

PROOF. By a simple argument with complementary sets, it suffices to prove the result in the case when $\mathbf{K} = \mathbf{\Sigma}^i_n$, so that $\mathbf{K}$ is closed under countable unions by Proposition 1.9.1(v), (vi). Suppose that a set $X \subseteq \mathbb{N} \times \mathbb{X}$ belongs to $\mathbf{K}$. Obviously, $X_j = f_j^{-1}[X]$, where $f_j : \mathbb{X} \to \mathbb{N} \times \mathbb{X}$ is the continuous map $f_j(x) = \langle j, x\rangle$. Thus, $X_j \in \mathbf{K}$ by Proposition 1.9.1(ix). To prove the converse, suppose that each set X_j belongs to $\mathbf{K}$. Then so does $Y_j = \mathbb{N} \times X_j$ (the preimage under the map $g(j, x) = x$). In addition, the set $U_j = \{j\} \times \mathbb{X}$ is $\mathbf{\Delta}^0_1$ (= clopen), hence, it belongs to $\mathbf{K}$. Therefore, the intersection $U_j \cap Y_j = \{j\} \times X_j$ is in $\mathbf{K}$ by Proposition 1.9.1(i). We conclude that $X = \bigcup_j (U_j \cap Y_j) \in \mathbf{K}$, since $\mathbf{K}$ is closed under countable unions by the assumption above. □

The next result is related to the boldface version of measurability of functions:

COROLLARY 1.9.3. *Let* $\mathbb{X}$ *be a Polish space and* $\mathbf{K} = \mathbf{\Gamma}^i_n$ *one of the Borel classes* $\mathbf{\Sigma}^0_\xi$ *or one of the projective classes beginning with* $\mathbf{\Delta}^1_1$. *A map* $f : \mathbb{X} \to \mathbb{N}^\mathbb{N}$ *is* $\mathbf{K}$*-measurable in the sense of Definition* 1.8.3 *iff all* f*-preimages of open sets* $U \subseteq \mathbb{N}^\mathbb{N}$ *belong to* $\mathbf{K}$.

In particular f *is* $\mathbf{\Sigma}^0_1$*-measurable iff it is continuous.*

PROOF. Suppose that $f : \mathbb{X} \to \mathbb{N}^\mathbb{N}$ is $\mathbf{K}$-measurable; that is, the set

$$\Gamma^*_f = \{\langle x, n, k\rangle : x \in \mathbb{X} \wedge f(x)(n) = k\} \subseteq \mathbb{X} \times \mathbb{N}^2$$

belongs to $\mathbf{K}$. Then all sets $X_{nk} = \{x \in \mathbb{X} : f(x)(n) = k\}$ are in $\mathbf{K}$ by Corollary 1.9.2. However, any open $U \subseteq \mathbb{N}^\mathbb{N}$ can be obtained from sets of the form $U_{nk} = \{a \in \mathbb{N}^\mathbb{N} : a(n) = k\}$ by finite intersections and countantable unions, and, accordingly, the f-preimage of U can be obtained from sets of the form X_{nk} by these operations under which $\mathbf{K}$ is closed.

Conversely, suppose that all f-preimages of open sets $U \subseteq \mathbb{N}^\mathbb{N}$ belong to $\mathbf{K}$. In particular so do all sets X_{nk}. But then $\Gamma^*_f \in \mathbf{K}$ by Corollary 1.9.2. □

REMARK 1.9.4. The criterion of measurability for boldface classes given in Corollary 1.9.3 is often considered as the definition of measurability; that is, f is called $\mathbf{K}$-measurable if all f-preimages of open sets belong to $\mathbf{K}$. But such a definition does not work for effective classes like $\varGamma^i_n$ because it lacks effectiveness. □

CHAPTER 2

Some theorems of descriptive set theory

It turns out that the whole amount of basic descriptive set theory employed in the study of Borel reducibility of equivalence relations can be summarized in a rather short list of definitions and basic theorems. Our goal here is to present these key theorems. This will not be a systematic treatment, and we give the books of MOSCHOVAKIS [**Mos80**] and KECHRIS [**Kec95**] as much broader sources in matters of descriptive set theory. Nevertheless, we present here such topics as reduction, Borel separation, uniformization, universal sets, reflection, the Gandy–Harrington topology, and sets with countable sections. This chapter ends with a summary of a coding system for Borel sets and countable ordinals, both instrumental in proofs and useful for understanding the meaning of theorems.

BLANKET AGREEMENT 2.0.1. Working with Borel and projective sets, we will consider only sets in product spaces. Recall that a *product space* is a space of the form $\mathbb{N}^k \times (\mathbb{N}^{\mathbb{N}})^\ell$. We will make frequent use of the *relational* style of characterization of sets in product spaces, so that if, say, $U \subseteq \mathbb{N}^1 \times (\mathbb{N}^{\mathbb{N}})^2$, then $U(n,x,y)$ has the same meaning as $\langle n,x,y\rangle \in U$. □

It has been known since the early years of descriptive set theory that many fundamental results related to Borel sets and sets in $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$ are deeply connected with well-founded trees in $\mathbb{N}^{<\omega}$. The first two sections introduce a related instrumentarium.

2.1. Trees and ranks

By definition, for every $\ell \ge 1$, the set $(\mathbb{N}^\ell)^{<\omega}$ consists of finite sequences whose terms are ℓ-sequences of natural numbers. Thus, every $\sigma \in (\mathbb{N}^\ell)^{<\omega}$ is a map from $m = \operatorname{lh}\sigma$ into $\mathbb{N}^\ell$, and we can write $\sigma(k) = \langle \sigma_0(k), \sigma_1(k), \dots, \sigma_{\ell-1}(k)\rangle$ for all $k < m$. Therefore, σ can be identified with the sequence $\langle \sigma_0, \sigma_1, \dots, \sigma_{\ell-1}\rangle$, where each σ_j $(j < \ell)$ belongs to $\mathbb{N}^m$. Thus, $(\mathbb{N}^\ell)^{<\omega}$ can be identified with the subset

$$\{\langle \sigma_0, \sigma_1, \dots, \sigma_{\ell-1}\rangle \in (\mathbb{N}^{<\omega})^\ell : \operatorname{lh}\sigma_0 = \operatorname{lh}\sigma_1 = \dots = \operatorname{lh}\sigma_{\ell-1}\}$$

of the entire cartesian product $(\mathbb{N}^{<\omega})^\ell$.

DEFINITION 2.1.1. A set $T \subseteq \mathbb{N}^{<\omega}$ is a *tree*, iff we have $t \in T$ whenever $s \in T$, $t \in \mathbb{N}^{<\omega}$, $t \subset s$. Then $\varnothing$ is a tree, and the *empty sequence* Λ belongs to each non-empty tree. For any tree $T \subseteq \mathbb{N}^{<\omega}$, we define

$$[T] = \{x \in \mathbb{N}^{\mathbb{N}} : \forall m\,(x \restriction m \in T)\},$$

this is a closed set in $\mathbb{N}^{\mathbb{N}}$. □

The notion of a tree $T \subseteq (\mathbb{N}^m)^{<\omega}$ and the definition of $[T]$ for such a tree are introduced similarly. For instance, a tree $T \subseteq (\mathbb{N}^2)^{<\omega}$ consists of pairs $\langle s, t\rangle$ such that $s, t \in \mathbb{N}^{<\omega}$ and $\mathtt{lh}\, s = \mathtt{lh}\, t$, and for every such tree T,

$$[T] = \{\langle x, y\rangle \in (\mathbb{N}^\mathbb{N})^2 : \forall m\, (\langle x \restriction m\,,\, y \restriction m\rangle \in T)\}.$$

Recall that a tree $T \subseteq \mathbb{N}^{<\omega}$ is *ill founded* iff it contains an infinite branch; that is, there exists $z \in \mathbb{N}^\mathbb{N}$ such that $z \restriction m \in T$, $\forall m$. Otherwise, it is *well founded*.

DEFINITION 2.1.2. Suppose that $T \subseteq \mathbb{N}^{<\omega}$ is a tree and $s \in T$. If the reduced tree $T_s = \{t \in \mathbb{N}^{<\omega} : s^\frown t \in T\}$[1] is well founded, then so is each tree of the form $T_{s^\frown k}$, $k \in \mathbb{N}$. This allows us to define an ordinal $|s|_T < \omega_1$ for every $s \in T$ such that T_s is well founded, so that

(i) $|s|_T = 0$ for all endpoints (*i.e.*, $\subseteq$-maximal in T elements) $s \in T$;

(ii) $|s|_T = \sup_{s^\frown k \in T}(|s^\frown k|_T + 1)$ for all $s \in T$ that are not endpoints in T.[2]

If T_s is ill founded, then put $|s|_T = \infty$. Thus, $|s|_T$ and is either an ordinal $< \omega_1$ or ∞, with the idea that $\xi < \infty$ for all $\xi < \omega_1$.

Finally, if $T \neq \varnothing$, then put $|T| = |\Lambda|_T$, where Λ is the empty sequence—the *rank* of a tree $T \neq \varnothing$. Obviously, T is well founded iff $|T| < \omega_1$ and ill founded iff $|T| = \infty$. Separately define $|T| = -1$ whenever $T = \varnothing$. □

The next theorem characterizes certain relations between trees in terms of definability in projective hierarchy. This requires an explanation. Recall that the enumeration $\mathbb{N}^{<\omega} = \{\mathbf{s}_n : n \in \mathbb{N}\}$ was introduced by Definition 1.1.6(ii).

DEFINITION 2.1.3. Say that a set $T \subseteq \mathbb{N}^{<\omega}$ belongs to a given class K iff the corresponding set $\{n : \mathbf{s}_n \in T\}$ does also. And, $\mathbb{X}$ being any product space, a set $X \subseteq \mathbb{X} \times \mathbb{N}^{<\omega}$ belongs to a given class K iff $\{\langle x, n\rangle : \langle x, \mathbf{s}_n\rangle \in X\}$ does too.

For a set $T \subseteq \mathbb{N}^{<\omega}$, let $\chi(T) \in 2^\mathbb{N}$ be the characteristic function of the set $\{n : \mathbf{s}_n \in T\}$. Thus, $\chi(T)(n) = 1$ provided $\mathbf{s}_n \in T$, and $= 0$ otherwise. Clearly, χ is a bijection of $\mathscr{P}(\mathbb{N}^{<\omega})$ onto $2^\mathbb{N} \subseteq \mathbb{N}^\mathbb{N}$.

Say that a set $X \subseteq (\mathscr{P}(\mathbb{N}^{<\omega}))^\ell$ belongs to a given class K iff the corresponding set $\{\langle \chi(T_1), \dots, \chi(T_\ell)\rangle : \langle T_1, \dots, T_\ell\rangle \in X\}$ does also as a subset of $(\mathbb{N}^\mathbb{N})^\ell$. Similarly, a set $X \subseteq \mathbb{X} \times \mathscr{P}(\mathbb{N}^{<\omega})$ belongs to a given class K iff the corresponding set $\{\langle x, \chi(T)\rangle : \langle x, T\rangle \in X\}$ does also as a subset of $\mathbb{X} \times \mathbb{N}^\mathbb{N}$. □

THEOREM 2.1.4. (i) *The set* $\mathbf{T}$ *of all trees* $T \subseteq \mathbb{N}^{<\omega}$ *is* Π^0_1.

(ii) *The set* **WFT** *of all well-founded trees is* Π^1_1, *and accordingly the set* **IFT** *of all ill-founded trees is* Σ^1_1.

(iii) *The set* $\{\langle S, T\rangle : S, T \in \mathbf{T} \wedge |S| \le |T|\}$ *is* Σ^1_1.

(iv) *There exists a* Σ^1_1 *set* $L \subseteq \mathbf{T} \times \mathbf{T}$ *such that if* $S, T \in \mathbf{T}$ *and* $|T| < \infty$, *then we have* $|S| < |T|$ *iff* $\langle S, T\rangle \in L$.

A comment on (iv). It is not true straightaway that the relation $|S| < |T|$ is Σ^1_1; in fact, it is strictly Π^1_1. However, it becomes Σ^1_1 under an extra condition that T is well founded.

[1] If $s, t \in \mathbb{N}^{<\omega}$, then $s^\frown t$ is the concatenation of s and t. If $k \in \mathbb{N}$, then $s^\frown k$ is the extension of s by $k \in \mathbb{N}$ as the right-most term, and $k^\frown s$ is the extension of s by k as the left-most term.

[2] For a set X of ordinals, $\sup X$ denotes the least ordinal $\ge$ than every $\xi \in X$.

PROOF. (i). To prove that $\mathbf{T}$ is Π^0_1, we show that the corresponding set

$$\mathbf{T}' = \{\chi(T) : T \in \mathbf{T}\} = \{\tau \in 2^{\mathbb{N}} : \{\mathbf{s}_n : \tau(n) = 1\} \in \mathbf{T}\}$$

is Π^0_1 in $\mathbb{N}^{\mathbb{N}}$. Indeed $\tau \in \mathbf{T}'$ iff $\forall n\,(\tau(n) = 0$ or $1)$ and

$$\forall k\,\forall n\,(\tau(n) = 1 \wedge \mathbf{s}_k \subset \mathbf{s}_n \Longrightarrow \tau(k) = 1).$$

Finally, replace the subformula $\mathbf{s}_k \subset \mathbf{s}_n$ by

$$\mathtt{lh}\,\mathbf{s}_k < \mathtt{lh}\,\mathbf{s}_n \wedge \forall j < \mathtt{lh}\,\mathbf{s}_k\,(\mathbf{s}_k(j) = \mathbf{s}_n(j)).$$

Leaving equally routine verification of (ii) as an exercise for the reader, we concentrate on the last two claims of the theorem.

(iii) If $S \subseteq \mathbb{N}^{<\omega}$, then a map $h : S \to \mathbb{N}^{<\omega}$ is a $\subset$-*homomorphism* iff it satisfies $s \subset t \Longrightarrow h(s) \subset h(t)$ for all $s, t \in S$.

LEMMA 2.1.5. *Suppose that $S, T \subseteq \mathbb{N}^{<\omega}$ are trees. Then $|S| \le |T|$ iff there exists a $\subset$-homomorphism $h : S \to T$.*

PROOF (Lemma). *From right to left.* This is easy: if $h : S \to T$ is a $\subset$-homomorphism, then by induction $|u|_S \le |h(u)|_T$ for all $u \in S$.

From left to right. We assert that if $u \in S$, $v \in T$, and $|u|_S \le |v|_T$, then there exists a $\subset$-homomorphism $h_{uv} : S_u \to T_v$. Applying this claim for $u = u = \Lambda$, we obtain the result required because $S_\Lambda = S$ and $T_\Lambda = T$.

The construction of h_{uv} goes on by transfinite induction on $|u|_S$. If $|u|_S = 0$, then u is an endpoint of S, hence $S_u = \{\Lambda\}$, and it suffices to define $h_{uv}(\Lambda) = \Lambda$ for every $v \in T$. Suppose that $0 < |u|_S \le |v|_T$. Then the set $N = \{n : u^\frown n \in S\}$ is non-empty and $|u^\frown n|_S < |u|_S$ for all $n \in N$. On the other hand, for every $n \in N$ there exists a number m_n such that $t^\frown m_n \in T$ and still $|u^\frown n|_S \le |v^\frown m_n|_T$. Applying the inductive hypothesis, we obtain a family of $\subset$-homomorphisms $h_{u^\frown n, v^\frown m_n} : S_{u^\frown n} \to T_{v^\frown m_n}$. Put $h_{uv}(\Lambda) = \Lambda$ and $h_{uv}(n^\frown s) = m_n{}^\frown h_{u^\frown n, v^\frown m_n}(s)$ whenever $n \in N$ and $s \in S_{u^\frown n}$. □ (Lemma)

We are ready to prove (iii). Given $z \in \mathbb{N}^{\mathbb{N}}$, we define $\hat{z} : \mathbb{N}^{<\omega} \to \mathbb{N}^{<\omega}$ by $\hat{z}(\mathbf{s}_k) = \mathbf{s}_n$ iff $z(k) = n$. Then, by Lemma 2.1.5, we have for any trees S, T:

$$|S| \le |T| \Longleftrightarrow \exists z\,(\hat{z} \restriction S \text{ is a } \subset\text{-homeomorphism to } T).$$

This formula can easily be presented in the form

$$\exists z\,(\text{ an arithmetic formula })$$

(see the proof that $\mathbf{T}$ is Π^0_1 above), which yields the result required by Example 1.5.2.

To finally prove (iv) we show that the set

$$L = \{\langle S, T\rangle \in \mathbf{T}^2 : \text{there is a } \subset\text{-homomorphism } h : S \to T \text{ such that } h(\Lambda) \ne \Lambda\}$$

works. That L is Σ^1_1 is verified as above. Let us prove that $|S| < |T| \Longleftrightarrow L(S, T)$ whenever S, T are trees and T is well founded. This is similar to Lemma 2.1.5. If h witnesses $L(S, T)$, then once again $|u|_S \le |h(u)|_T$ for all $u \in S$. In particular, $|S| = |\Lambda|_S \le |h(\Lambda)|_T$. However, $\Lambda \subset h(\Lambda)$ in T by the definition of L. As T is well founded, this implies $|T| = |\Lambda|_T > |h(\Lambda)|_T \ge |S|$, as required. If $|S| < |T|$, then there exists a 1-term sequence $t = \langle m\rangle \in T$ such that still $|S| = |\Lambda|_S \le |t|_T$. Then (see the proof of Lemma 2.1.5) there exists a $\subset$-homomorphism $h : S \to T$ such that $h(\Lambda) = t \ne \Lambda$, as required. □ (Theorem 2.1.4)

2.2. Trees and sets of the first projective level

Obviously a set $X \subseteq \mathbb{N}^{\mathbb{N}}$ is closed (that is, $\mathbf{\Pi}^0_1$) iff $X = [T]$ for a tree $T \subseteq \mathbb{N}^{<\omega}$. For instance, take $T = \{x \restriction m : x \in X \wedge m \in \mathbb{N}\}$. Do the same for sets in spaces $(\mathbb{N}^{\mathbb{N}})^m$ and trees in $(\mathbb{N}^m)^{<\omega}$. There is a more precise form of this assertion.

LEMMA 2.2.1. *Let $p \in \mathbb{N}^{\mathbb{N}}$. A set $X \subseteq (\mathbb{N}^{\mathbb{N}})^\ell$ belongs to $\Pi^0_1(p)$ iff there is a tree $T \subseteq (\mathbb{N}^\ell)^{<\omega}$ of class $\Delta^0_1(p)$ (in the sense of Definition 2.1.3) such that $X = [T]$.*

PROOF. Consider the case when $\ell = 1$ and the parameter p is absent; the general case does not differ much. Thus, we have to prove that a set $X \subseteq \mathbb{N}^{\mathbb{N}}$ belongs to Π^0_1 iff there exists a Δ^0_1 tree $T \subseteq \mathbb{N}^{<\omega}$ such that $X = [T]$. If such a tree T exists, then the set $N = \{n : \mathbf{s}_n \in T\}$ is Δ^0_1 as well by definition. Furthermore,

$$x \in X \iff \forall m \, \forall n \, (x \restriction m = \mathbf{s}_n \Longrightarrow n \in N),$$

and this implies $X \in \Pi^0_1$ because the right-hand side easily admits a transformation to Π^0_1 with the help of Table 1 of transformation rules on page 12.

To prove the converse, suppose that $X \subseteq \mathbb{N}^{\mathbb{N}}$ is a Π^0_1 set. Then its complement $Y = \mathbb{N}^{\mathbb{N}} \smallsetminus X$ is a Σ^0_1 set, and hence by Lemma 1.7.2 there exists a Σ^0_1 set $S \subseteq \mathbb{N}^{<\omega}$ such that $Y = \bigcup_{s \in S} \mathscr{O}_s(\mathbb{N}^{\mathbb{N}})$. Then by Lemma 1.7.1, S is recursively enumerable; therefore, there exists a Δ^0_1 function $f \in \mathbb{N}^{\mathbb{N}}$ such that $Y = \bigcup_n \mathscr{O}_{\mathbf{s}_{f(n)}}(\mathbb{N}^{\mathbb{N}})$. Let T be the tree of all $t \in \mathbb{N}^{<\omega}$ such that $\mathbf{s}_{f(n)} \not\subseteq t$ for all $n < \mathtt{lh}\, t$. One easily proves that T is Δ^0_1. To show that $X = [T]$, consider an arbitrary $x \in X$. Then $x \restriction m \in T$ for all m (therefore $x \in [T]$), since otherwise we have $\mathbf{s}_{f(n)} \subset x \restriction m$ for some n, then $x \in Y$, a contradiction. Conversely, consider any $x \in Y$. Then $\mathbf{s}_{f(n)} \subset x$ for some n. Take any m satisfying $m > n$ and $m > \mathtt{lh}\, \mathbf{s}_{f(n)}$. Then n witnesses that $x \restriction m \notin T$ and hence $x \notin [T]$, as required. □

The lemma just proved immediately implies similar representation for all higher projective classes. We give the next corollary as an example.

COROLLARY 2.2.2. *Suppose that $p \in \mathbb{N}^{\mathbb{N}}$. A set $X \subseteq (\mathbb{N}^{\mathbb{N}})^\ell$ belongs to $\Sigma^1_1(p)$ iff there is a $\Delta^0_1(p)$ tree $T \subseteq (\mathbb{N}^{\ell+1})^{<\omega}$ such that $X = \mathtt{pr}[T]$; that is,*

$$\langle x_1, \dots, x_\ell \rangle \in X \iff \exists y \, \forall m \, (\langle x_1 \restriction m, \dots, x_\ell \restriction m, y \restriction m \rangle \in T).$$

PROOF. First of all, by definition there is a $\Pi^0_1(p)$ set $P \subseteq (\mathbb{N}^{\mathbb{N}})^{\ell+1}$ satisfying

$$\langle x_1, \dots, x_\ell \rangle \in X \iff \exists y \, P(x_1, \dots, x_\ell, y).$$

And by Lemma 2.2.1 there exists a $\Delta^0_1(p)$ tree $T \subseteq (\mathbb{N}^{\ell+1})^{<\omega}$ such that

$$\langle x_1, \dots, x_\ell, y \rangle \in P \iff \forall m \, (\langle x_1 \restriction m, \dots, x_\ell \restriction m, y \restriction m \rangle \in T). \qquad \square$$

This leads us to an extremely useful representation of sets in $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$. In the spirit of Definition 1.8.3, a function $F : \mathbb{X} \to \mathscr{P}(\mathbb{N}^{<\omega})$ is called *K-measurable* if the sets $\{\langle x, s \rangle : s \in F(x)\}$ and $\{\langle x, s \rangle : s \notin F(x)\}$ (subsets of $\mathbb{X} \times \mathbb{N}^{<\omega}$) belong to K (see Definition 2.1.3).

THEOREM 2.2.3. *Suppose that $p \in \mathbb{N}^{\mathbb{N}}$ and X is a $\Sigma^1_1(p)$ set in a product space $\mathbb{X}$. Then there exists a $\Delta^0_1(p)$-measurable map $\rho : \mathbb{X} \to \mathscr{P}(\mathbb{N}^{<\omega})$ such that $X = \rho^{-1}[\mathbf{IFT}]$. Accordingly, if $Y \subseteq \mathbb{X}$ is a $\Pi^1_1(p)$ set, then there exists a $\Delta^0_1(p)$-measurable map $\rho : \mathbb{X} \to \mathscr{P}(\mathbb{N}^{<\omega})$ such that $Y = \rho^{-1}[\mathbf{WFT}]$.*

PROOF. We consider the case when $\mathbb{X} = \mathbb{N}^{\mathbb{N}}$ and the parameter p is absent. By Corollary 2.2.2, there exists a Δ^0_1 tree $T \subseteq (\mathbb{N}^2)^{<\omega}$ such that $X = \text{pr}[T]$; that is,

$$x \in X \iff \exists a \in \mathbb{N}^{\mathbb{N}} \, \forall m \, (\langle x \upharpoonright m, a \upharpoonright m \rangle \in T).$$

If $x \in \mathbb{N}^{\mathbb{N}}$, then put $\rho(x) = \{s \in \mathbb{N}^{<\omega} : \langle x \upharpoonright \text{lh}\, s, s \rangle \in T\}$, this is a tree in $\mathbb{N}^{<\omega}$. Moreover, the map $x \mapsto \rho(x)$ is Σ^0_1-measurable, hence Δ^0_1-measurable, by a rather obvious computation. Let us finally show that $x \in X$ iff $\rho(x)$ is ill founded.

Indeed if $x \notin X$, then there exists $a \in \mathbb{N}^{\mathbb{N}}$ such that $\langle x \upharpoonright m, a \upharpoonright m \rangle \in T$ for all m. Then $a \upharpoonright m \in \rho(x)$ for all m, and hence $\rho(x)$ is ill founded. Conversely, let $a \in \mathbb{N}^{\mathbb{N}}$ witness that $\rho(x)$ is ill founded, so that $\langle x \upharpoonright m, a \upharpoonright m \rangle \in T$ for all m, and hence $x \notin X$. □

2.3. Reduction and separation

The following key result opens our list of basic theorems. Recall that product spaces are those of the form $\mathbb{N}^k \times (\mathbb{N}^{\mathbb{N}})^\ell$. Let us fix a parameter $p \in \mathbb{N}^{\mathbb{N}}$.

THEOREM 2.3.1 (Reduction). *If* X, Y *are* $\Pi^1_1(p)$ *sets in a product space* $\mathbb{X}$ *then there exist* disjoint $\Pi^1_1(p)$ *sets* $X' \subseteq X$ *and* $Y' \subseteq Y$ *such that* $X' \cup Y' = X \cup Y$.

Sets X', Y' as in the theorem are said to *reduce* the pair X, Y.

PROOF. It follows from Theorem 2.2.3 that there exist $\Delta^0_1(p)$-measurable maps $f, g : \mathbb{X} \to \mathscr{P}(\mathbb{N}^{<\omega})$ such that $X = f^{-1}[\mathbf{WFT}]$ and $Y = g^{-1}[\mathbf{WFT}]$. The sets

$$\begin{aligned} X' &= \{x \in X : |f(x)| < |g(x)|\} &= \{x \in X : \neg\, |g(x)| \le |f(x)|\}, \\ Y' &= \{y \in Y : |g(y)| \le |f(y)|\} &= \{y \in Y : \neg\, |f(y)| < |g(y)|\} \end{aligned}$$

obviously reduce the given pair X, Y. So it remains to show that X', Y' are still $\Pi^1_1(p)$ sets. But this follows from Theorem 2.1.4. □

The following theorem is an elementary corollary of Theorem 2.3.1.

THEOREM 2.3.2 (Δ^1_1 Separation). *If* X, Y *are disjoint* $\Sigma^1_1(p)$ *sets in the same product space, then there is a* $\Delta^1_1(p)$ *set* Z *such that* $X \subseteq Z$ *and* $Y \cap Z = \varnothing$.

PROOF. Applying Theorem 2.3.1 to the complementary sets and then taking complements once again, we obtain a pair of mutually complementary Σ^1_1 sets X' and Y' that include X and Y, respectively. Then both X' and Y' are $\Delta^1_1(p)$ sets. □

A set Z as in Theorem 2.3.2 is said to *separate* X from Y. However, there is a bit more in *Separation*. The next theorem historically precedes Theorem 2.3.1 and is based on somewhat different ideas.

THEOREM 2.3.3 (Borel Separation). *If* X, Y *are disjoint* $\mathbf{\Sigma}^1_1$ *sets in a Polish space* $\mathbb{X}$, *then there is a Borel set* Z *such that* $X \subseteq Z$ *and* $Y \cap Z = \varnothing$.

PROOF. By Theorem 1.2.2(i) and 1.3.1, we can consider sets in $\mathbb{N}^{\mathbb{N}}$ *w.l.o.g.* There is a parameter $p \in \mathbb{N}^{\mathbb{N}}$ such that both X and Y are $\Sigma^1_1(p)$ sets. By Theorem 2.2.3 there exist $\Delta^0_1(p)$-measurable, hence continuous by Corollary 1.9.3, maps $f, g : \mathbb{X} \to \mathscr{P}(\mathbb{N}^{<\omega})$ such that $X = f^{-1}[\mathbf{IFT}]$ and $Y = g^{-1}[\mathbf{IFT}]$. Put, for $s, u \in \mathbb{N}^{<\omega}$,

$$X^u_s = \{x \in X : u \subset x \wedge s \in f(x)\} \quad \text{and} \quad Y^u_s = \{y \in Y : u \subset y \wedge s \in g(y)\}.$$

Clearly, $X_s^u = \bigcup_{n,i} X_{s^\frown n}^{u^\frown i}$, and the same for Y.

Now suppose toward the contrary that $X = X_\Lambda$ is *not* Borel separable from $Y = Y_\Lambda$ (Λ is the empty sequence). It follows that for at least one quadruple of $i_0, n_0, j_0, k_0 \in \mathbb{N}$, the set $X_{\langle n_0 \rangle}^{\langle i_0 \rangle}$ is not Borel separable from $Y_{\langle k_0 \rangle}^{\langle j_0 \rangle}$. (Indeed if a Borel set B_{nk}^{ij} separates $X_{\langle n \rangle}^{\langle i \rangle}$ from $Y_{\langle k \rangle}^{\langle j \rangle}$ whenever $i, n, j, k \in \mathbb{N}$, then the Borel set $B = \bigcup_{i,n} \bigcap_{j,k} B_{nk}^{ij}$ separates X from Y.) Iterating this process, we obtain infinite sequences $x, a, y, b \in \mathbb{N}^\mathbb{N}$ such that $X_{a \restriction m}^{x \restriction m}$ is not Borel separable from $Y_{b \restriction m}^{y \restriction m}$, $\forall m$.

Note that all sets $X_{a \restriction m}^{x \restriction m}$ are non-empty. (Indeed the empty set is Borel separable from any other set.) As f is continuous, it easily follows that $a \restriction m \in f(x)$ for all m. Similarly $b \restriction m \in g(y)$ for all m. We conclude that both $f(x)$ and $g(y)$ are ill founded, hence $x \in X$ and $y \in Y$. Therefore, $x \neq y$, let us say $s = x \restriction m \neq y \restriction m$. But then the Borel set $B = \{x' \in \mathbb{N}^\mathbb{N} : s \subset x'\}$ separates $X_{a \restriction m}^{x \restriction m}$ from $Y_{b \restriction m}^{y \restriction m}$, a contradiction. □

The following corollary is a famous result in classical descriptive set theory:

COROLLARY 2.3.4 (SOUSLIN). *A set in a Polish space is Borel iff it is* $\mathbf{\Delta}_1^1$. *A set in a product space is Borel iff it is* $\mathbf{\Delta}_1^1$ *iff it is* $\Delta_1^1(p)$ *for some* $p \in \mathbb{N}^\mathbb{N}$.

PROOF. To see that every $\mathbf{\Delta}_1^1$ set X is Borel, apply Theorem 2.3.3 to X and the complementary set. That all Borel sets are $\mathbf{\Delta}_1^1$, see Proposition 1.9.1(vi) for product spaces, and the result generalizes to all Polish spaces by Theorems 1.2.2(i) and 1.3.1. □

We complete this section with a corollary related to classification of functions. The result follows from Corollary 2.3.4 and Lemma 1.8.4 for product spaces, and Theorem 1.2.2(i) extends it to all Polish spaces.

COROLLARY 2.3.5. *Suppose that* $\mathbb{X}, \mathbb{Y}$ *are product spaces.* (Also true for arbitrary Polish spaces.) *Then the following classes of functions* $F : \mathbb{X} \to \mathbb{Y}$ *coincide*:

Borel maps: *those with Borel graphs* $\{\langle x, y \rangle : y = F(x)\}$;

$\mathbf{\Delta}_1^1$ **maps:** *those with* $\mathbf{\Delta}_1^1$ *graphs*;

Borel measurable maps: *such that all* F*-preimages of open sets are Borel.* □

2.4. Uniformization and Kreisel Selection

Let $\mathbb{X}, \mathbb{Y}$ be arbitrary spaces. If $P \subseteq \mathbb{X} \times \mathbb{Y}$, then they often write $P(x, y)$ instead of $\langle x, y \rangle \in P$. Recall that the set $\mathsf{pr}\, P = \mathsf{dom}\, P = \{x : \exists y\, P(x, y)\} \subseteq \mathbb{X}$ is the *projection* of P (onto $\mathbb{X}$). A set $P \subseteq \mathbb{X} \times \mathbb{Y}$ is called *uniform* if and only if for every $x \in \mathbb{X}$ there is at most one $y \in Y$ such that $P(x, y)$. This essentially means that P is the graph of a partial function $\mathbb{X} \to \mathbb{Y}$. If $P \subseteq Q \subseteq \mathbb{X} \times \mathbb{Y}$, P is uniform, and $\mathsf{pr}\, P = \mathsf{pr}\, Q$, then P is said to *uniformize* Q.

It follows from the axiom of choice that every set P can be uniformized by a suitable set Q. The true problem is to figure out how to "effectively" obtain a set Q that uniformizes a given set P, or, saying it differently, how simple a uniformizing set Q can be chosen in the sense of, say, being in a certain projective class. For instance, every closed set $P \subseteq \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}$ can be uniformized by picking the lexicographically least element in every cross-section $P_x = \{y : P(x, y)\}$, and the result will be a uniform $\mathbf{\Pi}_1^1$ set. Thus, every closed set can be uniformized by a $\mathbf{\Pi}^1_1$ set. Some much less trivial uniformization theorems are known, *e.g.*, for $\mathbf{\Sigma}^1_1$ sets, but the following theorem is the most profound result in this direction.

THEOREM 2.4.1 (NOVIKOV–KONDO–ADDISON). *Suppose that* $p \in \mathbb{N}^{\mathbb{N}}$ *and* $\mathbb{X}, \mathbb{Y}$ *are product spaces. Then every* $\Pi^1_1(p)$ *set* $P \subseteq \mathbb{X} \times \mathbb{Y}$ *can be uniformized by a set of the same class. (And the same for* $\mathbf{\Pi}^1_1$.*)*

PROOF. The proof is based on approximately the same ideas as the proof of Theorem 2.3.1 above, but it is significantly trickier. We suppose, for the sake of brevity, that $\mathbb{X} = \mathbb{Y} = \mathbb{N}^{\mathbb{N}}$, and we consider a Π^1_1 set $P \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$. It follows from Corollary 2.2.2 that there exists a Δ^0_1 tree $T \subseteq (\mathbb{N}^3)^{<\omega}$ such that $P = (\mathbb{N}^{\mathbb{N}})^2 \smallsetminus \mathrm{pr}[T]$; that is,

$$\langle x, y\rangle \in P \iff \forall z\, \exists m\, (\langle x \restriction m, y \restriction m, z \restriction m\rangle \notin T).$$

Put $T(x,y) = \{s \in \mathbb{N}^{<\omega} : \langle x \restriction \mathtt{lh}\, s, y \restriction \mathtt{lh}\, s, s\rangle \in T\}$. These sets are trees in $\mathbb{N}^{<\omega}$, and we have $P(x,y)$ iff $T(x,y)$ is well founded. (See the proof of Theorem 2.2.3.) Put $T(x,y)_s = \{t \in \mathbb{N}^{<\omega} : s^\frown t \in T(x,y)\}$ for $s \in \mathbb{N}^{<\omega}$. In particular, $T(x,y)_s = \varnothing$ and $|T(x,y)_s| = -1$ whenever $s \notin T(x,y)$, but $\Lambda \in T(x,y)_s$ and $|T(x,y)_s| \geq 0$ whenever $s \in T(x,y)$.

Define, by induction on n, a $\subseteq$-decreasing sequence of sets $P_n \subseteq P$ as follows: $P_0 = P$, and P_{n+1} consists of all pairs $\langle x, y\rangle \in P_n$ such that

$$\forall y'\, (P_n(x,y') \Longrightarrow [y(n) < y'(n) \vee (y(n) = y'(n) \wedge |T(x,y)_{\mathtt{s}_n}| \leq |T(x,y')_{\mathtt{s}_n}|)]).$$

Finally, put $Q = \bigcap_n P_n$. Let us prove that Q is a Π^1_1 set and Q uniformizes P.

That Q is Π^1_1 is especially surprising because the passage from P_n to P_{n+1} includes the quantifier $\forall y'\, (P_n(x,y') \Longrightarrow \dots)$, apparently leading to at least Π^1_2, since P_n is no better than Π^1_1. However, let us consider the set W of all $\langle n, x, y\rangle \in \mathbb{N}^1 \times (\mathbb{N}^{\mathbb{N}})^2$ such that

$$(*)\quad \begin{cases} \exists y'\, \big(\forall k < n\, (y(k) = y'(k) \wedge |T(x,y)_{\mathtt{s}_k}| = |T(x,y')_{\mathtt{s}_k}|) \\ \qquad\qquad \wedge [y'(n) < y(n) \vee (y'(n) = y(n) \wedge |T(x,y')_{\mathtt{s}_n}| < |T(x,y)_{\mathtt{s}_n}|)]\big). \end{cases}$$

We assert that

$$(\dagger)\qquad Q(x,y) \quad \text{iff} \quad P(x,y) \wedge \forall n \neg W(n,x,y).$$

Indeed, let $\langle x, y\rangle \in P \smallsetminus Q$. Then there exists n such that $\langle x, y\rangle \in P_n \smallsetminus P_{n+1}$, and hence there is y' such that $P_n(x,y')$, $y'(n) \leq y(n)$, and either $y'(n) < y(n)$ strictly or $|T(x,y)_{\mathtt{s}_n}| > |T(x,y')_{\mathtt{s}_n}|$. This y' witnesses $W(n,x,y)$: the first line of $(*)$ holds because we have both $P_{k+1}(x,y)$ and $P_{k+1}(x,y')$ for all $k < n$ in this case. Now suppose that y' witnesses $\langle n,x,y\rangle \in W$ for some n, and still $P(x,y)$. In particular $\langle x,y\rangle \in P_n$. Then, since the first line of $(*)$ holds for all $k < n$, we have $\langle x, y'\rangle \in P_n$ as well. But then it follows from the second line that $\langle x,y\rangle \notin P_{n+1}$, and hence $\langle x,y\rangle \notin Q$, as required. This proves $(\dagger)$.

Let us apply $(\dagger)$ to verify that Q is still Π^1_1. It suffices to show that in the assumption $\langle x,y\rangle \in P$ the relation $W(n,x,y)$ is equivalent to a Σ^1_1 relation. The equality $|T(x,y)_{\mathtt{s}_k}| = |T(x,y')_{\mathtt{s}_k}|$ is Σ^1_1 by Theorem 2.1.4(iii). And the inequality $|T(x,y')_{\mathtt{s}_n}| < |T(x,y)_{\mathtt{s}_n}|$ is Σ^1_1 by Theorem 2.1.4(iv), applicable because in the assumption $\langle x,y\rangle \in P$ the tree $T(x,y)$ is well founded, and hence so are all trees $T(x,y)_{\mathtt{s}_n}$. Thus, Q is indeed Π^1_1.

In order to prove that Q uniformizes P, let us come back to the definition of the sequence of sets P_n. For every $x \in \mathrm{pr}\, P_n$, let m^n_x denote the least m such that

$y(n) = m$ for some y with $\langle x, y\rangle \in P$. And let ξ_x^n be the least ordinal ξ such that $y(n) = m_x^n$ and $|T(x,y)_{\mathbf{s}_n}| = \xi$ for some y with $\langle x, y\rangle \in P$. And by definition, P_{n+1} consists of all pairs $\langle x, y\rangle \in P_n$ such that $y(n) = m_x^n$ and $|T(x,y)_{\mathbf{s}_n}| = \xi_x^n$. We conclude that $\operatorname{pr} P_n = \operatorname{pr} P_{n+1}$ for all n, and hence $\operatorname{pr} P_n = \operatorname{pr} P$, $\forall\, n$.

In addition, if $\langle x, y\rangle \in P_n$ and $k < n$, then still $\langle x, y\rangle \in P_{k+1}$ and $y(k) = m_x^k = y_x(k)$; therefore, we have $y \restriction n = y_x \restriction n$.

If $x \in \operatorname{pr} P$, then define $y_x \in \mathbb{N}^{\mathbb{N}}$ by $y_x(n) = m_x^n$ for all n. By definition we have $y(n) = m_x^n$ whenever $\langle x, y\rangle \in P_{n+1}$. It follows that $y = y_x$ whenever $\langle x, y\rangle \in Q$; therefore, Q is a uniform set. It remains to prove that $\operatorname{pr} Q = \operatorname{pr} P$ or, in other words, $\langle x, y_x\rangle \in Q$ for all $x \in \operatorname{pr} P$.

In fact we do not know yet that all pairs of the form $\langle x, y_x\rangle$, $x \in \operatorname{pr} P$, even belong to the original set P! This is based on the following lemma:

LEMMA 2.4.2. *Suppose that* $m, n \in \mathbb{N}$, $\mathbf{s}_n \subset \mathbf{s}_m$, *and* $\mathbf{s}_m$ *(and then* $\mathbf{s}_n$, *too) belongs to the tree* $T(x, y_x) = \{s \in \mathbb{N}^{<\omega} : \langle x \restriction \mathtt{lh}\, s, y_x \restriction \mathtt{lh}\, s, s\rangle \in T\}$. *Then* $\xi_x^m < \xi_x^n$.

PROOF. Recall that $x \in \operatorname{pr} P = \operatorname{pr} P_k$, $\forall\, k$. It follows that there is y satisfying $\langle x, y\rangle \in P_{m+1}$, and then $\in P_{n+1}$ as well because $n < m$ follows from $\mathbf{s}_n \subset \mathbf{s}_m$. Then $\xi_x^m = |T(x,y)_{\mathbf{s}_m}|$ and $\xi_x^n = |T(x,y)_{\mathbf{s}_n}|$ by the above. Note that $\mathbf{s}_m \in T(x,y)$. (Indeed, it suffices to check that $\langle x \restriction \ell,\, y \restriction \ell,\, \mathbf{s}_m\rangle \in T$, where $\ell = \mathtt{lh}\, \mathbf{s}_m$. But $y \restriction \ell = y_x \restriction \ell$ by the above because $\langle x, y\rangle \in P_m$ and $\ell = \mathtt{lh}\, \mathbf{s}_m \le m$. However, $\langle x \restriction \ell,\, y_x \restriction \ell,\, \mathbf{s}_m\rangle \in T$ since $\mathbf{s}_m \in T(x, y_x)$.) It follows that $\mathbf{s}_n \in T(x,y)$ as well because $\mathbf{s}_n \subset \mathbf{s}_m$. But then $|T(x,y)_{\mathbf{s}_m}| < |T(x,y)_{\mathbf{s}_n}|$ as required. □ (Lemma)

COROLLARY 2.4.3. *If* $x \in \operatorname{pr} P$, *then* $\langle x, y_x\rangle \in P$.

PROOF. Otherwise the tree $T(x, y_x)$ is ill founded; that is, there exists $a \in \mathbb{N}^{\mathbb{N}}$ such that $a \restriction j \in T(x, y_x)$ for all j. Let $n(j) \in \mathbb{N}$ be the number of $a \restriction j$, so that $a \restriction j = \mathbf{s}_{n(j)}$. Then $\xi_x^{n(j)} < \xi_x^{n(i)}$ for all $i < j$ by Lemma 2.4.2. Therefore, we obtain a descending chain of ordinals, a contradiction. □ (Corollary)

To accomplish the proof of the theorem we need yet another lemma.

LEMMA 2.4.4. *If* $x \in \operatorname{pr} P$ *and* $\mathbf{s}_n \in T(x, y_x)$, *then* $|T(x, y_x)_{\mathbf{s}_n}| \le \xi_x^n$.

PROOF. The tree $T(x, y_x)$ is well founded by Corollary 2.4.3; therefore, we can argue by induction on $|\mathbf{s}_n|_{T(x,y_x)}$. If $\mathbf{s}_n$ is an endpoint in $T(x, y_x)$, then $|T(x, y_x)_{\mathbf{s}_n}| = 0$, and there is nothing to prove. Let $\mathbf{s}_n$ be not an endpoint. If, toward the contrary, $|T(x, y_x)_{\mathbf{s}_n}| > \xi_x^n$, then there is a sequence $\mathbf{s}_m \in T(x, y_x)$ such that $\mathbf{s}_n \subset \mathbf{s}_m$ and $|T(x, y_x)_{\mathbf{s}_m}| \ge \xi_x^n$. But $\xi_x^m < \xi_x^n$ by Lemma 2.4.2. Therefore, $|T(x, y_x)_{\mathbf{s}_m}| \ge \xi_x^m$, contrary to the inductive hypothesis. □ (Lemma)

Now we are ready to show that $\langle x, y_x\rangle \in P_n$ by induction on n; hence $\langle x, y_x\rangle \in Q$ and we are done. Suppose that $\langle x, y_x\rangle \in P_n$. To prove $\langle x, y_x\rangle \in P_{n+1}$ it suffices to check that $y_x(n) = m_x^n$ and $|T(x, y_x)_{\mathbf{s}_n}| = \xi_x^n$. The first equality is fulfilled by definition. The second one follows from Lemma 2.4.4: indeed, $|T(x, y_x)_{\mathbf{s}_n}| \ge \xi_x^n$ since $\langle x, y_x\rangle \in P_n$ and $y_x(n) = m_x^n$. □ (Theorem 2.4.1)

As an immediate corollary, we obtain

THEOREM 2.4.5 (Kreisel Selection). *If* $\mathbb{X}$ *is a product space,* $P \subseteq \mathbb{X} \times \mathbb{N}$ *is a* $\Pi_1^1(p)$ *set, and* $X \subseteq \operatorname{pr} P$ *is a* $\Delta_1^1(p)$ *set, then there is a* $\Delta_1^1(p)$ *function* $F : X \to \mathbb{N}$ *such that* $\langle x, F(x)\rangle \in P$ *for all* $x \in X$.

PROOF. Let $Q \subseteq P$ be a $\Pi^1_1(p)$ set which uniformizes P. For every $x \in X$, let $F(x)$ be the only n with $\langle x, n\rangle \in Q$. Immediately, (the graph of) F is $\Pi^1_1(p)$. In addition, $F(x) = n \iff \forall m \neq n\,(F(x) \neq m)$ whenever $x \in X$, thus F is $\Sigma^1_1(p)$ as well. □

2.5. Universal sets

Recall that a *cross-section* of a set $P \subseteq \mathbb{A} \times \mathbb{X}$ is a set of the form

$$(P)_a = \{x \in \mathbb{X} : P(a, x)\}, \quad \text{where} \quad a \in \mathbb{A},$$

and $\mathbb{A}, \mathbb{X}$ are arbitrary underlying sets, *e.g.*, Polish spaces. But mainly just P_a is written instead of $(P)_a$ whenever this does not lead to confusion with indices.

DEFINITION 2.5.1. Let $\mathbb{X}$ be a product space.

A set $U \subseteq \mathbb{N} \times \mathbb{X}$ is a *universal* Σ^i_n *set* if U belongs to Σ^i_n, and for every Σ^i_n set $X \subseteq \mathbb{X}$ there exists an index $e \in \mathbb{N}$ such that $X = (U)_e$.

A set $U \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{X}$ is a Σ^i_n *set universal for* $\mathbf{\Sigma}^i_n$ if U belongs to Σ^i_n, and for every $\mathbf{\Sigma}^i_n$ set $X \subseteq \mathbb{X}$ there is an element $z \in \mathbb{N}^{\mathbb{N}}$ with $X = (U)_z$.

Similarly for Π^i_n and $\mathbf{\Pi}^i_n$. □

Thus there are two slightly different notions of universal sets. The first of them, sometimes called $\mathbb{N}$-*universal*, applies to lightface classes Σ^i_n, $\Sigma^i_n(a)$, Π^i_n, $\Pi^i_n(a)$, the second, sometimes called $\mathbb{N}^{\mathbb{N}}$-*universal*, applies to Borel and projective classes $\mathbf{\Sigma}^0_\xi$, $\mathbf{\Sigma}^1_n$, $\mathbf{\Pi}^0_\xi$, $\mathbf{\Pi}^1_n$. And there are no universal sets for Δ-classes; see below.

THEOREM 2.5.2. (i) *Each class of the form* $\mathbf{\Sigma}^0_\xi$, $\mathbf{\Sigma}^1_n$, $\mathbf{\Pi}^0_\xi$, $\mathbf{\Pi}^1_n$ *admits* $\mathbb{N}^{\mathbb{N}}$-*universal sets of the same class. Moreover, classes* $\mathbf{\Sigma}^i_n$ *and* $\mathbf{\Pi}^i_n$ *admit* $\mathbb{N}^{\mathbb{N}}$-*universal sets that belong to the corresponding lightface classes* Σ^i_n *and* Π^i_n.

(ii) *Each class of the form* Σ^i_n, Π^i_n *and* $\Sigma^i_n(a)$, $\Pi^i_n(a)$ $(a \in \mathbb{N}^{\mathbb{N}})$ *admits* $\mathbb{N}$-*universal sets of the same class.*

PROOF (Sketch). (i) Let us define a Σ^0_1 set $U \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ universal for $\mathbf{\Sigma}^0_1$. We make use of the enumeration $\mathbb{N}^{<\omega} = \{\mathbf{s}_n : n \in \mathbb{N}\}$ (see Definition 1.1.6(ii)). Put

$$U = \{\langle x, y\rangle \in \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} : \exists n\,(x(n) = 0 \wedge \mathbf{s}_n \subset y)\}.$$

The universality of this set for $\mathbf{\Sigma}^0_1$ is rather clear, while its class Σ^0_1 follows from the recursivity of the enumeration $\{\mathbf{s}_n\}$.

After this initial step, we proceed by induction. First of all, the complement of a universal $\mathbf{\Sigma}^1_1$ set is a universal $\mathbf{\Pi}^1_1$ set. Let us show how to pass from say $\mathbf{\Pi}^1_n$ to $\mathbf{\Sigma}^1_{n+1}$. Thus, suppose that $U \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ is a Π^1_n set universal for $\mathbf{\Pi}^1_n$. A suitable Δ^0_1 bijection of $\mathbb{N}^{\mathbb{N}}$ onto $(\mathbb{N}^{\mathbb{N}})^2$ produces a Π^1_n set $U' \subseteq \mathbb{N}^{\mathbb{N}} \times (\mathbb{N}^{\mathbb{N}})^2$ still universal for $\mathbf{\Pi}^1_n$. One can easily prove that

$$V = \{\langle x, z\rangle : \exists y\,(\langle x, y, z\rangle \in U')\}$$

is a Σ^1_{n+1} set universal for $\mathbf{\Sigma}^1_{n+1}$.

(ii) The existence of universal Σ^0_1 sets $U \subseteq \mathbb{N} \times \mathbb{N}^{\mathbb{N}}$ is a standard recursion-theoretic result, see *e.g.* [**Sho01**]. To get such a set, begin with an enumeration $\{\varphi_k(n, x)\}_{k \in \mathbb{N}}$ of all bounded formulas with two variables, recursive in a sense not to be described here, and then

$$U = \{\langle k, x\rangle : \exists n\, \varphi_k(n, x)\}$$

is a universal Σ_1^0 set. Then we proceed to higher classes by induction as above. □

COROLLARY 2.5.3 (The hierarchy theorem). *Each class Σ_n^i contains a set $x \subseteq \mathbb{N}$ that does not belong to the dual class Π_n^i, and that contains a set $X \subseteq \mathbb{N}^{\mathbb{N}}$ which does not belong to the dual boldface class $\mathbf{\Pi}_n^i$. And vice versa.*

PROOF. If $U \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ is a Σ_n^i set universal for $\mathbf{\Sigma}_n^i$, then U cannot belong to $\mathbf{\Pi}_n^i$. Indeed otherwise, $X = \{y : \langle y, y\rangle \notin U\}$ is still $\mathbf{\Sigma}_n^i$ as a continuous preimage of the $\mathbf{\Sigma}_n^i$ complement of the $\mathbf{\Pi}_n^i$ set U. (We make use of Proposition 1.9.1(ix).) By the universality of U there exists $x \in \mathbb{N}^{\mathbb{N}}$ such that $y \in X \iff \langle x, y\rangle \in U$ for all y. With $y = x$ we get a contradiction. □

REMARK 2.5.4. The same argument as in the proof of Corollary 2.5.3 implies that Theorem 2.5.2 fails for classes Δ and $\mathbf{\Delta}$. Indeed, if U is, say, $\mathbf{\Delta}_n^1$, then X, as in the proof of Corollary 2.5.3, also belongs to $\mathbf{\Delta}_n^1$ because these classes are self-dual. □

One more application is the following non-separation result:

THEOREM 2.5.5. *There are Π_1^1 sets $X, Y \subseteq \mathbb{N}^{\mathbb{N}}$ disjoint but Borel unseparable.*

PROOF. Let $U \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ be a Σ_n^i set universal for $\mathbf{\Sigma}_n^i$. The sets

$$U' = \{\langle (x)_0^2, y\rangle : \langle x, y\rangle \in U\} \quad \text{and} \quad U'' = \{\langle (x)_1^2, y\rangle : \langle x, y\rangle \in U\}$$

are still Σ_1^1 by different claims of Proposition 1.9.1. Indeed

$$\langle z, y\rangle \in U' \iff \exists x\, (z = (x)_1^2 \wedge \langle x, y\rangle \in U)$$

(see Definition 1.1.6 on $(x)_j^\ell$). And U', U'' form a $\mathbf{\Sigma}_1^1$-universal pair in the sense that if $X', X'' \subseteq \mathbb{N}^{\mathbb{N}}$ are $\mathbf{\Sigma}_1^1$, then there exists $x \in \mathbb{N}^{\mathbb{N}}$ with $X' = (U')_x$ and $X'' = (U'')_x$. (First pick x', x'' such that $X' = (U)_{x'}$ and $X'' = (U)_{x''}$, and then consider x such that $(x)_0^2 = x'$ and $(x)_1^2 = x''$.) It follows that the complementary Π_1^1 sets $V' = (\mathbb{N}^{\mathbb{N}})^2 \smallsetminus U'$ and $V'' = (\mathbb{N}^{\mathbb{N}})^2 \smallsetminus U''$ form a $\mathbf{\Pi}_1^1$-universal pair in the same sense. By Theorem 2.3.1 there exist disjoint Π_1^1 sets $P' \subseteq V'$ and $P'' \subseteq V''$ such that $P' \cup P'' = V' \cup V''$. We assert that P', P'' are Borel unseparable.

Suppose toward the contrary that $B \subseteq (\mathbb{N}^{\mathbb{N}})^2$ is a Borel set such that $P' \subseteq B$ and $P'' \cap B = \varnothing$. The set $X = \{a \in \mathbb{N}^{\mathbb{N}} : \langle a, a\rangle \notin B\}$ is Borel as well; therefore, both X and its complement $\mathbb{N}^{\mathbb{N}} \smallsetminus X$ are $\mathbf{\Pi}_1^1$. It follows that there exists $a \in \mathbb{N}^{\mathbb{N}}$ such that $X = (P')_a = \mathbb{N}^{\mathbb{N}} \smallsetminus (P'')_a$. Then

$$a \in X \iff \langle a, a\rangle \notin X \iff \langle a, a\rangle \in P'' \iff a \in (P'')_a \iff a \notin X\,,$$

a contradiction. □

COROLLARY 2.5.6 (Non-uniformization). *There exists a Π_1^0* (in particular, a closed) *set $P \subseteq (\mathbb{N}^{\mathbb{N}})^2$ non-uniformizable by a $\mathbf{\Sigma}_1^1$ set.*

PROOF. Let $X, Y \subseteq \mathbb{N}^{\mathbb{N}}$ be given by Theorem 2.5.5, and $X' = \mathbb{N}^{\mathbb{N}} \smallsetminus X$, $Y' = \mathbb{N}^{\mathbb{N}} \smallsetminus Y$, the complementary Σ_1^1 sets. Define $x, y \in \mathbb{N}^{\mathbb{N}}$ by $x(k) = 0$ and $y(k) = 1$ for all k. The set $Q = (X' \times \{x\}) \cup (Y' \times \{y\})$ is Σ_1^1 and the projection

$$\operatorname{pr} Q = \{a : \exists b\, (\langle a, b\rangle \in Q)\} = X' \cup Y'$$

is equal to $\mathbb{N}^{\mathbb{N}}$ since $X \cap Y = \varnothing$. We assert that Q is not uniformizable by a $\mathbf{\Sigma}_1^1$ set $U \subseteq (\mathbb{N}^{\mathbb{N}})^2$. Indeed otherwise, the sets

$$A' = \{a : \langle x, a\rangle \in U\} \quad \text{and} \quad B' = \{a : \langle y, a\rangle \in U\}$$

are Σ^1_1 and we have $A' \subseteq X'$, $B' \subseteq Y'$, and $A' \cup B' = X' \cup Y' = \operatorname{pr} Q = \mathbb{N}^{\mathbb{N}}$. In particular, A', B' are complementary Σ^1_1 sets, and hence they are Borel by Corollary 2.3.4. Yet, by definition, B' separates X from Y, a contradiction.

To end the proof, we note that there is a Π^0_1 set $P \subseteq (\mathbb{N}^{\mathbb{N}})^3 = (\mathbb{N}^{\mathbb{N}})^2 \times \mathbb{N}^{\mathbb{N}}$ whose projection onto $(\mathbb{N}^{\mathbb{N}})^2$ coincides with Q. Then P cannot be uniformized, in the sense of $\mathbb{N}^{\mathbb{N}} \times (\mathbb{N}^{\mathbb{N}})^2$, by a Σ^1_1 set $W \subseteq P$. Indeed if, toward the contrary, W is such a set, then $U = \{\langle a, b\rangle : \exists c(\langle a, b, c\rangle \in W)\}$ is still a Σ^1_1 set and U uniformizes Q, a contradiction. □

2.6. Good universal sets

This is a useful subpopulation of universal sets.

Definition 2.6.1. Let $\mathbb{X}$ be a product space. A universal Σ^i_n set $U \subseteq \mathbb{N} \times \mathbb{X}$ is *good* iff for every other Σ^i_n set $P \subseteq \mathbb{N} \times \mathbb{X}$ there is a Δ^0_1 function $f : \mathbb{N} \to \mathbb{N}$ such that $(P)_n = (U)_{f(n)}$ for all $n \in \mathbb{N}$. Similarly for Π^i_n. □

It is not difficult to get good universal sets beginning with arbitrary ones.

Theorem 2.6.2 (Universal sets). *For every product space $\mathbb{X}$ there exist a "good" universal Σ^1_1 set $U \subseteq \mathbb{N} \times \mathbb{X}$ and a "good" universal Π^1_1 set $V \subseteq \mathbb{N} \times \mathbb{X}$.* (In fact we can take $V = (\mathbb{N} \times \mathbb{X}) \smallsetminus U$.)

Proof. We begin with an arbitrary universal Σ^1_1 set $W \subseteq \mathbb{N} \times (\mathbb{N} \times \mathbb{N}^{\mathbb{N}})$. Put $U = \{\langle \pi(e, k), x\rangle : \langle e, k, x\rangle \in W\}$, where $\pi : \mathbb{N}^2 \xrightarrow{\text{onto}} \mathbb{N}$ is an arbitrary recursive bijection, say, $\pi(e, k) = 2^e(2k + 1) - 1$. This is a universal Σ^1_1 set for $\mathbb{X}$. Indeed, consider an arbitrary Σ^1_1 set $X \subseteq \mathbb{X}$. Then $Y = \{0\} \times X$ is still a Σ^1_1 set in $\mathbb{N} \times \mathbb{X}$. By the choice of W, there is an index e such that

$$x \in X \iff \langle 0, x\rangle \in Y \iff \langle e, 0, x\rangle \in W \iff \langle n, x\rangle \in U,$$

where $n = \pi(e, 0)$, as required. To show that U is good, consider a Σ^1_1 set $P \subseteq \mathbb{N} \times \mathbb{X}$. By the choice of W, there is an index e such that

$$\langle n, x\rangle \in P \iff \langle e, n, x\rangle \in W \iff \langle \pi(e, n), x\rangle \in U,$$

and hence $(P)_n = (U)_{f(n)}$ for all n, where $f(n) = \pi(e, n)$. □

To show how "good" universal sets work, we prove:

Proposition 2.6.3. *Let $\mathbb{X}$ be a product space and $U \subseteq \mathbb{N} \times \mathbb{X}$ a good universal Π^1_1 set. Then for every pair of Π^1_1 sets $P, Q \subseteq \mathbb{N} \times \mathbb{X}$, there are recursive functions $f, g : \mathbb{N} \to \mathbb{N}$ such that for each $m, n \in \mathbb{N}$ the pair of cross-sections $(U)_{f(m,n)}$, $(U)_{g(m,n)}$ reduces the pair $(P)_m, (Q)_n$.*

Thus, reduction of cross-sections can be maintained in a uniform way.

Proof. Consider the following Π^1_1 sets in $(\mathbb{N} \times \mathbb{N}) \times \mathbb{X}$:

$$A = \{\langle m, n, x\rangle : \langle m, x\rangle \in P \wedge n \in \mathbb{N}\}, \qquad B = \{\langle m, n, x\rangle : \langle n, x\rangle \in Q \wedge m \in \mathbb{N}\}.$$

By Theorem 2.3.1 (Reduction), there is a pair of Π^1_1 sets $A' \subseteq A$ and $B' \subseteq B$ that reduce the given pair A, B. Accordingly, the pair of sets $(A')_{mn}, (B')_{mn}$ reduces the cross-sections $(A)_{mn}, (B)_{mn}$ for each m, n. Finally, by the good universality there are recursive functions f, g such that $(A')_{mn} = (U)_{f(m,n)}$ and $(B')_{mn} = (U)_{g(m,n)}$ for all m, n. □

2.7. Reflection

Universal sets allow us to naturally classify *collections* of pointsets.

Indeed, suppose that $\mathbb{X}$ is a product space and $U \subseteq \mathbb{N} \times \mathbb{X}$ is a universal Π^1_1 set. A collection $\mathscr{A}$ of Π^1_1 sets $X \subseteq \mathbb{X}$ is Π^1_1 *in the codes* if $\{e : U_e \in \mathscr{A}\}$ is a Π^1_1 set in $\mathbb{N}$. Similarly, if $V \subseteq \mathbb{N} \times \mathbb{X}$ is a universal Σ^1_1 set, then a collection $\mathscr{A}$ of Σ^1_1 sets $X \subseteq \mathbb{X}$ is Π^1_1 *in the codes* if $\{e : V_e \in \mathscr{A}\}$ is a Π^1_1 set.

These notions appear to be dependent on the choice of the universal sets U, V. However, it is quite clear that the dependence does not exist for the category of *good* universal sets. In other words if $U, U' \subseteq \mathbb{N} \times \mathbb{X}$ are good universal Π^1_1 sets, then being Π^1_1 in the codes in the sense of U and being Π^1_1 in the codes in the sense of U' is one and the same. Thus "being Π^1_1 in the codes" is understood in the sense of any/every good universal Π^1_1 set. Similarly for Σ^1_1.

The following theorem is somewhat less transparent than most of the results in descriptive set theory presented in this chapter. But it is very useful in some applications because it allows us to considerably shorten sophisticated arguments with multiple applications of Separation and Kreisel Selection.

THEOREM 2.7.1 (Reflection). *Let $\mathbb{X}$ be a recursively presented Polish space.*

Π^1_1 form: *Suppose that a collection $\mathscr{A}$ of Π^1_1 sets $X \subseteq \mathbb{X}$ is Π^1_1 in the codes. (In the sense of a fixed good universal Π^1_1 set $U \subseteq \mathbb{N} \times \mathbb{X}$.) Then for every $Y \in \mathscr{A}$, there is a Δ^1_1 set $D \in \mathscr{A}$ with $D \subseteq Y$.*

Σ^1_1 form: *Suppose that a collection $\mathscr{A}$ of Σ^1_1 sets $X \subseteq \mathbb{X}$ is Π^1_1 in the codes. Then for every $Y \in \mathscr{A}$, there is a Δ^1_1 set $D \in \mathscr{A}$ with $Y \subseteq D$.*

One of the (generally, irrelevant here) consequences of this theorem is that the set of all codes of a strictly Π^1_1 set or properly Σ^1_1 set is never Π^1_1.

PROOF. We begin with a good universal Σ^0_1 set $R \subseteq \mathbb{N} \times (\mathbb{N} \times \mathbb{N})$. Put $\widehat{R} = \{\langle e, \mathbf{s}_k, \mathbf{s}_n\rangle : R(e,k,n)\}$ (a subset of $\mathbb{N} \times \mathbb{N}^{<\omega} \times \mathbb{N}^{<\omega}$). The set

$$U = \{\langle e, x\rangle \in \mathbb{N} \times \mathbb{N}^{\mathbb{N}} : \forall y \, \exists m \, \widehat{R}(e, x \restriction m, y \restriction m)\}$$

is then a universal Π^1_1 set in $\mathbb{N} \times \mathbb{N}^{\mathbb{N}}$ by Corollary 2.2.2. We assert that U is a good universal Π^1_1 set. To check this, consider an arbitrary Π^1_1 set $P \subseteq \mathbb{N} \times \mathbb{N}^{\mathbb{N}}$. Using a suitable version of Corollary 2.2.2, we obtain a Δ^0_1 set $\widehat{S} \subseteq \mathbb{N} \times \mathbb{N}^{<\omega} \times \mathbb{N}^{<\omega}$ such that

$$P = \{\langle e, x\rangle \in \mathbb{N} \times \mathbb{N}^{\mathbb{N}} : \forall y \, \exists m \, \widehat{S}(e, x \restriction m, y \restriction m)\}.$$

Then $S = \{\langle e, k, n\rangle : \widehat{S}(e, \mathbf{s}_k, \mathbf{s}_n)\}$ is a Δ^0_1 set as well, and since R is a good universal set, there is a Δ^0_1 function $a \in \mathbb{N}^{\mathbb{N}}$ such that $(S)_e = (R)_{a(e)}$ for all $e \in \mathbb{N}$. But then $(P)_e = (U)_{a(e)}$ for all e, as required.

After this preamble we prove the Π^1_1 form of the theorem in the case $\mathbb{X} = \mathbb{N}^{\mathbb{N}}$. In addition to U, we make use of a good universal Π^1_1 set $W \subseteq \mathbb{N} \times \mathbb{N}$. It follows from Theorem 2.2.3 that we can associate, by means of certain Δ^1_1 maps, a tree $T_{ne} \subseteq \mathbb{N}^{<\omega}$ to every $\langle n, e\rangle \in \mathbb{N}^2$ and a tree $S_{my} \subseteq \mathbb{N}^{<\omega}$ to every $\langle m, y\rangle \in \mathbb{N} \times \mathbb{N}^{\mathbb{N}}$ so that the following equivalences are fulfilled:

$$U(m,y) \iff |S_{my}| < \omega_1 \quad \text{and} \quad W(n,e) \iff |T_{ne}| < \omega_1.$$

By definition, the set $A = \{e : (U)_e \in \mathscr{A}\}$ is Π^1_1, and hence there is $\widehat{n}$ such that $A = (W)_{\widehat{n}}$; that is, $(U)_e \in \mathscr{A}$ iff $W(\widehat{n}, e)$. And as Y is a Π^1_1 subset of

$\mathbb{X} = \mathbb{N}^{\mathbb{N}}$, there exists $\widehat{m}$ such that $Y = (U)_{\widehat{m}}$; that is, $y \in Y$ iff $U(\widehat{m}, y)$. Note that the set $Q = \{\langle e, y\rangle \in \mathbb{N} \times Y : |S_{\widehat{m}y}| < |T_{\widehat{n}e}|\}$ is Π^1_1 by Theorem 2.1.4; the inequality $|S_{\widehat{m}y}| < |T_{\widehat{n}e}|$ is equivalent to the negation of $|T_{\widehat{n}e}| \le |S_{\widehat{m}y}|$. As U is a good universal set, there exists a Δ^0_1 map $a \in \mathbb{N}^{\mathbb{N}}$ such that $(Q)_e = (U)_{a(e)}$ for all $e \in \mathbb{N}$.

Return for a moment to the universal Σ^0_1 set $R \subseteq \mathbb{N} \times (\mathbb{N} \times \mathbb{N})$ considered in the beginning of the proof. As $a \in \mathbb{N}^{\mathbb{N}}$ is Δ^1_1, that is, recursive map, the recursion theorem of recursion theory implies the existence of a number ε such that $(R)_\varepsilon = (R)_{a(\varepsilon)}$. Then obviously $(Q)_\varepsilon = (U)_{a(\varepsilon)} = (U)_\varepsilon$ by the above.

We assert that the set $(U)_\varepsilon = (Q)_\varepsilon$ belongs to $\mathscr{A}$. Indeed, if $(U)_\varepsilon \notin \mathscr{A}$, then $\langle \widehat{n}, \varepsilon\rangle \notin W$ by the choice of n, and hence $|T_{\widehat{n}\varepsilon}| = \infty$. It follows that $(Q)_\varepsilon = Y$, indeed,

$$y \in Y \iff U(\widehat{m}, y) \iff |S_{\widehat{m}y}| < \omega_1 \iff |S_{\widehat{m}y}| < |T_{\widehat{n}\varepsilon}| \iff \langle \varepsilon, y\rangle \in Q.$$

Therefore, $(Q)_\varepsilon = Y \in \mathscr{A}$. However, $(Q)_\varepsilon = (U)_\varepsilon$, a contradiction. Thus, in fact $(U)_\varepsilon \in \mathscr{A}$.

Note that by definition $(Q)_\varepsilon = \{y \in Y : |S_{\widehat{m}y}| < |T_{\widehat{n}\varepsilon}|\} \subseteq Y$. (Although it may not be true that $(Q)_\varepsilon = Y$ as it was the case above under the assumption $(U)_\varepsilon \notin \mathscr{A}$.) And the set $(Q)_\varepsilon = (U)_\varepsilon$ belongs to $\mathscr{A}$. It remains to show that $(Q)_\varepsilon$ is Δ^1_1. We have $\langle \widehat{n}, \varepsilon\rangle \in W$ just because $(U)_\varepsilon \in \mathscr{A}$. Therefore, $|T_{\widehat{n}\varepsilon}| < \omega_1$. It follows that $(Q)_\varepsilon = \{y \in \mathbb{N}^{\mathbb{N}} : |S_{\widehat{m}y}| < |T_{\widehat{n}\varepsilon}|\}$ simply because if $y \notin Y = (U)_{\widehat{m}}$, then $\langle \widehat{m}, y\rangle \notin U$ and $|S_{\widehat{m}y}| = \infty \not< |T_{\widehat{n}\varepsilon}|$. However, the inequality $|S_{\widehat{m}y}| < |T_{\widehat{n}\varepsilon}|$, as a unary relation with $\widehat{m}, \widehat{n}, \varepsilon$ being fixed natural numbers and $|T_{\widehat{n}\varepsilon}| < \omega_1$, can be expressed by a Σ^1_1 formula and a Π^1_1 formula by Theorem 2.1.4. □

2.8. Enumeration of Δ^1_1 sets

Recall that universal sets provide enumeration of all sets in a given class by their cross-sections. For instance, a universal Σ^1_1 set $U \subseteq \mathbb{N} \times \mathbb{N}^{\mathbb{N}}$ provides enumeration of all Σ^1_1 sets $X \subseteq \mathbb{N}^{\mathbb{N}}$ in the form $X = (U)_n = \{x : \langle n, x\rangle \in U\}$. Universal Δ^1_1 sets do not exist, see Remark 2.5.4. Yet there exists a useful enumeration of Δ^1_1 sets only slightly more complicated than Δ^1_1:

THEOREM 2.8.1 (Δ^1_1 Enumeration). *If $\mathbb{X}$ is a product space, then there exist Π^1_1 sets $\mathsf{Cod}(\Delta^1_1) \subseteq \mathbb{N}$ and $W \subseteq \mathbb{N} \times \mathbb{X}$ and a Σ^1_1 set $W' \subseteq \mathbb{N} \times \mathbb{X}$ such that $(W)_e = (W')_e$ for all $e \in \mathsf{Cod}(\Delta^1_1)$, and a set $X \subseteq \mathbb{X}$ is Δ^1_1 iff there is $e \in \mathsf{Cod}(\Delta^1_1)$ such that $X = (W)_e = (W')_e$.* □

Here, as usual, $(W)_e = \{x : W(e, x)\}$ and similarly for $(W')_e$.

PROOF. We begin with a universal Π^1_1 set $U \subseteq \mathbb{N} \times \mathbb{X}$. It leads to a pair

$$A = \{\langle n, x\rangle : U((n)^2_0, x)\} \quad \text{and} \quad B = \{\langle n, x\rangle : U((n)^2_1, x)\}$$

of Π^1_1 sets, double universal in the sense that for every pair of Π^1_1 sets $X, Y \subseteq \mathbb{X}$ there exists n such that $X = (A)_n$ and $Y = (B)_n$. (See Definition 1.1.6(i) on $(n)^k_i$.) Theorem 2.3.1 implies the existence of disjoint Π^1_1 sets $A' \subseteq A$ and $B' \subseteq B$ with $A' \cup B' = A \cup B$. The set

$$D = \{n : A'_n \cup B'_n = \mathbb{X}\} = \{n : \forall x\, (\langle n, x\rangle \in A' \vee \langle n, x\rangle \in B')\}$$

is still Π^1_1. On the other hand, by the double universality for every Δ^1_1 set $X \subseteq \mathbb{X}$, there exists n such that $X = A_n$ and $\mathbb{X} \smallsetminus X = B_n$—and then obviously $X = (A')_n$

and $\mathbb{X} \smallsetminus X = (B')_n$. It follows that the sets $\mathsf{Cod}(\Delta^1_1) = D$, $W = A'$, and $W' = (\mathbb{N} \times \mathbb{X}) \smallsetminus B'$ are as required. □

There is a generalization useful for relativised classes of the form $\Delta^1_1(a)$. The proof (a minor modification of the proof of Theorem 2.8.1) is left for the reader.

THEOREM 2.8.2 (Relativized Δ^1_1 Enumeration). *If $\mathbb{X}$ is product space, then there exist Π^1_1 sets $\mathsf{Cod}(\Delta^1_1) \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}$ and $W \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N} \times \mathbb{X}$ and a Σ^1_1 set $W' \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N} \times \mathbb{X}$ such that $(W)_{ae} = (W')_{ae}$ for all $\langle a, e\rangle \in \mathsf{Cod}(\Delta^1_1)$ and, for every $a \in \mathbb{N}^{\mathbb{N}}$, a set $X \subseteq \mathbb{X}$ is $\Delta^1_1(a)$ iff there is e such that $\langle a, e\rangle \in \mathsf{Cod}(\Delta^1_1)$ and $X = (W)_{ae} = (W')_{ae}$.* (Here $(W)_{ae} = \{x : W(a, e, x)\}$ and similarly for $(W')_{ae}$.) □

Let us derive a corollary related to Δ^1_1 elements of $\mathbb{N}^{\mathbb{N}}$.

COROLLARY 2.8.3. *There exist Π^1_1 sets $E \subseteq \mathbb{N}$ and $W \subseteq \mathbb{N} \times \mathbb{N}^2$ and a Σ^1_1 set $W' \subseteq \mathbb{N} \times \mathbb{N}^2$ such that for every $e \in E$ the sets*

$$(W)_e = \{\langle k, n\rangle : W(e, k, n)\} \quad \textit{and} \quad (W')_e = \{\langle k, n\rangle : W'(e, k, n)\}$$

coincide with some (one and the same) $u_e \in \mathbb{N}^{\mathbb{N}}$, and $\{u_e : e \in E\}$ is exactly the set of all Δ^1_1 points of $\mathbb{N}^{\mathbb{N}}$.

PROOF. Consider a triple of sets $D = \mathsf{Cod}(\Delta^1_1)$, W, W' satisfying Theorem 2.8.1 for $\mathbb{X} = \mathbb{N}^2$. The set

$$\begin{aligned} E &= \{e \in D : (W)_e \in \mathbb{N}^{\mathbb{N}}\} \\ &= \{e \in D : \forall k\, \exists n\, W(e, k, n) \wedge \forall k\, \forall n \neq m\, (W'(e, k, n) \Longrightarrow \neg W'(e, k, m))\} \end{aligned}$$

is still Π^1_1, and it proves the corollary. □

There is a more general relativized version:

COROLLARY 2.8.4. *There exist Π^1_1 sets $E \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}$ and $W \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N} \times \mathbb{N}^2$ and a Σ^1_1 set $W' \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N} \times \mathbb{N}^2$ such that for every $\langle a, e\rangle \in E$, the sets*

$$W_{ae} = \{\langle k, n\rangle : W(a, e, k, n)\} \quad \textit{and} \quad W'_{ae} = \{\langle k, n\rangle : W'(a, e, k, n)\}$$

coincide with some (one and the same) $u_{ae} \in \mathbb{N}^{\mathbb{N}}$, and for each $a \in \mathbb{N}^{\mathbb{N}}$ the set $\{u_{ae} : \langle a, e\rangle \in E\}$ is exactly the set of all $\Delta^1_1(a)$ points of $\mathbb{N}^{\mathbb{N}}$. □

The next corollary shows that being Δ^1_1 is a Π^1_1 notion.

COROLLARY 2.8.5. *The set $D = \{\langle a, x\rangle \in \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} : x \text{ is } \Delta^1_1(a)\}$ is Π^1_1. In particular, the set $\{x \in \mathbb{N}^{\mathbb{N}} : x \text{ is } \Delta^1_1\}$ is Π^1_1.*

PROOF. Let E, W, W', u_{ae} be as in Corollary 2.8.4. Then

$$\begin{aligned} \langle a, x\rangle \in D &\iff \exists e\, (\langle a, e\rangle \in E \wedge x = u_{ae}) \\ &\iff \exists e\, (\langle a, e\rangle \in E \wedge \forall k\, W(a, e, k, x(k))) . \end{aligned}$$ □

There is another important corollary of the enumeration given by Corollary 2.8.4. The quantifier $\exists x$ (x runs over $\mathbb{N}^{\mathbb{N}}$) applied to a Δ^1_1 relation leads to a Σ^1_1 relation. But the quantifier $\exists x \in \Delta^1_1$ does something different!

COROLLARY 2.8.6. *If $A \subseteq (\mathbb{N}^{\mathbb{N}})^3$ is a Δ^1_1 set, then the following set B is Π^1_1:*

$$B = \{\langle a, y\rangle \in (\mathbb{N}^{\mathbb{N}})^2 : \exists x\, (x \textit{ is } \Delta^1_1(a) \wedge A(a, x, y))\}.$$

PROOF. Let E, W, W', u_{ae} be as in Corolllary 2.8.4. Then

$$\begin{aligned}\langle a,y\rangle \in B &\iff \exists e\,(\langle a,e\rangle \in E \wedge A(a,u_{ae},y))\\ &\iff \exists e\,(\langle a,e\rangle \in E \wedge \forall x\,(x=u_{ae} \Longrightarrow A(a,x,y)))\,.\end{aligned}$$

It remains to replace the equality $x = u_{ae}$ by $\forall k\, W'(a,e,k,x(k))$ and make use of Table 1, the transformation table, on page 12 (or of Proposition 1.9.1). □

2.9. Coding Borel sets

This is an important application of the Δ^1_1 enumeration theorems. Recall that Borel sets $X \subseteq \mathbb{N}^{\mathbb{N}}$ are the same as $\mathbf{\Delta}^1_1$ sets, that is, sets in $\Delta^1_1(p)$ for some $p \in \mathbb{N}^{\mathbb{N}}$. Theorem 2.8.2 allows us to code all Borel sets. Naturally, a code cannot be a natural number by an obvious cardinality argument. But we can code Borel sets by "reals"; that is, here we use elements of $\mathbb{N}^{\mathbb{N}}$ as codes.

Let us fix a product space $\mathbb{X}$ in this section.

Let Π^1_1 sets $\mathtt{Cod}(\Delta^1_1) \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}$ and $W \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N} \times \mathbb{X}$ and a Σ^1_1 set $W' \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N} \times \mathbb{X}$ be as in Theorem 2.8.2. Put

$$\begin{aligned} C &= \{c \in \mathbb{N}^{\mathbb{N}} : \langle c^-, c(0)\rangle \in \mathtt{Cod}(\Delta^1_1)\} \quad \text{where} \quad c^-(k) = c(k+1),\ \forall k\,,\\ V &= \{\langle c,x\rangle \in \mathbb{N}^{\mathbb{N}} \times \mathbb{X} : \langle c^-, c(0), x\rangle \in W\}\,,\\ V' &= \{\langle c,x\rangle \in \mathbb{N}^{\mathbb{N}} \times \mathbb{X} : \langle c^-, c(0), x\rangle \in W'\}\,.\end{aligned}$$

And we immediately obtain

PROPOSITION 2.9.1. *The sets C and V are Π^1_1; the set V' is Σ^1_1. If $c \in C$, then the sets $(V)_c = \{x \in \mathbb{X} : \langle c,x\rangle \in V\}$ and $(V')_c$ are equal to each other. If $c \in C$, then $(V)_c$ is a Borel subset of $\mathbb{X}$, and, conversely, for every Borel $X \subseteq \mathbb{X}$ there exists a code $c \in C$ such that $X = (V)_c$.* □

There is another, much more transparent system of coding of Borel sets. In the case $\mathbb{X} = \mathbb{N}^{\mathbb{N}}$ (and with minor modifications for other Polish spaces) it works as follows. We define a set of codes $K \subseteq \mathbb{N}^{\mathbb{N}}$ and a set $\mathbf{B}_c \subseteq \mathbb{N}^{\mathbb{N}}$ for every $c \in K$. The definition consists of three items that can be formulated in a slightly different way, for instance, in matters of the operations involved in 2) below, but leading to essentially the same goal.

1) If $c \in \mathbb{N}^{\mathbb{N}}$ is such that $c(k) = 0$ for all k except for some k_0, and $c(k_0) \neq 0$, then $c \in K$ and $\mathbf{B}_c = \{a \in \mathbb{N}^{\mathbb{N}} : a(k_0) = c(k_0) - 1\}$.

2) If $c \in \mathbb{N}^{\mathbb{N}}$ and $(c)_n \in K$ for all n, then $c \in K$ and $\mathbf{B}_c = \mathbb{N}^{\mathbb{N}} \smallsetminus \bigcup_n \mathbf{B}_{(c)_n}$.

3) There are no elements in K except those obtained by a (finite or countable) transfinite iteration of 1) and 2) above.

For this coding, it is immediately clear that all coded sets are Borel and every Borel set is coded. Yet this coding has basically the same definability properties as those in Proposition 2.9.1. See more detail on the coding of Borel sets in **[Sol70]**, **[Jec71]**, or **[KL04]**.

2.10. Choquet property of Σ^1_1 and the Gandy–Harrington topology

It is sometimes very useful to have a mechanism that forces infinite decreasing sequences of sets of a certain type to have a non-empty intersection. For instance, the whole class of complete spaces is characterized by the property that decreasing sequences of sets, whose diameter shrinks to 0, have a non-empty intersection. This property, generally speaking, fails for sequences of open sets, let alone more complicated ones. However, the non-emptiness can sometimes be saved in the presence of appropriate additional requirements.

For instance, if $\{X_n\}_{n\in\mathbb{N}}$ is a $\subseteq$-decreasing sequence of non-empty open sets with diameters $\to 0$ in a complete space, then for $\bigcap X_n \neq \varnothing$ it suffices to require that each X_n includes the closure of X_{n+1}. It turns out that decreasing unions of Σ^1_1 sets in $\mathbb{N}^\mathbb{N}$ can also be made non-empty under certain restrictions!

The *Choquet game* $C_\mathbb{X}$ associated to a topological space $\mathbb{X}$ is played by two players I and II so that player I begins and plays an open set $\varnothing \neq U_1 \subseteq \mathbb{X}$ and player II responds by an open set $\varnothing \neq V_1 \subseteq U_1$. Then player I once again plays an open set $\varnothing \neq U_2 \subseteq V_1$, player II responds by an open set $\varnothing \neq V_2 \subseteq U_2$, and so on. The result of the run is defined as follows: player II wins if and only if the intersection $\bigcap_n U_n = \bigcap_n V_n$ is non-empty; otherwise, player I wins. And finally a space $\mathbb{X}$ is said to be a *Choquet space* if and only if player II has a winning strategy in this game.

Every complete metric space is Choquet by obvious reasons. (Indeed, player II can play by picking each V_n as an open non-empty set of diameter $< n^{-1}$, whose closure is included in U_n.) The converse fails; see an example below. Yet Choquet spaces share the following property with complete ones.

PROPOSITION 2.10.1. *Every Choquet space is Baire; that is, all comeager sets are dense.*

PROOF. Suppose that $\mathbb{X}$ is a Choquet space, and D_n, $n \in \mathbb{N}$, are dense open sets. Let $\varnothing \neq U \subseteq \mathbb{X}$ be an arbitrary open set. Consider a run in the game $C_\mathbb{X}$ in which player II follows the winning strategy while player I begins with $U_0 = U$ and plays so that $U_{n+1} \subseteq V_n \cap D_n$ for all n. This is consistent because of the density of D_n. Winning the run, player II gets a point in $U \cap \bigcap_n D_n$. □

The following is an example of a non-Polish Choquet space.

DEFINITION 2.10.2. Let $\mathbb{X} = \mathbb{N}^k \times (\mathbb{N}^\mathbb{N})^\ell$ be a product space. The *Gandy–Harrington topology* on $\mathbb{X}$ consists of all unions of Σ^1_1 sets $S \subseteq \mathbb{X}$. □

This topology extends the Polish topology on $\mathbb{X}$ but is not Polish itself. Indeed, by Corollary 2.5.3 there exists a Π^1_1 set $P \subseteq \mathbb{X}$ that is not $\mathbf{\Sigma}^1_1$. Note that P is closed in the Gandy–Harrington topology. Assume toward the contrary that the topology is Polish. Closed sets in Polish topologies are $\mathbf{G}_\delta$, that is, $P = \bigcap_n \bigcup_m S_{mn}$, where all sets S_{mn} are Σ^1_1. Then obviously P is $\mathbf{\Sigma}^1_1$, a contradiction.

The proof that the Gandy–Harrington topology is Choquet involves another property, perhaps of a somewhat more general nature.

DEFINITION 2.10.3. A family $\mathscr{F}$ of sets in a topological space is *Polish-like* if there exists a countable collection $\{\mathscr{D}_n : n \in \mathbb{N}\}$ of open dense subsets $\mathscr{D}_n \subseteq \mathscr{F}$ such that we have $\bigcap_n F_n \neq \varnothing$ whenever $F_0 \supseteq F_1 \supseteq F_2 \supseteq \cdots$ is a decreasing sequence of sets $F_n \in \mathscr{F}$ that intersects every $\mathscr{D}_n$.

Here, a set $\mathscr{D} \subseteq \mathscr{F}$ is *open dense* if $\forall F \in \mathscr{F}\, \exists D \in \mathscr{D}\,(D \subseteq F)$ and

$$\forall F \in \mathscr{F}\, \forall D \in \mathscr{D}\,(F \subseteq D \Longrightarrow F \in \mathscr{D}).^3 \qquad \square$$

For instance, if $\mathbb{X}$ is a Polish space, then the collection of all its non-empty closed sets is Polish-like. Indeed, we can take $\mathscr{D}_n$ to be all closed sets of diameter $\le n^{-1}$. The next result is much less trivial:

THEOREM 2.10.4 (see e.g. [**HKL90**], [**Kan96**], [**Hjo00a**]). (i) *Let $\mathbb{X}$ be a product space. The collection $\mathbb{P}$ of all non-empty Σ_1^1 subsets of $\mathbb{X}$ is Polish-like.*
(ii) *It follows that the Gandy–Harrington topology on $\mathbb{X}$ is Choquet and Baire.*

PROOF. (i) For the sake of simplicity consider the case $\mathbb{X} = \mathbb{N}^{\mathbb{N}}$. Recall that $\operatorname{pr} P = \{x : \exists y\, P(x,y)\}$ (the projection) for any set $P \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$. If $P \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ and $s, t \in \mathbb{N}^{<\omega}$, then let $P_{st} = \{\langle x,y\rangle \in P : s \subset x \wedge t \subset y\}$. Let $\mathscr{D}(P,s,t)$ be the collection of all Σ_1^1 sets $\varnothing \ne X \subseteq \mathbb{N}^{\mathbb{N}}$ such that either $X \cap \operatorname{pr} P_{st} = \varnothing$ or $X \subseteq \operatorname{pr} P_{s^\frown i, t^\frown j}$ for some i, j. (Note that in the "or" case, i is unique but j may be not unique.) Let $\{\mathscr{D}_n : n \in \mathbb{N}\}$ be an arbitrary enumeration of all sets of the form $\mathscr{D}(P,s,t)$, where $P \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ is Π_1^0. Note that in this case, all sets of the form $\operatorname{pr} P_{st}$ are Σ_1^1 subsets of $\mathbb{N}^{\mathbb{N}}$; therefore, $\mathscr{D}(P,s,t)$ is easily an open dense subset of $\mathbb{P}$ so that all sets $\mathscr{D}_n$ are open dense subsets of $\mathbb{P}$.

Now consider a decreasing sequence $X_0 \supseteq X_1 \supseteq \cdots$ of non-empty Σ_1^1 sets $X_k \subseteq \mathbb{N}^{\mathbb{N}}$, which intersects every $\mathscr{D}_n$; prove that $\bigcap_n X_n \ne \varnothing$. Call a set $X \subseteq \mathbb{N}^{\mathbb{N}}$ *positive* if there is n such that $X_n \subseteq X$. For every n, fix a Π_1^0 set $P^n \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ such that $X_n = \operatorname{pr} P^n$. For every $s, t \in \mathbb{N}^{<\omega}$, if $\operatorname{pr} P^n_{st}$ is positive, then, by the choice of the sequence of X_n, there is a unique i and some j such that $\operatorname{pr} P^n_{s^\frown i, t^\frown j}$ is also positive. It follows that there is a unique $x = x_n \in \mathbb{N}^{\mathbb{N}}$ and some $y = y_n \in \mathbb{N}^{\mathbb{N}}$ (perhaps not unique) such that $\operatorname{pr} P^n_{x\restriction k, y\restriction k}$ is positive for every k. As P^n is closed, we have $P^n(x,y)$; hence, $x_n = x \in X_n$.

It remains to show that $x_m = x_n$ for $m \ne n$. To see this, note that if both $\operatorname{pr} P_{st}$ and $\operatorname{pr} Q_{s't'}$ are positive, then either $s \subseteq s'$ or $s' \subseteq s$.

(ii) The density of the sets $\mathscr{D}_n$ allows player II to play so that $V_n \in \mathscr{D}_n$ for all n, where by (i) $\{\mathscr{D}_n : n \in \mathbb{N}\}$ is a countable collection of open dense subsets of the set $\mathbb{P}$ of all Σ_1^1 sets $\varnothing \ne X \subseteq \mathbb{X}$. $\square$

The following corollary can be established by more elementary tools, but it becomes really easy on the basis of Theorem 2.10.4.

COROLLARY 2.10.5. *If a Σ_1^1 set $X \subseteq \mathbb{N}^{\mathbb{N}}$ contains an element $x \notin \Delta_1^1$, then there exists a continuous injection $f : 2^{\mathbb{N}} \to X$, and hence X is a set of cardinality continuum, formally, $\operatorname{card} X = \mathfrak{c}$. More generally, if $p \in \mathbb{N}^{\mathbb{N}}$ and a $\Sigma_1^1(p)$ set $X \subseteq \mathbb{N}^{\mathbb{N}}$ contains an element $x \notin \Delta_1^1(p)$, then there is a continuous injection $f : 2^{\mathbb{N}} \to X$, and hence $\operatorname{card} X = \mathfrak{c}$. Thus, all finite and countable $\Sigma_1^1(p)$ sets contain only $\Delta_1^1(p)$ elements.*

PROOF. It can be assumed by Corollary 2.8.5 that X does not contain Δ_1^1 elements at all. Then each Σ_1^1 set $\varnothing \ne Y \subseteq X$ contains at least two elements. (Indeed if Y is a Σ_1^1 singleton, then its only element is Δ_1^1 by Lemma 1.8.6.) Therefore, such Y contains two disjoint non-empty Σ_1^1 subsets. This allows us to

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[3] Sets $\mathscr{D}$ satisfying only the first condition are called *dense*. Note that if $\mathscr{D} \subseteq \mathscr{F}$ is dense, then $\mathscr{D}' = \{F \in \mathscr{F} : \exists D \in \mathscr{D}\,(F \subseteq D)\}$ is open dense.

define a splitting system $\{X_s\}_{s\in 2^{<\omega}}$ of non-empty Σ^1_1 sets $X_s \subseteq X$ satisfying the following conditions:

(1) $X_{s^\frown i} \subseteq X_s$ and $X_{s^\frown 0} \cap X_{s^\frown 1} = \varnothing$;

(2) if $\mathtt{lh}\, s = n$, then the diameter of X_s as a set in $\mathbb{N}^\mathbb{N}$ is less than n^{-1};

(3) $X_s \in \mathscr{D}_{\mathtt{lh}\, s}$, where $\{\mathscr{D}_n : n \in \mathbb{N}\}$ is any countable collection of open dense subsets of the set $\mathbb{P}$ of all Σ^1_1 sets $\varnothing \neq Y \subseteq \mathbb{N}^\mathbb{N}$ that witnesses the Polish-likeness of $\mathbb{P}$ (we refer to Theorem 2.10.4).

Then for each $a \in 2^\mathbb{N}$, the intersection $X_a = \bigcap_n X_{a\restriction n}$ is non-empty by (3) above. Therefore, $X_a = \{f(a)\}$ is a singleton by (2), and, still by (2), $f : 2^\mathbb{N} \to X$ is continuous. Finally, $X_a \cap X_b = \varnothing$ whenever $a \neq b$ by (1); thus, f is an injection.

To prove the p-version, one has to employ the corresponding p-version of the Gandy–Harrington topology generated by non-empty $\Sigma^1_1(p)$ sets. □

2.11. Sets with countable sections

This topic belongs to a direction in descriptive set theory that studies *sets with special sections*. We consider *planar* sets, those situated in spaces of the form $\mathbb{X} \times \mathbb{Y}$. A *cross-section* of a set $P \subseteq \mathbb{X} \times \mathbb{Y}$ is a set of the form

$$(P)_x = \{y \in \mathbb{Y} : P(x,y)\}, \quad \text{where} \quad x \in \mathbb{X}.$$

Some of these sets are empty, in fact $(P)_x$ is non-empty iff

$$x \in \operatorname{pr} P = \{x \in \mathbb{X} : \exists\, y\, P(x,y)\}.$$

Sets with special cross-sections is a generic name for various categories of planar sets distinguished by this or another property of their cross-sections.

For instance, the requirement that every $(P)_x$ contains at most one element characterizes uniform sets considered in Section 2.4. Sets with countable, compact, σ-compact cross-sections are considered as well. Another category is formed by sets with "large" cross-sections; for instance, it is required that all non-empty sections $(P)_x$ are non-meager sets. See **[Kec95]** on related results and methods.

We are not going to present this branch of descriptive set theory in any generality. On the other hand, to make the exposition self-contained, we prove the next four theorems. They will be used in the following chapters of the book.

Below in this section, $\mathbb{X}$ and $\mathbb{Y}$ are arbitrary product spaces, *i.e.*, those of the form $\mathbb{N}^k \times (\mathbb{N}^\mathbb{N})^\ell$, and $p \in \mathbb{N}^\mathbb{N}$ is an arbitrary parameter.

THEOREM 2.11.1 (Countable-to-1 Projection). *If $P \subseteq \mathbb{X} \times \mathbb{Y}$ is a $\Delta^1_1(p)$ set and for each $x \in \mathbb{X}$ the cross-section $(P)_x$ is at most countable, then $\operatorname{pr} P$ is a $\Delta^1_1(p)$ set.*

COROLLARY 2.11.2. *Suppose that X, Y are sets in product spaces and $f : X \xrightarrow{\text{onto}} Y$ a $\Delta^1_1(p)$ map. Then $X = \operatorname{dom} f$ is a $\Delta^1_1(p)$ set, and moreover, if $Y' \subseteq Y$ is a $\Delta^1_1(p)$ set, then so is its f-preimage $f^{-1}[Y'] = \{x \in X : f(x) \in Y'\}$.*

In addition, if f is countable-to-1, then $Y = \operatorname{ran} f$ is Δ^1_1 set as well. And in this case if $X' \subseteq X$ is $\Delta^1_1(p)$, then so is its f-image $f[X'] = \{f(x) : x \in X'\}$.

Thus, preimages of $\Delta^1_1(p)$ sets via $\Delta^1_1(p)$ maps are $\Delta^1_1(p)$ sets, and images of $\Delta^1_1(p)$ sets via countable-to-1, in particular, 1-to-1 $\Delta^1_1(p)$ maps are $\Delta^1_1(p)$ sets, too. (But images of $\Delta^1_1(p)$ sets via arbitrary $\Delta^1_1(p)$ maps are, generally speaking, arbitrary $\Sigma^1_1(p)$ sets.)

PROOF (Corollary). To prove the first part, apply Theorem 2.11.1 for the set $P = \{\langle x, y\rangle : x \in X \wedge f(x) = y\}$ (the graph of f). To prove the second part, apply the theorem for the inverse graph $P^{-1} = \{\langle y, x\rangle : x \in X \wedge f(x) = y\}$. □

THEOREM 2.11.3 (Countable-to-1 Enumeration). *If P is as in Theorem 2.11.1, then there is a $\Delta^1_1(p)$ map $F : \mathbb{N} \times \mathrm{pr}\, P \to \mathbb{Y}$ such that $(P)_x = \{F(e, x) : e \in \mathbb{N}\}$ for all $x \in \mathrm{pr}\, P$.*

THEOREM 2.11.4 (Borel Extension). *If $P \subseteq \mathbb{X}\times\mathbb{Y}$ is a $\Sigma^1_1(p)$ set, and $(P)_x = \{y : P(x, y)\}$ is at most countable for every $x \in \mathbb{X}$, then there is a $\Delta^1_1(p)$ superset $Q \supseteq P$ with all cross-sections $(Q)_x$ at most countable. Similarly, if $P \subseteq \mathbb{X} \times \mathbb{Y}$ is a uniform $\Sigma^1_1(p)$ set, then there is a uniform $\Delta^1_1(p)$ superset $Q \supseteq P$.*

THEOREM 2.11.5 (Countable-to-1 Uniformization). *If P is as in Theorem 2.11.1, then P can be uniformized by a $\Delta^1_1(p)$ set $Q \subseteq P$.*

Classical forms of these theorems (that is, for the boldface classes $\mathbf{\Sigma}^1_1$ and $\mathbf{\Delta}^1_1$ = Borel, see Section 2.12) were established in the late 1920s by LUZIN and P. NOVIKOV, and the original geometric-style proofs were quite complicated (see e.g. LUZIN [**Lus72**]). Methods of effective descriptive theory allow us to prove the theorems by very short and transparent arguments.

PROOF (Theorem 2.11.1). Assume for the sake of brevity that $\mathbb{X} = \mathbb{Y} = \mathbb{N}^{\mathbb{N}}$ and that the parameter p is absent. The set $X = \mathrm{pr}\, P$ is Σ^1_1 anyway. And if $x \in \mathbb{X}$, then the cross-section $(P)_x$ is a countable $\Delta^1_1(x)$ set; hence, all elements of $(P)_x$ are $\Delta^1_1(x)$ by Corollary 2.10.5. Therefore, $X = \{x \in \mathbb{X} : \exists y \in \Delta^1_1(x)\, P(x, y)\}$. But then X is Π^1_1 by Corollary 2.8.6. □

PROOF (Theorem 2.11.4, countable sections). Assume that $\mathbb{X} = \mathbb{Y} = \mathbb{N}^{\mathbb{N}}$. If all cross-sections $(P)_x$ are at most countable, then $y \in \Delta^1_1(x)$ whenever $\langle x, y\rangle \in P$ (see the proof of Theorem 2.11.1). It follows that P is a subset of the set

$$W = \{\langle x, y\rangle \in \mathbb{X} \times \mathbb{Y} : y \text{ is } \Delta^1_1(x)\}.$$

However, W is Π^1_1 by Corollary 2.8.5. Therefore, by Theorem 2.3.2 (Separation) there exists a Δ^1_1 set Q such that $P \subseteq Q \subseteq W$. The set Q is as required. □

PROOF (Theorem 2.11.4, uniform sets). If $P \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ is a uniform Σ^1_1 set, then

$$A = \{\langle x, y\rangle : \forall z\, (P(x, z) \Longrightarrow y = z)\}$$

is a Π^1_1 set and $P \subseteq A$. Therefore, by Theorem 2.3.2 there exists a Δ^1_1 set B such that $P \subseteq B \subseteq A$. Furthermore,

$$C = \{\langle x, y\rangle \in B : \forall z\, (B(x, z) \Longrightarrow y = z)\}$$

is a Π^1_1 set and still $P \subseteq C$. Once again, by Theorem 2.3.2 there exists a Δ^1_1 set Q such that $P \subseteq Q \subseteq C$. Such a set Q is as required. □

PROOF (Theorem 2.11.3). Still assume that $\mathbb{X} = \mathbb{Y} = \mathbb{N}^{\mathbb{N}}$. We have $y \in \Delta^1_1(x)$ whenever $\langle x, y\rangle \in P$ (see the proof of Theorem 2.11.4 above). Let E, W, W', u_{ae} be as in Corollary 2.8.4. Then in particular for each pair $\langle x, y\rangle \in P$ there is a number e with $\langle x, e\rangle \in E$ such that $y = u_{xe}$. It follows that P is equal to the union of all sets

$$Q(e) = \{\langle x, y\rangle \in P : \langle x, e\rangle \in E \wedge y = u_{xe}\}, \quad e \in \mathbb{N},$$

and all $Q(e)$ are uniform sets. In addition, $Q(e)$ are Π^1_1 sets, and even the set $Q = \{\langle x, y, e\rangle : \langle x, y\rangle \in Q(e)\}$ is Π^1_1 because such is the set E, while the equality $y = u_{xe}$ is equivalent to $\forall k\, W(x, e, k, y(k))$ whenever $\langle x, e\rangle \in E$.

By Theorem 2.4.5 Q can be uniformized by a Δ^1_1 set $R \subseteq Q$ in the sense of $(\mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}) \times \mathbb{N}$. In other words, for each $x, y \in \mathbb{N}^\mathbb{N}$,

$$\exists\, e\, Q(x, y, e) \Longrightarrow \exists!\, e\, R(x, y, e).$$

Then the sets $R(e) = \{\langle x, y\rangle : R(x, y, e)\}$ are pairwise disjoint Δ^1_1, and $\bigcup_e R(e) = \bigcup_e Q(e) = P$. Moreover, clearly $R(e) \subseteq Q(e)$, and hence $R(e)$ are uniform sets.

Note that the sets $D(e) = \operatorname{pr} R(e)$ (subsets of $\mathbb{N}^\mathbb{N}$) are Δ^1_1 by Theorem 2.11.1, and $\bigcup_e D(e) = \operatorname{pr} P$. It suffices to define $F(e, x)$ as follows. If $x \in D(e)$, then $F(e, x)$ is the only y with $\langle x, y\rangle \in R(e)$. If $x \in \operatorname{pr} P \smallsetminus D(e)$, then take the least e' such that $x \in D(e')$ and let $F(e, x)$ be the only y with $\langle x, y\rangle \in R(e')$. □

PROOF (Theorem 2.11.5). Let F be as in Theorem 2.11.3. Then the set $Q = \{\langle x, y\rangle : y = F(0, x)\}$ is as required. □

2.12. Applications for Borel sets

Theorems 2.11.1–2.11.5 admit rather obvious modifications for Borel (that is, $\mathbf{\Delta}^1_1$) and $\mathbf{\Sigma}^1_1$ sets, established as theorems of classical descriptive set theory in the 1920s. They are as follows. Let $\mathbb{X}, \mathbb{Y}$ be arbitrary Polish spaces.

THEOREM 2.12.1 (Countable-to-1 Projection). *If $P \subseteq \mathbb{X} \times \mathbb{Y}$ is Borel and for each $x \in \mathbb{X}$ the cross-section $(P)_x$ is at most countable, then the projection $\operatorname{pr} P = \{x \in \mathbb{X} : \exists\, y\, (\langle x, y\rangle \in P)\}$ is Borel.*

COROLLARY 2.12.2. *In Polish spaces, preimages of Borel sets via Borel maps are still Borel sets. And images of Borel sets via countable-to-1, in particular, 1-to-1 Borel maps, are still Borel.*

THEOREM 2.12.3 (Countable-to-1 Enumeration). *If P is as in Theorem 2.12.1, then there is a Borel map $F : \mathbb{N} \times \operatorname{pr} P \to \mathbb{Y}$ such that $(P)_x = \{F(e, x) : e \in \mathbb{N}\}$ for all $x \in \operatorname{pr} P$.*

THEOREM 2.12.4 (Borel Extension). *If $P \subseteq \mathbb{X} \times \mathbb{Y}$ is a $\mathbf{\Sigma}^1_1$ set, and the cross-section $(P)_x = \{y : P(x, y)\}$ is at most countable for every $x \in \mathbb{X}$, then there is a Borel superset $Q \supseteq P$ with all cross-sections $(Q)_x$ at most countable. Similarly, if $P \subseteq \mathbb{X} \times \mathbb{Y}$ is a uniform $\mathbf{\Sigma}^1_1$ set, then there is a uniform Borel superset $Q \supseteq P$.*

THEOREM 2.12.5 (Countable-to-1 Uniformization). *If P is as in Theorem 2.12.1, then P can be uniformized by a Borel set $Q \subseteq P$.*

PROOF (Theorems 2.12.1–2.12.5). The results for product spaces $\mathbb{X}, \mathbb{Y}$ routinely follow from Theorems 2.11.1–2.11.5 because $\mathbf{\Delta}^1_1 = \bigcup_{p \in \mathbb{N}^\mathbb{N}} \Delta^1_1(p)$ and $\mathbf{\Sigma}^1_1 = \bigcup_{p \in \mathbb{N}^\mathbb{N}} \Sigma^1_1(p)$. The generalization to all Polish spaces immediately follows from Theorem 1.2.2(i) (with the help of Theorem 1.3.1 whenever we deal with $\mathbf{\Sigma}^1_1$ sets). □

We derive a similar routine consequence of Corollary 2.10.5. This is the following theorem of classical descriptive set theory, first obtained for Borel sets independently by HAUSDORFF and ALEKSANDROV in 1916 and then for $\mathbf{\Sigma}^1_1$ sets (a wider class) by SOUSLIN in 1917.

THEOREM 2.12.6. *If X is an uncountable Σ^1_1 set in a product space (in fact the result is true for all Polish spaces, see 13.6 and 14.13 in* [**Kec95**]*), then there exists a continuous injection $f : 2^{\mathbb{N}} \to X$, and hence* $\mathtt{card}\, X = \mathfrak{c}$.

PROOF. There is a parameter p such that X is $\varSigma^1_1(p)$. Apply Corollary 2.10.5. □

It follows that the cardinality $\mathtt{card}\, X$ of a Borel, or more generally Σ^1_1, set X in a Polish space is either a natural n or $\aleph_0$ or the cardinality of continuum $\mathfrak{c}$.

Now a few more words in extension of Corollary 2.12.2. It asserts that if f is a Borel *bijection* (1-to-1 map), then both f-images and f-preimages of Borel sets are Borel sets. Thus, Borel bijections can be called *Borel isomorphisms*. Corollary 2.12.2 implies further interesting corollaries. First of all,

COROLLARY 2.12.7. *If X, Y are Borel sets in Polish spaces and $f : X \xrightarrow{\text{onto}} Y$ is a Borel bijection, then so is f^{-1}.* □

COROLLARY 2.12.8. *If X, Y are uncountable Borel sets in product spaces (also true for all Polish spaces by Theorem* 1.2.2(i)*), then there is a Borel bijection $f : X \xrightarrow{\text{onto}} Y$. Thus, all uncountable Borel sets are Borel isomorphic.*

PROOF. It follows from Theorem 2.12.6 that there exists a continuous, hence, Borel injection $f : 2^{\mathbb{N}} \to X$. On the other hand, obviously there is a Borel injection $g : \mathbb{N}^{\mathbb{N}} \to 2^{\mathbb{N}}$. Viewing Y as a subset of $\mathbb{N}^{\mathbb{N}}$, the restriction $f \restriction Y$ is still a Borel injection into $2^{\mathbb{N}}$. Thus, the superposition $\varphi = f \circ g$ is a Borel injection $Y \to X$. By the same reasoning, there is a Borel injection $\psi : X \to Y$. In this case, following a common proof of the Schroeder–Bernstein theorem, we obtain a bijection h between X and Y that consists of certain fragments of φ and ψ. Moreover, closer inspection of the construction shows that those fragments, as well as all intermediate sets involved in the construction, are Borel by Corollary 2.12.2, as required. □

We finish with the following result in a sense dual to Corollary 2.12.2.

PROPOSITION 2.12.9. *If X is a Borel set in a product space (also true for all Polish spaces, see* 13.7 *in* [**Kec95**]*), then there exists a closed set $P \subseteq \mathbb{N}^{\mathbb{N}}$ and a continuous* 1*-to-*1 *map $f : P \xrightarrow{\text{onto}} X$.*

Thus, Borel sets are continuous 1-to-1 images of closed sets!

PROOF. We argue by transfinite induction on the construction of X from closed sets by countable sums and countable intersections. If $X = \bigcup_n X_n$, sets $P_n \subseteq \mathbb{N}^{\mathbb{N}}$ are closed and $f_n : P_n \xrightarrow{\text{onto}} X_n$ continuous and 1-to-1, then put $P = \{\langle n, a\rangle : a \in P_n\}$ and $f(n,a) = f_n(a)$. If $X = \bigcap_n X_n$ and P_n, f_n are as above, then put

$$P = \{z \in (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}} : \forall n\, (z(n) \in P_n) \wedge \forall m, n\, (f_m(z(m)) = f_n(z(n)))\}$$

and define $f(z) = f_0(z(0))$ for all $z \in P$. □

CHAPTER 3

Borel ideals

This chapter does not offer any sort of broad introduction into Borel ideals. Instead we consider some issues close to the content of the book, including the Rudin–Blass reducibility, P-ideals, polishable ideals, LSC submeasures, summable, density, and Fréchet ideals. Finally, a proof of SOLECKI's theorem, that characterizes P-ideals in terms of LSC submeasures and polishability and shows that $\mathscr{I}_1$ is the least Borel non-polishable ideal, will be given.

3.1. Introduction to ideals

Recall that an *ideal* $\mathscr{I}$ on a set A (called the *underlying set* of $\mathscr{I}$) is any non-empty set $\mathscr{I} \subseteq \mathscr{P}(A)$ closed under $\cup$ and satisfying $x \in \mathscr{I} \Longrightarrow y \in \mathscr{I}$ whenever $y \subseteq x \subseteq A$. Thus, every ideal contains the empty set $\varnothing$. Usually, they consider only *non-trivial* ideals; *i.e.*, those that contain all singletons $\{a\}$, $a \in A$, and do not contain A, *i.e.*, $\mathscr{P}_{\texttt{fin}}(A) \subseteq \mathscr{I} \subsetneqq \mathscr{P}(A)$. But sometimes the ideal $\{\varnothing\}$, whose only element is the empty set $\varnothing$, is considered and often denoted by 0.

If A is a countable set, then by identifying $\mathscr{P}(A)$ with 2^A via characterictic functions we equip $\mathscr{P}(A)$ with the Polish product topology.[1] In this sense, a *Borel* ideal on A is any ideal that is a Borel subset of $\mathscr{P}(A)$ in this topology. Let us give several important examples of Borel ideals:

- $\texttt{Fin} = \{x \subseteq \mathbb{N} : x \text{ is finite}\}$, the ideal of all finite sets;
- $\mathscr{I}_1 = \{x \subseteq \mathbb{N}^2 : \{k : (x)_k \neq \varnothing\} \in \texttt{Fin}\}$, where $(x)_a = \{b : \langle a, b\rangle \in x\}$;
- $\mathscr{I}_2 = \{x \subseteq \mathbb{N} : \sum_{n\in x,\ n\geq 1} \frac{1}{n}\} < +\infty$, the *summable ideal*;
- $\mathscr{I}_3 = \{x \subseteq \mathbb{N}^2 : \forall k\, ((x)_k \in \texttt{Fin})\}$;
- $\mathscr{Z}_0 = \{x \subseteq \mathbb{N} : \texttt{lim}_{n\to+\infty} \frac{\texttt{card}\,(x\cap[0,n))}{n} = 0\}$, the *density ideal*.

Given an ideal $\mathscr{I}$ on a set A, we define $\mathscr{I}^+ = \mathscr{P}(A) \smallsetminus \mathscr{I}$ ($\mathscr{I}$*-positive* sets) and $\mathscr{I}^{\complement} = \{X : \complement X \in \mathscr{I}\}$ (*the dual filter*). Clearly, $\varnothing \neq \mathscr{I}^{\complement} \subseteq \mathscr{I}^+$.

If $B \subseteq A$, then we put $\mathscr{I} \restriction B = \{x \cap B : x \in \mathscr{I}\}$.

3.2. Reducibility of ideals

There are different methods of reduction of an ideal $\mathscr{I}$ on a set A to an ideal $\mathscr{J}$ on a set B, where the reducibility means that $\mathscr{I}$ is in some sense simpler (in a non-strict way) than $\mathscr{J}$.

Rudin–Keisler order: $\mathscr{I} \leq_{\text{RK}} \mathscr{J}$ iff there exists a function $\beta : B \to A$ (a Rudin–Keisler reduction) such that $x \in \mathscr{I} \Longleftrightarrow \beta^{-1}[x] \in \mathscr{J}$.

[1] This topology on $\mathscr{P}(A)$ is generated by all sets of the form $B_{uv} = \{x \subseteq A : u \subseteq x \wedge v \cap x \neq \varnothing\}$, where $u, v \subseteq A$ are finite disjoint sets.

Rudin–Blass order: $\mathscr{I} \le_{\rm RB} \mathscr{J}$ iff there is a *finite-to-*1 function $\beta : B \to A$ (a Rudin–Blass reduction) with the same property.

A version: $\mathscr{I} \le^{+}_{\rm RB} \mathscr{J}$ allows the map β to be defined on a proper subset of B. In other words, there exist pairwise disjoint finite non-empty sets $w_a = \beta^{-1}[\{a\}]$, $a \in A$, such that $x \in \mathscr{I} \iff w_x = \bigcup_{a\in x} w_a \in \mathscr{J}$.

Another version: $\mathscr{I} \le^{++}_{\rm RB} \mathscr{J}$, applicable in the case when $A = B = \mathbb{N}$, requires in addition that the sets w_a satisfy $\max w_a < \min w_{a+1}$.

The next result known from [**JN76**], [**Mat75**], [**Tal80**] shows that the ideal $\mathtt{Fin}$ of all finite sets is a $\le_{\rm RB}$-least ideal!

THEOREM 3.2.1. (i) *If $\mathscr{I}$ is a* (non-trivial) *ideal on $\mathbb{N}$ and $\mathscr{I}$ has the Baire property in the topology of $\mathscr{P}(\mathbb{N})$, then $\mathtt{Fin} \le^{++}_{\rm RB}$ and $\le_{\rm RB} \mathscr{I}$.*

(ii) *If $\mathscr{I} \le^{+}_{\rm RB} \mathscr{J}$ are Borel ideals and there is an infinite set $Z \subseteq \mathtt{dom}\,\mathscr{I}$ such that $\mathscr{I} \restriction Z = \mathscr{P}_{\tt fin}(Z)$, then $\mathscr{I} \le_{\rm RB} \mathscr{J}$.*

PROOF. (i) First of all, the ideal $\mathscr{I}$ must be a meager set in $\mathscr{P}(\mathbb{N})$. (Otherwise, $\mathscr{I}$ would be comeager somewhere, easily leading to a contradiction with the non-triviality.) Thus, $\mathscr{I} = \mathscr{P}(\mathbb{N}) \setminus \bigcap_k D_k$, where all D_k are dense open subsets of $\mathscr{P}(\mathbb{N})$ with $D_{k+1} \subseteq D_k$, $\forall\, k$. Now we prove:

LEMMA 3.2.2. *If $n, k \in \mathbb{N}$, then there exist $m > n$ and a set $u \subseteq [n, m)$ such that all sets $x \in \mathscr{P}(\mathbb{N})$ satisfying $x \cap [n, m) = u$ belong to D_k.*

PROOF. This is a rather typical argument. First of all, fix an arbitrary enumeration $\{s_j : j < J\}$ of all sets $s \subseteq [0, n)$, where obviously $J = 2^n$. Consider the set s_0. As D_k is open, there exist $n_1 > n$ and a set $u_1 \subseteq [n, n_1)$ such that all sets $x \in \mathscr{P}(\mathbb{N})$ satisfying $x \cap [0, n_1) = s_0 \cup u_1$ belong to D_k. Now consider the set s_1. By the same reasoning there exist $n_2 > n_1$ and a set $u_2 \subseteq [n_1, n_2)$ such that all sets $x \in \mathscr{P}(\mathbb{N})$ satisfying $x \cap [0, n_2) = s_1 \cup u_1 \cup u_2$ still belong to D_k.

Following this inductive construction, we define a finite sequence of numbers $n < n_1 < n_2 < \dots < n_J$ and sets $u_j \subseteq [n_j, n_{j+1})$ such that if $j < J$, then all sets $x \in \mathscr{P}(\mathbb{N})$ satisfying $x \cap [0, n_{j+1}) = s_j \cup u_1 \cup u_2 \cup \dots \cup u_j$ belong to D_k. We claim that $m = n_J$ and the set $u = u_1 \cup u_2 \cup \dots \cup u_J$ are as required.

Indeed suppose that $x \in \mathscr{P}(\mathbb{N})$ satisfies $x \cap [n, m) = u$. The set $s = x \cap [0, n)$ coincides with some s_j, $j < J$. Then obviously $x \cap [0, n_{j+1}) = s_j \cup u_1 \cup u_2 \cup \dots \cup u_j$, and hence $x \in D_k$ by the construction. □ (Lemma)

Coming back to the proof of the theorem, note that the lemma allows us to define a sequence of non-empty finite sets $w_k \subseteq \mathbb{N}$ with $\max w_k < \min w_{k+1}$ such that every union x of infinitely many of them belongs accordingly to infinitely many sets D_k; therefore, it belongs to all of them because $D_{k+1} \subseteq D_k$, and hence belongs to $\mathscr{P}(\mathbb{N}) \setminus \mathscr{I}$. It follows that the map $x \mapsto w_x = \bigcup_{a\in x} w_a$ witnesses $\mathtt{Fin} \le^{++}_{\rm RB} \mathscr{I}$. To derive $\mathtt{Fin} \le_{\rm RB} \mathscr{I}$, let us cover each w_k by a finite set u_k such that $\bigcup_{k\in\mathbb{N}} u_k = \mathbb{N}$ and still $u_k \cap u_l = \varnothing$ for $k \ne l$.

(ii) Assume *w.l.o.g.* that $\mathscr{I}, \mathscr{J}$ are ideals over $\mathbb{N}$. Let pairwise disjoint finite sets $w_k \subseteq \mathbb{N}$ witness $\mathscr{I} \le^{+}_{\rm RB} \mathscr{J}$. Put $Z' = \mathbb{N} \setminus Z$, $X = \bigcup_{k\in Z} w_k$, and $Y = \bigcup_{k\in Z'} w_k$. The reduction via $\{w_k\}$ reduces $\mathscr{P}_{\tt fin}(Z)$ to $\mathscr{J} \restriction X$ and $\mathscr{I} \restriction Z'$ to $\mathscr{J} \restriction Y$. Keeping the latter, replace the former by a $\le_{\rm RB}$-like reduction of $\mathscr{P}_{\tt fin}(Z)$ to $\mathscr{J} \restriction Y'$, where $Y' = \mathbb{N} \setminus Y$, which exists by Theorem 3.2.1(i). □

Another type of reducibility is connected with Δ-homomorphisms. Suppose that $\mathscr{I}, \mathscr{J}$ are ideals on sets A, B, respectively. The power sets $\mathscr{P}(A)$, $\mathscr{P}(B)$ can be considered to be groups with Δ as the operation and $\varnothing$ as the neutral element. Then a *Δ-homomorphism* is any map $\vartheta : \mathscr{P}(A) \to \mathscr{P}(B)$ such that $\vartheta(x) \mathbin{\Delta} \vartheta(y) = \vartheta(x \mathbin{\Delta} y)$ for all $x, y \subseteq A$.

The quotient $\mathscr{P}(A)/\mathscr{I}$ consists of $\mathscr{I}$-classes $[x]_{\mathscr{I}} = \{x \mathbin{\Delta} a : a \in \mathscr{I}\}$ of sets $x \subseteq A$; it is endowed by the group operation $[x]_{\mathscr{I}} \mathbin{\Delta} [y]_{\mathscr{I}} = [x \mathbin{\Delta} y]_{\mathscr{I}}$. Similarly, $\mathscr{P}(B)/\mathscr{J}$. For a map $\vartheta : \mathscr{P}(A) \to \mathscr{P}(B)$ to induce in an obvious way a group homomorphism of $\mathscr{P}(A)/\mathscr{I}$ to $\mathscr{P}(B)/\mathscr{J}$, it is necessary and sufficient that

1) $(\vartheta(x) \mathbin{\Delta} \vartheta(y)) \mathbin{\Delta} \vartheta(x \mathbin{\Delta} y) \in \mathscr{J}$ for all $x, y \subseteq A$, and

2) $x \in \mathscr{I} \iff \vartheta(x) \in \mathscr{J}$ for all $x \subseteq A$.

Let us call every such a map an *$(\mathscr{I}, \mathscr{J})$-approximate Δ-homomorphism.*

Borel Δ-reducibility: $\mathscr{I} \leq_{\text{B}}^{\Delta} \mathscr{J}$ iff there is a Borel $(\mathscr{I}, \mathscr{J})$-approximate Δ-homomorphism $\vartheta : \mathscr{P}(A) \to \mathscr{P}(B)$.

Note that if a map $\beta : B \to A$ witnesses, say, $\mathscr{I} \leq_{\text{RK}} \mathscr{J}$, then the map $\vartheta(x) = \beta^{-1}[x]$ obviously witnesses $\mathscr{I} \leq_{\text{B}}^{\Delta} \mathscr{J}$.

Isomorphism: $\mathscr{I} \cong \mathscr{J}$ of ideals $\mathscr{I}, \mathscr{J}$ on sets A, B, respectively, means that there is a bijection $\beta : A \xrightarrow{\text{onto}} B$ such that $x \in \mathscr{I} \iff \beta[x] \in \mathscr{J}$ for all $x \subseteq A$.

The following notion belongs to a somewhat different category since it does not allow us to really define $\mathscr{I}$ in terms of $\mathscr{J}$.

Reducibility via inclusion (see [JKL02]): $\mathscr{I} \leq_{\text{I}} \mathscr{J}$ iff there is a map $\beta : B \to A$ such that $x \in \mathscr{I} \implies \beta^{-1}[x] \in \mathscr{J}$. (Note $\implies$ instead of $\iff$!)

In particular, if $\mathscr{I} \subseteq \mathscr{J}$ (and $B = A$), then $\mathscr{I} \leq_{\text{I}} \mathscr{J}$ via $\beta(a) = a$. It follows that this order is not fully compatible with the Borel reducibility $\leq_{\text{B}}$.

3.3. P-ideals and submeasures

Many important Borel ideals belong to the class of P-ideals.

DEFINITION 3.3.1. An ideal $\mathscr{I}$ on $\mathbb{N}$ is a *P-ideal* if for every countable sequence of sets $x_n \in \mathscr{I}$ there is a set $x \in \mathscr{I}$ such that $x_n \subseteq^* x$ (that is, $x_n \smallsetminus x \in \mathtt{Fin}$) for all n. □

EXERCISE 3.3.2. Prove that the ideals $\mathtt{Fin}$, $\mathscr{I}_2$, $\mathscr{I}_3$, $\mathscr{Z}_0$ are P-ideals while $\mathscr{I}_1$ is not a P-ideal. □

This class admits several apparently different but equivalent characterizations, one of which is connected with submeasures.

DEFINITION 3.3.3. A *submeasure* on a set A is any map $\varphi : \mathscr{P}(A) \to [0, +\infty]$, satisfying $\varphi(\varnothing) = 0$, $\varphi(\{a\}) < +\infty$ for all a, and $\varphi(x) \leq \varphi(x \cup y) \leq \varphi(x) + \varphi(y)$.

A submeasure φ on $\mathbb{N}$ is *lower semicontinuous*, or LSC for brevity, if we have $\varphi(x) = \sup_n \varphi(x \cap [0, n))$ for all $x \in \mathscr{P}(\mathbb{N})$. □

To be a (finitely additive) *measure*, a submeasure φ has to satisfy, in addition, that $\varphi(x \cup y) = \varphi(x) + \varphi(y)$ whenever x, y are disjoint. Note that every σ-additive measure is LSC, but if φ is LSC, then φ_∞ is not necessarily LSC itself.

Suppose that φ is a submeasure on $\mathbb{N}$. Define the *tail submeasure* $\varphi_\infty(x) = ||x||_\varphi = \mathtt{inf}_n(\varphi(x \cap [n, \infty)))$. The following ideals are considered:

$$\begin{aligned} \mathtt{Fin}_\varphi &= \{x \in \mathscr{P}(\mathbb{N}) : \varphi(x) < +\infty\}; \\ \mathtt{Null}_\varphi &= \{x \in \mathscr{P}(\mathbb{N}) : \varphi(x) = 0\}; \\ \mathtt{Exh}_\varphi &= \{x \in \mathscr{P}(\mathbb{N}) : \varphi_\infty(x) = 0\} \quad = \quad \mathtt{Null}_{\varphi_\infty}. \end{aligned}$$

Some ideals are of the form $\mathtt{Exh}_\varphi$ for appropriate LSC submeasures φ:

EXERCISE 3.3.4. Prove the following:

(i) $\mathrm{Fin} = \mathtt{Exh}_\varphi = \mathtt{Null}_\varphi$, where $\varphi(x) = 1$, $\forall x \neq \varnothing$.

(ii) $\mathscr{I}_3 = \mathtt{Exh}_\psi$, where $\psi(x) = \sum_k 2^{-k} \varphi(\{l : \langle k, l\rangle \in x\})$ is an LSC submeasure and φ is defined as in (i).

(iii) $\mathscr{I}_2 = \mathtt{Exh}_\varphi$, where $\varphi(x) = \sum_{n \in x,\, n \geq 1} \frac{1}{n}$ is an LSC submeasure. □

The next example is somewhat more complicated:

LEMMA 3.3.5. *$\mathscr{Z}_0 = \mathtt{Exh}_\varphi$, where $\varphi(x) = \mathtt{sup}_{y \subseteq x,\, y \text{ finite}} \frac{\mathtt{card}\, y}{1 + \mathtt{max}\, y}$ is an* LSC *submeasure.*

PROOF. Indeed, that φ is LSC is rather easy. (Note that a simpler definition, like $\varphi(x) = \frac{\mathtt{card}\, x}{1+\mathtt{max}\, x}$, fails to satisfy $x \subseteq y \Longrightarrow \varphi(x) \leq \varphi(y)$.) Suppose that $z \in \mathscr{Z}_0$ and $\varepsilon > 0$. There exists n_ε such that $\frac{\mathtt{card}\,(z \cap [0,n))}{n} < \varepsilon$ for all $n > n_\varepsilon$. Then obviously $\varphi(x) < \varepsilon$ for all finite sets $x \subseteq z \cap [n, \infty)$. This proves that $\varphi_\infty(z) = 0$.

To prove the converse, suppose that $z \in \mathtt{Exh}_\varphi$ and $\varepsilon > 0$. Then there is n_ε such that $\varphi(x) < \varepsilon$ for every finite $x \subseteq z \cap [n_\varepsilon, \infty)$. Consider an arbitrary $n > n_\varepsilon \varepsilon^{-1}$. Easily, $\frac{\mathtt{card}\,(z \cap [0,n))}{n} \leq \frac{n_\varepsilon}{n} + \varphi(x)$, where $x = z \cap [n_\varepsilon, n]$, which is less than 2ε by the choice of n_ε and n. This proves $z \in \mathscr{Z}_0$. □

It turns out (SOLECKI, see Theorem 3.5.1 below) that Σ^1_1 P-ideals are the same as ideals of the form $\mathtt{Exh}_\varphi$, where φ is an LSC submeasure on $\mathbb{N}$. This implies that every $\mathbf{\Sigma}^1_1$ (in particular every Borel) P-ideal is in fact $\mathbf{\Pi}^0_3$.

3.4. Polishable ideals

There is one more useful characterization of Borel P-ideals. Let T be the ordinary Polish product topology on $\mathscr{P}(\mathbb{N})$ as described in Section 3.1. Then $\mathscr{P}(\mathbb{N})$ is a Polish group in the sense of T and the symmetric difference $\triangle$ as the operation, and each ideal $\mathscr{I}$ on $\mathbb{N}$ is a subgroup of $\mathscr{P}(\mathbb{N})$.

DEFINITION 3.4.1. An ideal $\mathscr{I}$ on $\mathbb{N}$ is *polishable* if there is a Polish group topology τ on $\mathscr{I}$ that produces the same Borel subsets of $\mathscr{I}$ as $T \restriction \mathscr{I}$. □

The same SOLECKI's theorem (Theorem 3.5.1) proves that, for analytic ideals, to be a P-ideal is the same as to be polishable. It follows (see Exercise 3.3.4) that, for instance, Fin and $\mathscr{I}_3$ are polishable, but $\mathscr{I}_1$ is not. The latter will be shown directly after the next lemma.

LEMMA 3.4.2. *Suppose that an ideal $\mathscr{I} \subseteq \mathscr{P}(\mathbb{N})$ is polishable. Then there is a unique Polish group topology τ on $\mathscr{I}$. This topology refines $T \restriction \mathscr{I}$ and is metrizable by a Δ-invariant metric. If $Z \in \mathscr{I}$, then $\tau \restriction \mathscr{P}(Z)$ coincides with $T \restriction \mathscr{P}(Z)$. In addition, $\mathscr{I}$ itself is T-Borel.*

PROOF. Let τ witness that $\mathscr{I}$ is polishable. The identity map

$$f(x) = x : \quad \langle \mathscr{I} ; \tau \rangle \to \langle \mathscr{P}(\mathbb{N}) ; T \rangle$$

is a Δ-homomorphism and is Borel-measurable because all $(T \restriction \mathscr{I})$-open sets are τ-Borel. Therefore, by the Pettis theorem (see *e.g.* KECHRIS [**Kec95**]), f is continuous. It follows that all $(T \restriction \mathscr{I})$-open subsets of $\mathscr{I}$ are τ-open, and that $\mathscr{I}$ is T-Borel in $\mathscr{P}(\mathbb{N})$ because 1-to-1 continuous images of Borel sets are Borel.

A similar "identity map" argument shows that τ is unique if it exists.

It is known (see *e.g.* KECHRIS [**Kec95**]) that every Polish group topology admits a left-invariant compatible (not necessarily complete) metric, which, in this case, is right-invariant as well since Δ is an abelian operation.

Let $Z \in \mathscr{P}(\mathbb{N})$. Then $\mathscr{P}(Z)$ is T-closed, hence, τ-closed by the above, subgroup of $\mathscr{I}$, and $\tau \restriction \mathscr{P}(Z)$ is a Polish group topology on $\mathscr{P}(Z)$. Yet $T \restriction \mathscr{P}(Z)$ is another Polish group topology on $\mathscr{P}(Z)$, with the same Borel sets. The same "identity map" argument proves that T and τ coincide on $\mathscr{P}(Z)$. □

EXAMPLE 3.4.3. The ideal $\mathscr{I}_1$ is not polishable. Indeed, we have $\mathscr{I}_1 = \bigcup_n W_n$, where $W_n = \{x : x \subseteq \{0, 1, \dots, n\} \times \mathbb{N}\}$. Let, on the contrary, τ be a Polish group topology on $\mathscr{I}_1$. Then τ and the ordinary topology T coincide on each set W_n by the lemma, in particular, each W_n remains τ-nowhere dense in W_{n+1}, hence, in $\mathscr{I}_1$, a contradiction with the Baire category theorem for τ. □

3.5. Characterization of polishable ideals

The next theorem of SOLECKI [**Sol96, Sol99**] proves that the ideal $\mathscr{I}_1$ is the $\le_{\mathrm{RB}}$-least among all Borel non-polishable ideals.

THEOREM 3.5.1. *Suppose that $\mathscr{I} \subseteq \mathbb{N}$ is an ideal. The following conditions are equivalent:*

(i) *$\mathscr{I}$ has the form Exh_φ, where φ is a* LSC *submeasure on $\mathbb{N}$;*

(ii) *$\mathscr{I}$ is a polishable ideal;*

(iii) *$\mathscr{I}$ is a Σ^1_1 P-ideal;*

(iv) *$\mathscr{I}$ is a Σ^1_1 ideal such that all countable unions of $\mathscr{I}$-small sets are $\mathscr{I}$-small,* where a set $X \subseteq \mathscr{P}(\mathbb{N})$ is *$\mathscr{I}$-small* if there is $A \in \mathscr{I}$ such that $X \restriction A = \{x \cap A : x \in X\} \subseteq \mathscr{P}(A)$ is meager in $\mathscr{P}(A)$;

(v) *$\mathscr{I}$ is a Σ^1_1 ideal satisfying $\mathscr{I}_1 \not\le_{\mathrm{RB}} \mathscr{I}$.*

PROOF (sketch). We first establish rather elementary (but tricky in some points) equivalences (i) $\Longleftrightarrow$ (ii) and (iii) $\Longleftrightarrow$ (v) $\Longleftrightarrow$ (iv) and the implication (i) $\Longrightarrow$ (iii), and then carry out the hard part, the implication (iv) $\Longrightarrow$ (i).

(i) $\Longrightarrow$ (ii). If $\varphi(\{n\}) > 0$ for all n, then the required metric on $\mathscr{I} = \mathrm{Exh}_\varphi$ can be defined by $d_\varphi(x, y) = \varphi(x \,\Delta\, y)$. Then each set $U \subseteq \mathscr{I}$ open, in the sense of the ordinary topology (the one inherited from $\mathscr{P}(\mathbb{N})$), is d_φ-open, while each d_φ-open set is Borel in the ordinary sense. In the general case we assemble the required metric of d_φ on the domain $\{n : \varphi(\{n\}) > 0\}$ and the ordinary Polish metric on $\mathscr{P}(\mathbb{N})$ on the complementary domain.

(ii) $\Longrightarrow$ (i). Let τ be a Polish group topology on $\mathscr{I}$, generated by a Δ-invariant compatible metric d. It can be shown (see [**Sol99**, p. 60]) that $\varphi(x) = \sup_{y\in\mathscr{I},\, y\subseteq x} d(\varnothing, x)$ is an LSC submeasure with $\mathscr{I} = \mathtt{Exh}_\varphi$. The key observation is that for every $x \in \mathscr{I}$ the sequence $\{x \cap [0,n)\}_{n\in\mathbb{N}}$ d-converges to x by the last statement of Lemma 3.4.2, which implies both that φ is LSC (because the supremum above can be restricted to finite sets y) and that $\mathscr{I} = \mathtt{Exh}_\varphi$ (where the inclusion $\supseteq$ needs another "identity map" argument).

(i) $\Longrightarrow$ (iii). That every ideal of the form $\mathscr{I} = \mathtt{Exh}_\varphi$, φ being LSC, is a P-ideal, is an easy exercise: if $x_1, x_2, x_3, \dots \in \mathscr{I}$, then define an increasing sequence of numbers $n_i \in x_i$ with $\varphi(x_i \cap [n_i, \infty)) \le 2^{-n}$ and put $x = \bigcup_i (x_i \cap [n_i, \infty))$.

(iii) $\Longrightarrow$ (v). This is because $\mathscr{I}_1$ easily does not satisfy (iii).

(v) $\Longrightarrow$ (iv). Suppose that sets $X_n \subseteq \mathscr{P}(\mathbb{N})$ are $\mathscr{I}$-small, so that $X_n \restriction A_n$ is meager in $\mathscr{P}(A_n)$ for some $A_n \in \mathscr{I}$, but $X = \bigcup_n X_n$ is not $\mathscr{I}$-small, and prove $\mathscr{I}_1 \le_{\mathrm{RB}} \mathscr{I}$. Arguing as in the proof of Theorem 3.2.1, we use the meagerness to find, for every n, a sequence of pairwise disjoint non-empty finite sets $w^n_k \subseteq A_n$, $k \in \mathbb{N}$, and subsets $u^n_k \subseteq w^n_k$, such that

(a) if $x \subseteq \mathbb{N}$ and $\exists^\infty k\, (x \cap w^n_k = u^n_k)$, then $x \notin X_n$.

Dropping some sets w^n_k and re-enumerating the rest, we can strengthen the disjointness to the following: $w^n_k \cap w^m_l = \varnothing$ unless both $n = m$ and $k = l$.

Now put $w^n_{ij} = w^n_{2^i(2j+1)-1}$. The sets $\overline{w}_{ij} = \bigcup_{n\le i} w^n_{ij}$ are still pairwise disjoint and satisfy the following two properties:

(b) $\bigcup_j \overline{w}_{ij} \subseteq A_0 \cup \dots \cup A_i$, hence, $\in \mathscr{I}$, for all i;

(c) if a set $Z \subseteq \mathbb{N}\times\mathbb{N}$ does not belong to $\mathscr{I}_1$, i.e., $\exists^\infty i\, \exists j\, (\langle i,j\rangle \in Z)$, then $\forall n\, \exists^\infty k\, (w^n_k \subseteq \overline{w}_Z)$, where $\overline{w}_Z = \bigcup_{\langle i,j\rangle \in Z} \overline{w}_{ij})$.

We assert that the map $\langle i,j\rangle \mapsto \overline{w}_{ij}$ witnesses $\mathscr{I}_1 \le^+_{\mathrm{RB}} \mathscr{I}$. (Then a simple argument, as in the proof of Theorem 3.2.1, gives $\mathscr{I}_1 \le_{\mathrm{RB}} \mathscr{I}$.)

Indeed, if $Z \subseteq \mathbb{N}\times\mathbb{N}$ belongs to $\mathscr{I}_1$, then $\overline{w}_Z \in \mathscr{I}$ by (b). Suppose that $Z \notin \mathscr{I}_1$. It suffices to show that $X_n \restriction \overline{w}_Z$ is meager in $\mathscr{P}(\overline{w}_Z)$ for each n. Note that by (c) the set $K = \{k : w^n_k \subseteq \overline{w}_Z\}$ is infinite and in fact $\overline{w}_Z \cap A_n = \bigcup_{k\in K} w^n_k$. Therefore, every $x \subseteq \overline{w}_Z$ satisfying $x \cap w^n_k = u^n_k$ for infinitely many $k \in K$, does not belong to X_n by (a). Now the meagerness of $X_n \restriction \overline{w}_Z$ is clear.

(iv) $\Longrightarrow$ (iii). This also is quite easy: if a sequence of sets $Z_n \in \mathscr{I}$ witnesses that $\mathscr{I}$ is not a P-ideal, then the union of $\mathscr{I}$-small sets $\mathscr{P}(Z_n)$ is not $\mathscr{I}$-small.

(iv) $\Longrightarrow$ (i). This is the hard part of Theorem 3.5.1. A couple of definitions precede the key lemma.

- Let $C(\mathscr{I})$ be the collection of all hereditary (*i.e.*, $y \subseteq x \in K \Longrightarrow y \in K$) compact $\mathscr{I}$-large sets $K \subseteq \mathscr{P}(\mathbb{N})$. (By definition a set $K \subseteq \mathscr{P}(\mathbb{N})$ is *$\mathscr{I}$-large* iff it is not $\mathscr{I}$-small in the sense of (iv) of Theorem 3.5.1.)

Note that if $K \subseteq \mathscr{P}(\mathbb{N})$ is hereditary and compact, then for $K \in C(\mathscr{I})$ it is necessary and sufficient that for every $A \in \mathscr{I}$ there is n such that $A \cap [n,\infty) \in K$.

- Given sets $X, Y \subseteq \mathscr{P}(\mathbb{N})$, let $X + Y = \{x \cup y : x \in X \wedge y \in Y\}$.

LEMMA 3.5.2. *Assume that $\mathscr{I}$ is of type* (iv) *of Theorem* 3.5.1. *Then there is a countable sequence of sets $K_m \in C(\mathscr{I})$ such that for every set $K \in C(\mathscr{I})$ there exist numbers m, n with $K_m + K_n \subseteq K$.*

PROOF. As $\mathscr{I}$ is a Σ^1_1 subset of $\mathscr{P}(\mathbb{N})$, there exists a continuous map $f : \mathbb{N}^{\mathbb{N}} \xrightarrow{\text{onto}} \mathscr{I}$. For any $s \in \mathbb{N}^{<\omega}$, we define

$$N_s = \{a \in \mathbb{N}^{\mathbb{N}} : s \subset a\} \quad \text{and} \quad B_s = f[N_s] \quad \text{(the } f\text{-image of } N_s\text{)}.$$

Consider the set $T = \{s : B_s \text{ is } \mathscr{I}\text{-large}\}$. As $\mathscr{I}$ itself is clearly $\mathscr{I}$-large, $\Lambda \in T$. On the other hand, the assumption (iv) easily implies that T has no endpoints and no isolated branches; hence, $P = \{a \in \mathbb{N}^{\mathbb{N}} : \forall n\, (a \restriction n \in T)\}$ is a perfect set. Moreover, $F_s = f[(P \cap N_s)]$ is $\mathscr{I}$-large for every $s \in T$ because $B_s \smallsetminus F_s$ is a countable union of $\mathscr{I}$-small sets.

Now consider any set $K \in C(\mathscr{I})$. By definition, if $x, y \in \mathscr{I}$, then $z = x \cup y \in \mathscr{I}$, thus, $K \restriction z$ is not meager in $\mathscr{P}(z)$. Hence, by the compactness, $K \restriction z$ includes a basic nbhd of $\mathscr{P}(z)$, and hence, by the hereditarity, there is a number n such that $Z \cap [n, \infty) \in K$. We conclude that $P^2 = \bigcup_n Q_n$, where each $Q_n = \{\langle a, b\rangle \in P^2 : (f(a) \cup f(b)) \cap [n, \infty) \in K\}$ is closed in P because so is K and f is continuous. Thus, there are $s, t \in T$ such that $P^2 \cap (N_s \times N_t) \subseteq Q_n$; in other words, $(F_s + F_t) \restriction [n, \infty) \subseteq K$, hence, $(\widehat{F_s + F_t}) \restriction [n, \infty) \subseteq K$, where $\widehat{\ldots}$ denotes the topological closure of the hereditary hull. Thus we can take, as the collection of sets K_m, all sets of the form $K_{sn} = \widehat{F_s} \restriction n$. □

As $C(\mathscr{I})$ is obviously a filter, the sequence of sets given by the lemma can be transformed (still in the assumption that $\mathscr{I}$ is of type (iv)) into a $\subseteq$-decreasing sequence of sets $K_n \in C(\mathscr{I})$ such that

(1) for every $K \in C(\mathscr{I})$ there is n with $K_n \subseteq K$,

and $K_{n+1} + K_{n+1} \subseteq K_n$ for every n. Taking any other term of the sequence, we can strengthen the latter requirement to

(2) for every n: $K_{n+1} + K_{n+1} + K_{n+1} \subseteq K_n$.

This is a starting point for the construction of an LSC submeasure φ with $\mathscr{I} = \mathrm{Exh}_\varphi$. Assuming that, in addition, $K_0 = \mathscr{P}(\mathbb{N})$, put, for every $x \in \mathscr{P}_{\mathtt{fin}}(\mathbb{N})$,

$$\begin{aligned} \varphi_1(x) &= \mathtt{inf}\,\{\, 2^{-n} : x \in K_n \,\}, \text{ and} \\ \varphi_2(x) &= \mathtt{inf}\,\{\, \textstyle\sum_{i=1}^m \varphi_1(x_i) : m \geq 1 \wedge x_i \in \mathscr{P}_{\mathtt{fin}}(\mathbb{N}) \wedge x \subseteq \bigcup_{i=1}^m x_i \,\}. \end{aligned}$$

Then put $\varphi(x) = \mathtt{sup}_n\, \varphi_2(x \cap [0, n))$ for every $x \subseteq \mathbb{N}$. A routine verification shows that φ is a submeasure and that $\mathscr{I} = \mathrm{Exh}_\varphi$. (See SOLECKI [Sol99]. To check that every $x \in \mathrm{Exh}_\varphi$ belongs to $\mathscr{I}$, use the following observation: $x \in \mathscr{I}$ iff for every $K \in C(\mathscr{I})$ there is n such that $x \cap [n, \infty) \in K$.) □ (Theorem 3.5.1)

3.6. Summable and density ideals

Any sequence $\{r_n\}_{n\in\mathbb{N}}$ of positive reals r_n with $\sum r_n = +\infty$ defines the ideal

$$\mathscr{I}_{\{r_n\}} = \{x \subseteq \mathbb{N} : \sum_{n\in x} r_n < +\infty\} = \{x : \mu_{\{r_n\}}(x) < +\infty\},\text{[2]}$$

[2] The particular case $r_n = \frac{1}{n}$ will be frequently considered below. Obviously, r_0 is not defined, and hence the definition is amended to $\mathscr{I}_{\{1/n\}} = \{x \subseteq \mathbb{N} : \sum_{n\in x,\, x\geq 1} r_n < +\infty\}$. This is the same as putting additionally $r_0 = 0$.

where $\mu_{\{r_n\}}(X) = \sum_{n\in X} r_n$. These ideals are called *summable ideals*; all of them are $\mathbf{F}_\sigma$ in the product Polish topology on $\mathscr{P}(\mathbb{N})$. Every summable ideal is easily a P-ideal: indeed, $\mathscr{S}_{\{r_n\}} = \mathtt{Exh}_\varphi$, where $\varphi(X) = \sum_{n\in X} r_n$ is a σ-additive measure. Summable ideals are perhaps the easiest to study among all P-ideals.

LEMMA 3.6.1. *Assume that* $r_n \geq 0$, $r_n \to 0$, $\sum_n r_n = +\infty$. *Then* $\langle \mathscr{S}_{\{r_n\}}\,;\triangle\rangle$ *is a Polish group, and every summable ideal* $\mathscr{I}$ *satisfies* $\mathscr{I} \leq^{++}_{\mathtt{RB}} \mathscr{S}_{\{r_n\}}$.

PROOF. Show that $\langle \mathscr{S}_{\{r_n\}}\,;\triangle\rangle$ is a Polish group with the distance $d_{\{r_n\}}(a,b) = \varphi_{\{r_n\}}(a \triangle b)$, where

$$\varphi_{\{r_n\}}(x) = \sum_{n\in x} r_n \text{ for } x \in \mathscr{P}(\mathbb{N}), \text{ hence } \mathscr{S}_{\{r_n\}} = \{x : \varphi_{\{r_n\}}(x) < +\infty\}.$$

To prove that the operation is continuous, let $x, y \in \mathscr{P}(\mathbb{N})$. Fix a real $\delta > 0$, and let $\varepsilon = \frac{\delta}{2}$. If x', y' belong to the ε-nbhds of x, y in $\mathscr{S}_{\{r_n\}}$ with the distance $d_{\{r_n\}}$, then $(x' \triangle y') \triangle (x \triangle y) \subseteq (x \triangle x') \cup (y \triangle y')$; therefore

$$d_{\{r_n\}}(x' \triangle y', x \triangle y) \leq d_{\{r_n\}}(x, x') + d_{\{r_n\}}(y, y') = \delta.$$

To prove the second claim, let $\mathscr{I} = \mathscr{S}_{\{p_n\}}$, where $p_n \geq 0$ (no other requirements!). Under the assumptions of the lemma we can associate a finite set $w_n \subseteq \mathbb{N}$ to every n so that $\mathtt{max}\, w_n < \mathtt{min}\, w_{n+1}$ and $|r_n - \sum_{j\in w_n} r_i| < 2^{-n}$. □

For more on summable ideals, see [**Mat72, Maz91, Far00**].

FARAH [**Far00**, 1.10] defines a non-summable $\mathbf{F}_\sigma$ P-ideal as follows. Let $I_k = [2^k, 2^{k+1})$ and $\psi_k(s) = k^{-2}\,\mathtt{min}\{k, \mathtt{card}\, s\}$ for all k and $s \subseteq I_k$, and then

$$\psi(X) = \sum_{k=0}^{\infty} \psi_k(X \cap I_k) \quad \text{and} \quad \mathscr{I} = \mathtt{Fin}_\psi\,;$$

it turns out that $\mathscr{I}$ is an $\mathbf{F}_\sigma$ P-ideal, but not summable. To show that $\mathscr{I}$ is not of the form $\mathscr{S}_{\{r_n\}}$, FARAH notes that there is a set X (which depends on $\{r_n\}$) such that the differences $|\mu_{\{r_n\}}(X \cap I_k) - \psi_k(X \cap I_k)|$, $k = 0, 1, 2, \dots$, are unbounded.

There exist other important types of Borel P-ideals. Every sequence $\{r_n\}_{n\in\mathbb{N}}$ of positive reals r_n with $\sum r_n = +\infty$ defines the ideal

$$EU_{\{r_n\}} = \left\{ x \subseteq \mathbb{N} : \lim_{n\to+\infty} \frac{\sum_{i\in x\cap[0,n)} r_i}{\sum_{i\in[0,n)} r_i} = 0 \right\}.$$

These ideals are called ERDÖS–ULAM (or EU) ideals. Two examples are $\mathscr{Z}_0 = EU_{\{1\}}$ and $\mathscr{Z}_{\mathtt{log}} = EU_{\{1/n\}}$.

This definition can be generalized. Given a measure μ on $\mathbb{N}$, we put $\mathtt{supp}\,\mu = \{n : \mu(\{n\}) > 0\}$. Measures μ, ν are *orthogonal* if we have $\mathtt{supp}\,\mu \cap \mathtt{supp}\,\nu = \varnothing$. Now suppose that $\vec{\mu} = \{\mu_n\}_{n\in\mathbb{N}}$ is a sequence of pairwise orthogonal measures on $\mathbb{N}$, with finite sets $\mathtt{supp}\,\mu_i$. Define $\varphi_{\vec{\mu}}(X) = \sup_n \mu_n(X)$: this is an LSC submeasure on $\mathbb{N}$. Finally let $\mathscr{D}_{\vec{\mu}} = \mathtt{Exh}(\varphi_{\vec{\mu}}) = \{X : ||X||_{\varphi_\mu} = 0\}$. Ideals of this form are called *density ideals*. This class includes all EU ideals (although this is not immediately transparent) and some other ideals: for instance, $\mathscr{I}_3$ is a density but non-EU ideal. Generally density ideals are more complicated than summables. We obtain an even wider class if the requirement, that the sets $\mathtt{supp}\,\mu_n$ are finite, is dropped: this wider family includes all summmable ideals, too. See [JK84] or [**Far00**, § 1.13] on density ideals.

3.7. Operations on ideals and Fréchet ideals

Suppose that A is any non-empty set, and $\mathscr{J}_a$ is an ideal on a set B_a for all $a \in A$. The following two operations on such a family of ideals are defined.

Disjoint sum: $\sum_{a\in A} \mathscr{J}_a$ is the ideal on the set $B = \{\langle a, b\rangle : a \in A \wedge b \in B_a\}$ that consists of all sets $x \subseteq B$ such that $(x)_a \in \mathscr{J}_a$ for all $a \in A$, where $(x)_a = \{b : \langle a, b\rangle \in x\}$ (the cross-section). If the sets B_a are pairwise disjoint, then $\sum_{a\in A} \mathscr{I}_a$ can be defined equivalently as the ideal on $B = \bigcup_{a\in A} B_a$ that consists of all sets of the form $\bigcup_{a\in A} x_a$, where $x_a \in \mathscr{I}_a$ for all a.

In the case of two summands, the disjoint sum $\mathscr{I} \oplus \mathscr{J}$ of ideals $\mathscr{I}, \mathscr{J}$ on disjoint sets A, B is equal to $\{x \cup y : x \in \mathscr{I} \wedge y \in \mathscr{J}\}$.

Fubini sum and product: Suppose in addition that $\mathscr{I}$ is an ideal on A. The *Fubini sum* $\sum_{a\in A} \mathscr{J}_a / \mathscr{I}$ of the ideals $\mathscr{J}_a$ modulo $\mathscr{I}$ is the ideal on the set B (defined as above) which consists of all sets $y \subseteq B$ such that the set $\{a : (x)_a \notin \mathscr{J}_a\}$ belongs to $\mathscr{I}$. This ideal obviously coincides with the plain disjoint sum $\sum_{a\in A} \mathscr{J}_a$ in the case when $\mathscr{I} = \{\varnothing\}$.

In particular, the *Fubini product* $\mathscr{I} \otimes \mathscr{J}$ of two ideals $\mathscr{I}, \mathscr{J}$ on sets A, B, respectively, is equal to $\sum_{a\in A} \mathscr{J}_a / \mathscr{I}$, where $\mathscr{J}_a = \mathscr{J}$, $\forall a$. Thus $\mathscr{I} \otimes \mathscr{J}$ consists of all sets $y \subseteq A \times B$ such that $\{a : (y)_a \notin \mathscr{J}\} \in \mathscr{I}$.

Coming back to the ideals defined in Section 3.1, $\mathscr{I}_1$ and $\mathscr{I}_3$ coincide with $\mathtt{Fin} \times 0$ and $0 \times \mathtt{Fin}$, respectively, where, we recall, 0 denotes the least ideal $0 = \{\varnothing\}$.

The operations of the Fubini sum and product lead to an important family of *Fréchet ideals*. This family consists of ideals $\mathtt{Fr}_\xi$, $\xi < \omega_1$, defined by transfinite induction as follows:

- $\mathtt{Fr}_1 = \mathtt{Fin}$ and $\mathtt{Fr}_{\xi+1} = \mathtt{Fin} \otimes \mathtt{Fr}_\xi$ for all ξ,
- $\mathtt{Fr}_\lambda = \sum_{\xi<\lambda} \mathtt{Fr}_\xi / \{\varnothing\}$ for all limit ordinals $\lambda < \omega_1$.

Limit steps can be treated differently; for instance, by

$$\mathtt{Fr}_\lambda = \textstyle\sum_{\xi<\lambda} \mathtt{Fr}_\xi / \mathtt{Fin}_\lambda, \quad \text{or by} \quad \mathtt{Fr}_\lambda = \sum_{\xi<\lambda} \mathtt{Fr}_\xi / \mathtt{Bou}_\lambda,$$

where $\mathtt{Fin}_\lambda$ is the ideal of all finite subsets of λ and $\mathtt{Bou}_\lambda$ is the ideal of all bounded subsets of λ, and also by $\mathtt{Fr}_\lambda = \sum_{n\in\mathbb{N}} \mathtt{Fr}_{\xi_n} / \mathtt{Fin}$, where $\{\xi_n\}$ is a once and for all fixed cofinal increasing sequence of ordinals below λ, as in [**JKL02**], with the understanding that the result is independent of the choice of ξ_n, modulo a certain equivalence relation.

3.8. Some other ideals

We consider two interesting families of Borel ideals (mainly, non-P-ideals), united by their relation to countable ordinals. Note that the underlying sets of these ideals are countable sets different from $\mathbb{N}$.

Indecomposable ideals. Let $\mathtt{otp}\,X$ be the order type of $X \subseteq \mathtt{Ord}$. For any ordinals ξ, $\vartheta < \omega_1$ define:

$$\mathscr{I}^\xi_\vartheta = \{A \subseteq \vartheta : \mathtt{otp}\,A < \omega^\xi\} \quad (\text{nontrivial only if } \vartheta \geq \omega^\xi).$$

To see that the sets $\mathscr{I}_\vartheta^\xi$ are really ideals, note that ordinals of the form ω^ξ and only those ordinals are *indecomposable*, i.e., are not sums of a pair of smaller ordinals, hence, the set $\{A \subseteq \vartheta : \mathtt{otp}\, A < \gamma\}$ is an ideal iff $\gamma = \omega^\xi$ for some ξ.

Weiss ideals. Let $|X|_{\mathrm{CB}}$ be the Cantor–Bendixson rank of $X \subseteq \mathtt{Ord}$, equal to the least ordinal α such that $X^{(\alpha)} = \varnothing$. Here $X^{(\alpha)}$ is defined by induction on $\alpha \in \mathtt{Ord}$: $X^{(0)} = X$, $X^{(\lambda)} = \bigcap_{\alpha<\lambda} X^{(\alpha)}$ at limit steps λ, and finally $X^{(\alpha+1)} = (X^{(\alpha)})'$, where A', the Cantor–Bendixson derivative, is the set of all ordinals $\gamma \in x$ which are limit points of X in the interval topology. For any pair of ordinals $\xi, \vartheta < \omega_1$, define the Weiss ideal:

$$\mathscr{W}_\vartheta^\xi = \{A \subseteq \vartheta : |A|_{\mathrm{CB}} < \omega^\xi\}$$

(nontrivial only if $\vartheta \ge \omega^{\omega^\xi}$). It is less transparent that all sets of the form $\mathscr{W}_\vartheta^\xi$ are ideals (see FARAH [**Far00**, 1.14]) while $\{A \subseteq \vartheta : |A|_{\mathrm{CB}} < \gamma\}$ is not an ideal whenever γ is not of the form ω^ξ.

Ideals on finite sequences. The set $\mathbb{N}^{<\omega}$ of all finite sequences of natural numbers is countable, yet its own order structure is quite different from that of $\mathbb{N}$. We can exploit this in several ways, for instance, with ideals of sets $X \subseteq \mathbb{N}^{<\omega}$ which intersect every branch in $\mathbb{N}^{<\omega}$ by a set which belongs to a given ideal on $\mathbb{N}$.

CHAPTER 4

Introduction to equivalence relations

Recall that an *equivalence relation* (ER, for brevity) on a set A is a reflexive, transitive, and symmetric binary relation on A. Suppose that E is an equivalence relation on a set X. Then

$$\begin{aligned} [y]_{\mathsf{E}} &= \{x \in X : y \mathrel{\mathsf{E}} x\} \quad \text{for } y \in X \quad (\text{the } \mathsf{E}\text{-}class \text{ of } x), \text{ and} \\ [Y]_{\mathsf{E}} &= \textstyle\bigcup_{y \in Y} [y]_{\mathsf{E}} \quad \text{for } Y \subseteq X \quad (\text{the } \mathsf{E}\text{-}saturation \text{ of } Y). \end{aligned}$$

- A set $Y \subseteq X$ is E-*invariant* if $[Y]_{\mathsf{E}} = Y$.
- A subset $Y \subseteq X$ is *pairwise* E-*equivalent*, resp., *pairwise* E-*inequivalent*, if $x \mathrel{\mathsf{E}} y$, resp., $x \not\mathrel{\mathsf{E}} y$, holds for all $x \neq y$ in Y.
- If X, Y are sets and E is a binary relation, then $X \mathrel{\mathsf{E}} Y$ means that we have both $\forall x \in X \, \exists y \in Y \, (x \mathrel{\mathsf{E}} y)$ and $\forall y \in Y \, \exists x \in X \, (x \mathrel{\mathsf{E}} y)$.

We introduce several Borel equivalence relations in Section 4.1. We then discuss operations on equivalence relations in Section 4.2, orbit ERs of Borel actions of Borel and Polish groups in Sections 4.3 and 4.4, and discuss connections between Borel equivalence relations and Borel measures in Sections 4.5 and 4.6.

4.1. Some examples of Borel equivalence relations

Let Δ_X denote the equality on a set X, considered to be an equivalence relation on X. This is the most elementary type of equivalence relation. A much more diverse family consists of equivalence relations $\mathsf{E}_{\mathscr{I}}$ generated by Borel ideals.

- If $\mathscr{I}$ is an ideal on a set A, then $\mathsf{E}_{\mathscr{I}}$ is an equivalence relation on $\mathscr{P}(A)$, defined so that $x \mathrel{\mathsf{E}_{\mathscr{I}}} y$ iff $x \mathbin{\Delta} y \in \mathscr{I}$.

Equivalently, $\mathsf{E}_{\mathscr{I}}$ can be considered to be an equivalence relation on 2^A defined so that $f \mathrel{\mathsf{E}_{\mathscr{I}}} g$ iff $f \mathbin{\Delta} g \in \mathscr{I}$, where $f \mathbin{\Delta} g = \{a \in A : f(a) \neq g(a)\}$. Note that $\mathsf{E}_{\mathscr{I}}$ is Borel provided $\mathscr{I}$ is Borel. We obtain the following important equivalence relations:[1]

$\mathsf{E}_0 = \mathsf{E}_{\mathtt{Fin}}$: is an ER on $\mathscr{P}(\mathbb{N})$, and $x \mathrel{\mathsf{E}_0} y$ iff $x \mathbin{\Delta} y \in \mathtt{Fin}$.

$\mathsf{E}_1 = \mathsf{E}_{\mathscr{I}_1}$: is an ER on $\mathscr{P}(\mathbb{N} \times \mathbb{N})$, and $x \mathrel{\mathsf{E}_1} y$ iff $(x)_k = (y)_k$ for all but finite k, where $(x)_k = \{n : \langle k, n\rangle \in x\}$ for $x \subseteq \mathbb{N} \times \mathbb{N}$.

$\mathsf{E}_2 = \mathsf{E}_{\mathscr{I}_2}$: is an ER on $\mathscr{P}(\mathbb{N})$, and $x \mathrel{\mathsf{E}_2} y$ iff $\sum_{k \in x \Delta y,\, k \geq 1} \frac{1}{k} < \infty$.

$\mathsf{E}_3 = \mathsf{E}_{\mathscr{I}_3}$: is an ER on $\mathscr{P}(\mathbb{N} \times \mathbb{N})$, and $x \mathrel{\mathsf{E}_3} y$ iff $(x)_k \mathrel{\mathsf{E}_0} (y)_k$, $\forall k$.

$\mathsf{Z}_0 = \mathsf{E}_{\mathscr{Z}_0}$: is an ER on $\mathscr{P}(\mathbb{N})$, and $x \mathsf{Z}_0 y$ iff $\mathtt{lim}_{n \to \infty} \frac{\mathtt{card}\,((x \Delta y) \cap [0, n))}{n} = 0$.

---

[1] The notational system we follow is not the only one used in modern texts. For instance $\mathsf{E}_1, \mathsf{E}_2, \mathsf{E}_3$ are sometimes denoted differently; see e.g. [**Gao06**].

Alternatively, E_0 can be viewed as an equivalence relation on $2^{\mathbb{N}}$ defined as $a\ \mathsf{E}_0\ b$ iff $a(k) = b(k)$ for all but finite k. Similarly, E_1 can be viewed as an equivalence relation on $\mathscr{P}(\mathbb{N})^{\mathbb{N}}$, or even on $(2^{\mathbb{N}})^{\mathbb{N}}$, defined as $x\,\mathsf{E}_1\,y$ iff $x(k) = y(k)$ for all but finite k, for all $x, y \in \mathscr{P}(\mathbb{N})^{\mathbb{N}}$. And finally E_3 can be viewed as an equivalence relation on $\mathscr{P}(\mathbb{N})^{\mathbb{N}}$, or on $(2^{\mathbb{N}})^{\mathbb{N}}$, defined as $x\,\mathsf{E}_3\,y$ iff $x(k)\,\mathsf{E}_0\,y(k)$ for all k.

DEFINITION 4.1.1. Generalizing the definition of E_0 and E_1, we define for any set $W \neq \varnothing$ an equivalence relation $\mathsf{E}_0(W)$ on the set $W^{\mathbb{N}}$ (of all infinite sequences of elements of W) so that $x\ \mathsf{E}_0(W)\ y$ iff $x(k) = y(k)$ for all but finite k, for all x, $y \in W^{\mathbb{N}}$. Thus, E_0 is $\mathsf{E}_0(2)$, while E_1 is $\mathsf{E}_0(\mathscr{P}(\mathbb{N}))$. □

The next type includes equivalence relations induced by actions of (the additive groups of) some Banach spaces; see below on group actions. The following Banach spaces are well known from textbooks:

$$
\begin{array}{rcll}
\ell^p & = & \{x \in \mathbb{R}^{\mathbb{N}} : \sum_n |x(n)|^p < \infty\}\ \ (p \geq 1); & \|x\|_p = (\sum_n |x(n)|^p)^{\frac{1}{p}}\,; \\
\ell^\infty & = & \{x \in \mathbb{R}^{\mathbb{N}} : \mathtt{sup}_n\, |x(n)| < \infty\}; & \|x\|_\infty = \mathtt{sup}_n\, |x(n)|\,; \\
\mathsf{c} & = & \{x \in \mathbb{R}^{\mathbb{N}} : \mathtt{lim}_n\, x(n) < \infty \text{ exists}\}; & \|x\| = \mathtt{sup}_n\, |x(n)|\,; \\
\mathsf{c}_0 & = & \{x \in \mathbb{R}^{\mathbb{N}} : \mathtt{lim}_n\, x(n) = 0\}; & \|x\| = \mathtt{sup}_n\, |x(n)|\,.
\end{array}
$$

Note that ℓ^p, c, c_0 are separable spaces while ℓ^∞ is non-separable. The domain of each of these spaces consists of infinite sequences $x = \{x(n)\}_{n\in\mathbb{N}}$ of reals, and is a subgroup of the group $\mathbb{R}^{\mathbb{N}}$ (with the component-wise addition). The latter can be naturally equipped with the Polish product topology, so that ℓ^p, ℓ^∞, c, c_0 are Borel subgroups of $\mathbb{R}^{\mathbb{N}}$. (However, these are not topological subgroups since the distances are different. The metric definitions as in ℓ^p or ℓ^∞ do not work for $\mathbb{R}^{\mathbb{N}}$.)

Each of the four mentioned Banach spaces induces an *orbit equivalence relation*, a Borel equivalence relation on $\mathbb{R}^{\mathbb{N}}$ also denoted by ℓ^p, ℓ^∞, c, c_0, respectively. For instance, $x\ \ell^p\ y$ if and only if $\sum_k |x(k) - y(k)|^p < +\infty$ (for all $x, y \in \mathbb{R}^{\mathbb{N}}$).

There is one more important equivalence relation:

T_2: often called *the equality of countable sets of reals*, is an equivalence relation defined on $(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$ so that $g\ \mathsf{T}_2\ h$ iff $\mathtt{ran}\, g = \mathtt{ran}\, h$ $(g, h \in (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}})$.

There is no reasonable way to turn $\mathscr{P}_{\mathtt{ctbl}}(\mathbb{N}^{\mathbb{N}})$, the set of all at most countable subsets of $\mathbb{N}^{\mathbb{N}}$, into a Polish space, in order to directly define the equality of countable sets of reals in terms of Δ_X for a suitable Polish X. However, nonempty members of $\mathscr{P}_{\mathtt{ctbl}}(\mathbb{N}^{\mathbb{N}})$ can be identified with equivalence classes in $(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}/\mathsf{T}_2$ (see Chapter 12 on the whole series of equivalence relations T_α, $\alpha < \omega_1$).

We finish with yet another significant equivalence relation:

E_∞: *the universal countable Borel equivalence relation.* The countability here means that all E-equivalence classes $[x]_{\mathsf{E}}$ are at most countable sets. The notion of universality will be explained below.

See Example 4.4.5 on an exact definition of E_∞.

4.2. Operations on equivalence relations

The following operations over equivalence relations are in part parallel to the operations on ideals in Section 3.7.

Suppose that A is a non-empty and *at most countable* set, and F_a is an equivalence relation on a set X_a for all $a \in A$. The following operations on such a family of equivalence relations are defined.

(o1) *Union* $\bigcup_{a\in A} \mathsf{F}_a$ (if it results in an equivalence relation) and *intersection* $\bigcap_{a\in A} \mathsf{F}_a$ (it always results in an equivalence relation) — in the case when all F_a are equivalence relations on one and the same set $X = X_a$, $\forall a$.

(o2) *Countable disjoint union* $\bigvee_{a\in A} \mathsf{F}_a$ is an equivalence relation E on the set $X = \bigcup_a(\{a\} \times X_a)$ defined as follows: $\langle a, x\rangle \mathrel{\mathsf{E}} \langle b, y\rangle$ iff $a = b$ and $x \mathrel{\mathsf{E}_a} y$.
If the sets X_a are pairwise disjoint, then we can equivalently define an equivalence relation $\mathsf{E} = \bigvee_a \mathsf{F}_a$ on the set $Y = \bigcup_a X_a$ so that $x \mathrel{\mathsf{E}} y$ iff x, y belong to the same X_a and $x \mathrel{\mathsf{F}_a} y$.

(o3) *Product* $\prod_{a\in A} \mathsf{F}_a$ is an equivalence relation E on the cartesian product $\prod_{a\in A} X_a$ defined so that $x \mathrel{\mathsf{E}} y$ iff $x(a) \mathrel{\mathsf{F}_a} y(a)$ for all $a \in A$.
In particular the product $\mathsf{F}_1 \times \mathsf{F}_2$ of equivalence relations E, F on sets X_1, X_2, respectively, is an equivalence relation E on $X_1 \times X_2$ defined so that $\langle x_1, x_2\rangle \mathrel{\mathsf{E}} \langle y_1, y_2\rangle$ iff $x_1 \mathrel{\mathsf{F}_1} y_1$ and $x_2 \mathrel{\mathsf{F}_2} y_2$.
If $X_a = X$ and $\mathsf{F}_a = \mathsf{F}$ for all a, then the power notation F^A can be used instead of $\prod_{a\in A} \mathsf{F}_a$.

(o4) The *Fubini product* (ultraproduct) $\prod_{a\in A} \mathsf{F}_a \,/\, \mathscr{I}$ modulo an ideal $\mathscr{I}$ on A is the equivalence relation on the product space $\prod_a X_a$ defined as follows: $x \mathrel{\mathsf{E}} y$ iff the set $\{a : x(a) \not\mathrel{\mathsf{F}_a} y(a)\}$ belongs to $\mathscr{I}$.
If $X_a = X$ and $\mathsf{F}_a = \mathsf{F}$ for all a, then the ultrapower notation $\mathsf{F}^A/\mathscr{I}$ can be used instead of $\prod_{a\in A} \mathsf{F}_a \,/\, \mathscr{I}$.

(o5) *Countable power* of an equivalence relation F on a set X is an equivalence relation F^+ defined on the set $X^{\mathbb{N}}$ as follows:

$$x \mathrel{\mathsf{F}^+} y \quad \text{iff} \quad \{[x(k)]_{\mathsf{E}} : k \in \mathbb{N}\} = \{[y(k)]_{\mathsf{E}} : k \in \mathbb{N}\},$$

so that for every k there is l with $x(k) \mathrel{\mathsf{F}} y(l)$ and for every l there is k with $x(k) \mathrel{\mathsf{F}} y(l)$.

EXAMPLE 4.2.1. In these terms, the equivalence relations E_1 and E_3 coincide with $(\Delta_{2^{\mathbb{N}}})^{\mathbb{N}}/\mathsf{Fin}$ and ${\mathsf{E}_0}^{\mathbb{N}}$, respectively, modulo obvious bijections between the spaces considered. Generally, the operations on ideals introduced in Section 3.7 transform in some regular way into operations on the corresponding equivalence relations. For instance, $\mathsf{E}_{\sum_{a\in A} \mathscr{I}_a / \mathscr{I}}$ is equal to $\prod_{a\in A} \mathsf{E}_{\mathscr{I}_a} / \mathscr{I}$, while $\mathsf{E}_{\mathscr{I}\otimes\mathscr{J}}$ is equal to $(\mathsf{E}_{\mathscr{J}})^A/\mathscr{I}$, where A is the domain of $\mathscr{I}$.
Accordingly, $\mathsf{E}_{\sum_a \mathscr{I}_a}$ is equal to $\prod_a \mathsf{E}_{\mathscr{I}_a}$. In particular if $\mathscr{I}, \mathscr{J}$ are ideals on disjoint sets A, B, then $\mathsf{E}_{\mathscr{I}\oplus\mathscr{J}}$ is equal to $\mathsf{E}_{\mathscr{I}} \times \mathsf{E}_{\mathscr{J}}$. □

EXERCISE 4.2.2. Show that the equivalence relation T_2 defined in Section 4.1 coincides with ${\Delta_{\mathbb{N}^{\mathbb{N}}}}^+$. □

Iterating these operations, we obtain a number of interesting equivalence relations starting just with very primitive ones.

EXAMPLE 4.2.3. Iterating the operation of countable power, H. FRIEDMAN defines the sequence of equivalence relations T_ξ, $1 \le \xi < \omega_1$, in **[Fri00]** as follows.[2]

[2] HJORTH **[Hjo00b]** uses F_ξ instead of T_ξ.

Let $\mathsf{T}_1 = \Delta_{\mathbb{N}^{\mathbb{N}}}$, the equality relation on $\mathbb{N}^{\mathbb{N}}$. Put $\mathsf{T}_{\xi+1} = \mathsf{T}_\xi{}^+$ for all $\xi < \omega_1$. If $\lambda < \omega_1$ is a limit ordinal, then put $\mathsf{T}_\lambda = \bigvee_{\xi<\lambda} \mathsf{T}_\xi$. The definition for the second term T_2 coincides with the separate definition of T_2 in Section 4.1 by Exercise 4.2.2. □

4.3. Orbit equivalence relations of group actions

An *action* of a group $\mathbb{G}$ on a space $\mathbb{X}$ is a map $\mathfrak{a} : \mathbb{G} \times \mathbb{X} \to \mathbb{X}$, usually written as $\mathfrak{a}(g,x) = g{\cdot}x$, such that 1) $e{\cdot}x = x$, and 2) $g{\cdot}(h{\cdot}x) = (gh){\cdot}x$. Then, for every $g \in \mathbb{G}$, the map $x \mapsto g{\cdot}x$ is a bijection of $\mathbb{X}$ onto $\mathbb{X}$ with $x \mapsto g^{-1}{\cdot}x$ being the inverse map. A $\mathbb{G}$-*space* is a pair $\langle \mathbb{X}\,; \mathfrak{a}\rangle$, where $\mathfrak{a}$ is an action of $\mathbb{G}$ on $\mathbb{X}$. In this case, $\mathbb{X}$ itself is also called a $\mathbb{G}$-space, and the *orbit equivalence relation*, or *equivalence relation induced by the action*, $\mathsf{E}^{\mathbb{X}}_{\mathfrak{a}} = \mathsf{E}^{\mathbb{X}}_{\mathbb{G}}$ is defined on $\mathbb{X}$ so that $x \,\mathsf{E}^{\mathbb{X}}_{\mathbb{G}}\, y$ iff there is $g \in \mathbb{G}$ with $y = g{\cdot}x$. $\mathsf{E}^{\mathbb{X}}_{\mathbb{G}}$-classes are the same as $\mathbb{G}$-orbits, that is,

$$[x]_{\mathbb{G}} = [x]_{\mathsf{E}^{\mathbb{X}}_{\mathbb{G}}} = \{y : \exists\, g \in \mathbb{G}\ (g{\cdot}x = y)\}\,.$$

DEFINITION 4.3.1. A group is *Polish*, resp., *Borel* iff its underlying set is a Polish space, resp., a Borel set in a Polish space, and the operations are continuous, resp., Borel maps. A Borel group is *Polishable* if there is a Polish topology on the underlying set which 1) produces the same Borel sets as the original topology and 2) makes the group Polish.[3] □

If both $\mathbb{X}$ and $\mathbb{G}$ are Polish and the action continuous, then $\langle \mathbb{X}\,; \mathfrak{a}\rangle$ (and also $\mathbb{X}$) is called a *Polish* $\mathbb{G}$-space. If both $\mathbb{X}$ and $\mathbb{G}$ are Borel and the action is a Borel map, then $\langle \mathbb{X}\,; \mathfrak{a}\rangle$ (and also $\mathbb{X}$) is called a *Borel* $\mathbb{G}$-space.

PROPOSITION 4.3.2. *If $\mathbb{G}$ is a Borel group and $\langle \mathbb{X}\,; a\rangle$ is a Borel $\mathbb{G}$-space, then the induced equivalence relation $\mathsf{E} = \mathsf{E}^{\mathbb{X}}_{\mathbb{G}}$ is a $\mathbf{\Sigma}^1_1$ relation.*

PROOF. According to Theorem 1.2.2, we can assume that both $\mathbb{G}$ (with its operations) and $\mathbb{X}$ are Borel sets in product spaces. This will allow us to apply Proposition 1.9.1. By definition $x \,\mathsf{E}\, y$ is equivalent to $\exists\, a\,(a \in \mathbb{G} \wedge y = a \cdot x)$. Both relations in brackets are Borel since the group and the action are Borel; hence, $\mathbf{\Delta}^1_1$ by Proposition 1.9.1(vi) and hence, $\mathbf{\Sigma}^1_1$. Adding the quantifier $\exists\, a$ preserves $\mathbf{\Sigma}^1_1$ by Proposition 1.9.1(iv). □

Are Polish actions any better? The next theorem (too special to be proved here) shows that the type of the group is more important than the class of the action: roughly, every Borel action of a Polish group $\mathbb{G}$ is a Polish action of $\mathbb{G}$.

THEOREM 4.3.3 ([**BK96**, 5.2.1]). *Suppose that $\mathbb{G}$ is a Polish group and $\langle \mathbb{X}\,; a\rangle$ is a Borel $\mathbb{G}$-space. Then $\mathbb{X}$ admits a Polish topology that 1) produces the same Borel sets as the original topology, and 2) makes the action Polish.* □

There are cases when $\mathsf{E}^{\mathbb{X}}_{\mathbb{G}}$ is even Borel, not merely $\mathbf{\Sigma}^1_1$:

EXERCISE 4.3.4. Prove that if $\mathbb{G}$ is a countable group and the action is Borel then $\mathsf{E}^{\mathbb{X}}_{\mathbb{G}}$ is Borel. Prove the same in the case when $\mathbb{G} = \mathscr{I} \subseteq \mathscr{P}(\mathbb{N})$ is a Borel ideal, considered to be a Δ-group acting on $\mathbb{X} = \mathscr{P}(\mathbb{N})$ by Δ, as in Example 4.4.1). *Hint.* $\mathsf{E}^{\mathscr{P}(\mathbb{N})}_{\mathbb{G}} = \mathsf{E}_{\mathscr{I}}$ is Borel since $x \,\mathsf{E}^{\mathscr{P}(\mathbb{N})}_{\mathbb{G}}\, y$ is equivalent to $x \,\Delta\, y \in \mathscr{I}$. □

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[3] It is known [**Hjo00b**, 8.3, 8.4] that there exist non-polishable Borel groups and even Borel groups that cannot be realized as Borel subgroups of Polish groups.

Several much less trivial cases when $\mathsf{E}^{\mathbb{X}}_{\mathbb{G}}$ is Borel are described in [**BK96**, Chapter 7]; for instance, if all $\mathsf{E}^{\mathbb{X}}_{\mathbb{G}}$-classes are Borel sets of bounded rank, then $\mathsf{E}^{\mathbb{X}}_{\mathbb{G}}$ is Borel [**BK96**, 7.1.1]. Yet rather surprisingly equivalence classes generated by Borel actions are always Borel.

THEOREM 4.3.5 (see [**Kec95**, 15.14]). *If* $\mathbb{G}$ *is a Polish group and* $\langle \mathbb{X} ; \mathfrak{a} \rangle$ *is a Borel* $\mathbb{G}$*-space, then every equivalence class of* $\mathsf{E}^{\mathbb{X}}_{\mathbb{G}}$ *is Borel.*

The first notable case of this theorem was established by SCOTT [**Sco64**] in the course of the proof that for every countable order type t (not necessarily well ordered) the set of all sets $x \subseteq \mathbb{Q}$ of order type t is Borel in $\mathscr{P}(\mathbb{Q})$.

PROOF. It can be assumed, by Theorem 4.3.3, that the action is continuous. Then for every $x \in \mathbb{X}$ the *stabilizer* $\mathbb{G}_x = \{g : g \cdot x = x\}$ is a closed subgroup of $\mathbb{G}$.[4] We can consider $\mathbb{G}_x$ to be continuously acting on $\mathbb{G}$ by $g \cdot h = gh$ for all $g, h \in \mathbb{G}$. Let F denote the induced orbit equivalence relation. Then every F-class $[g]_{\mathsf{F}} = g\,\mathbb{G}_x$ is a shift of $\mathbb{G}_x$, hence, $[g]_{\mathsf{F}}$ is closed. On the other hand, the saturation $[\mathscr{O}]_{\mathsf{F}}$ of every open set $\mathscr{O} \subseteq \mathbb{G}$ is obviously open. Therefore (see Lemma 7.2.1(iv) below), F admits a Borel *transversal* $S \subseteq \mathbb{G}$; that is, S has exactly one element in common with each F-class. Yet $g \longmapsto g \cdot x$ is a Borel 1-to-1 map of S, a Borel set, onto $[x]_{\mathsf{E}}$. We conclude that $[x]_{\mathsf{E}}$ is a Borel set by Theorem 2.12.1. □

It follows that *not* all $\mathbf{\Sigma}^1_1$ equivalence relations are induced by Borel actions of Polish groups. Indeed, take a non-Borel $\mathbf{\Sigma}^1_1$ set $X \subseteq \mathbb{N}^{\mathbb{N}}$; define $x \mathrel{\mathsf{E}} y$ if either $x = y$ or $x, y \in X$—this is a $\mathbf{\Sigma}^1_1$ equivalence relation with a non-Borel class X. On the other hand, it is known that every $\mathbf{\Sigma}^1_1$ equivalence relation is induced by a Borel action of a Borel group; see [**BK96**, 6.2] or [**Hjo00b**, 8.5].

4.4. Some examples of orbit equivalence relations

EXAMPLE 4.4.1 ($\triangle$-action of ideals). Every ideal $\mathscr{I} \subseteq \mathscr{P}(\mathbb{N})$ is a group with the symmetric difference $\triangle$ as the group operation and the empty set $\varnothing$ as the neutral element. If $\mathscr{I}$ is a polishable ideal (see Section 3.4), then $\langle \mathscr{I} ; \triangle \rangle$ is a Polish group in an appropriate Polish topology compatible with the Borel structure of $\mathscr{I}$. Given such a topology, the $\triangle$-action of $\mathscr{I}$ on $\mathscr{P}(\mathbb{N})$ is Polish too.

For instance, show that the $\triangle$-action of a summable ideal $\mathscr{I} = \mathscr{S}_{\{r_n\}}$ (see Section 3.6) on $\mathscr{P}(\mathbb{N})$ is continuous in the sense of the $d_{\{r_n\}}$-topology of $\mathscr{S}_{\{r_n\}}$ (as in the proof of Lemma 3.6.1) and the ordinary Polish product topology on $\mathscr{P}(\mathbb{N})$. Suppose that $g \in \mathscr{S}_{\{r_n\}}$, $x \in \mathscr{P}(\mathbb{N})$, and fix a Polish nbhd $V = \{y \in \mathscr{P}(\mathbb{N}) : y \cap n = (g \cdot x) \cap n\}$ of $g \cdot x$ in $\mathscr{P}(\mathbb{N})$, where $n \in \mathbb{N}$. Consider the corresponding nbhd $U = \{x' \in \mathscr{P}(\mathbb{N}) : x' \cap n = x \cap n\}$ of x. Let $\varepsilon = \mathtt{min}\{r_k : k < n\}$. Then every element $g' \in \mathscr{S}_{\{r_n\}}$ of the ε-nbhd of g in the $d_{\{r_n\}}$-topology satisfies $g \triangle g' \subseteq [n, \infty)$; therefore, $g' \triangle x' \in V$ for all $x' \in U$. □

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[4] KECHRIS [**Kec95**, 9.17] gives an independent proof. Both $\mathbb{G}_x$ and its topological closure, say, G', are subgroups. Moreover, G' is a closed subgroup; hence, we can assume that $G' = \mathbb{G}$. In other words, $\mathbb{G}_x$ is dense in $\mathbb{G}$, and the goal is to prove that $\mathbb{G}_x = \mathbb{G}$. By a simple argument, $\mathbb{G}_x$ is either comeager or meager in $\mathbb{G}$. But a comeager subgroup easily coincides with the whole group; hence, assume that $\mathbb{G}_x$ is meager (and dense) in $\mathbb{G}$ and draw a contradiction.

Let $\{V_n\}_{n \in \mathbb{N}}$ be a basis of the topology of $\mathbb{X}$, and $A_n = \{g \in \mathbb{G} : g \cdot x \in V_n\}$. Easily $A_n h = A_n$ for all $h \in \mathbb{G}_x$. It follows, because $\mathbb{G}_x$ is dense, that every A_n is either meager or comeager. Now, if $g \in \mathbb{G}$, then $\{g\} = \bigcap_{n \in N(g)} A_n$, where $N(g) = \{n : g \cdot x \in V_n\}$; thus, at least one of sets A_n containing g is meager. It follows that $\mathbb{G}$ is meager, a contradiction.

EXAMPLE 4.4.2 (E_0 as an orbit equivalence relation). Define an action of $\mathbb{G} = \mathscr{P}_{\tt fin}(\mathbb{N})$, a countable subgroup of $\langle \mathscr{P}(\mathbb{N})\,;\triangle\rangle$, on $2^{\mathbb{N}}$ as follows: $(w{\cdot}x)(n) = x(n)$ if $n \notin w$, and $(w{\cdot}x)(n) = 1 - x(n)$ otherwise. The orbit equivalence relation $\mathsf{E}^{\mathbb{X}}_{\mathbb{G}}$ of this action is E_0. This action is Polish (given $\mathbb{G} = \mathscr{P}_{\tt fin}(\mathbb{N})$ the discrete topology) and *free*: $x = w{\cdot}x$ implies $w = \varnothing$ (the neutral element) for any $x \in 2^{\mathbb{N}}$. □

Remarkably, E_0 also can be induced by a Borel action of $\mathbb{Z}$; see Remark 8.1.3.

LEMMA 4.4.3 (0-1 law for category). (i) *If $X \subseteq 2^{\mathbb{N}}$ is a Borel E_0-invariant set* (so that $[X]_{\mathsf{E}_0} = X$), *then X is either meager or comeager.*

(ii) *Every Borel pairwise E_0-inequivalent set $T \subseteq 2^{\mathbb{N}}$ is meager.*

(iii) E_0 *does not have a Borel transversal* (= a pairwise E_0-inequivalent set T satisfying $[T]_{\mathsf{E}_0} = 2^{\mathbb{N}}$).

PROOF. (i) If X is not meager, then it is comeager on a basic clopen set $\mathscr{O}_s(2^{\mathbb{N}}) = \{x \in 2^{\mathbb{N}} : s \subset x\}$, where $s \in 2^{<\omega}$. Consider any other $t \in 2^{<\omega}$ with ${\tt lh}\, t = n$, where $n = {\tt lh}\, s$. Put $w = \{i < n : s(i) \neq t(i)\}$. Then $w \in \mathbb{G}$, and the action of w maps $X \cap \mathscr{O}_s(2^{\mathbb{N}})$ onto $X \cap \mathscr{O}_t(2^{\mathbb{N}})$. Thus X is comeager on $\mathscr{O}_t(2^{\mathbb{N}})$ as well, and hence comeager in $2^{\mathbb{N}}$ in general.

(ii) Note that $X = [T]_{\mathsf{E}_0} = \bigcup_{w \in \mathbb{G}} w{\cdot}T$ is a Borel set: indeed, if $w \in 2^{<\omega}$, then $w{\cdot}T$ is Borel by Theorem 2.12.1 since the map $x \mapsto w \cdot x$ is Borel and 1-to-1. And X is E_0-invariant. Therefore, X is either meager or comeager. If X is meager, then there is nothing to prove. If X is comeager, then T is not meager, hence T is comeager on a set $\mathscr{O}_s(2^{\mathbb{N}})$, $s \in 2^{<\omega}$. Put $w = \{n\}$, where $n = {\tt lh}\, s$. Then $w \in \mathbb{G}$ and the action $x \mapsto w \cdot x$ is a homeomorphism of $\mathscr{O}_s(2^{\mathbb{N}})$. Moreover, T and $w \cdot T$ are disjoint sets by the choice of T, both comeager on $\mathscr{O}_s(2^{\mathbb{N}})$, a contradiction.

(iii) If T is a Borel transversal, then $X = [T]_{\mathsf{E}_0} = 2^{\mathbb{N}}$, but on the other hand X must be meager in $2^{\mathbb{N}}$; see the proof of (ii). This is a contradiction. □

EXAMPLE 4.4.4 (The shift action). The *canonical* (or *shift*) *action* of a group $\mathbb{G}$ on a set of the form $B^{\mathbb{G}}$ (B being any set) is defined so that if $g \in \mathbb{G}$ and $x \in B^{\mathbb{G}}$, then $g \cdot x \in B^{\mathbb{G}}$ and $(g \cdot x)(f) = x(g^{-1}f)$ for all $f \in \mathbb{G}$. This is clearly a Polish action provided $\mathbb{G}$ is countable, B is a Polish space (for instance, a finite or countable discrete set), and $B^{\mathbb{G}}$ given the product topology. The equivalence relation on $B^{\mathbb{G}}$ induced on the space $B^{\mathbb{G}}$ by this action is denoted by $\mathsf{E}(\mathbb{G}, B)$.

The *free domain* $(B)^{\mathbb{G}}$ of this action consists of all points $x \in B^{\mathbb{G}}$ such that

$$\forall g \in \mathbb{G}\,(g \neq 1 \Longrightarrow g \cdot x \neq x).$$

If $\mathbb{G}$ is at most countable, then $(B)^{\mathbb{G}}$ is a Borel set in $B^{\mathbb{G}}$, and $\mathbb{G}$-invariant so that if $x \in (B)^{\mathbb{G}}$ and $g \in \mathbb{G}$, then $g \cdot x \in (B)^{\mathbb{G}}$ either. Note that the action of $\mathbb{G}$ restricted to $(B)^{\mathbb{G}}$ is free. Put $\mathsf{Fr}(\mathbb{G}, B) = \mathsf{E}(\mathbb{G}, B) \restriction (B)^{\mathbb{G}}$, the *free part* of $\mathsf{E}(\mathbb{G}, B)$. □

EXAMPLE 4.4.5 (The free group). The free group of two generators F_2 consists of finite irreducible words composed of the symbols a, b, a^{-1}, b^{-1}, including the empty word (the neutral element 1). The group operation is the concatenation of words (followed by reduction, if necessary, *e.g.* $ab \cdot b^{-1}a = aa$).

The shift action of F_2 on the compact space 2^{F_2} is defined in accordance with the general scheme of Example 4.4.4 so that if $x \in 2^{F_2}$ and $w \in F_2$, then $(w{\cdot}x)(u) = x(w^{-1}u)$ for all $u \in F_2$. Put, for $x, y \in 2^{F_2}$, $x \mathrel{\mathsf{E}_\infty} y$ iff $x = w{\cdot}y$ for some $w \in F_2$. Thus E_∞ is $\mathsf{E}(F_2, 2)$, in the sense of Example 4.4.4. In addition, let $\mathsf{E}_{\infty T} = \mathsf{Fr}(F_2, 2) = \mathsf{E}(F_2, 2) \restriction (2)^{F_2}$, the free part of E_∞. □

EXAMPLE 4.4.6 (Vitali equivalence). The additive group of rationals $\mathbb{Q}$ acts on $\mathbb{R}$ by addition: $q \cdot x = q + x$ (where $q \in \mathbb{Q}$ and x is a real). The induced equivalence relation is the *Vitali equivalence relation* Vit, also denoted by $\mathsf{E}(\mathbb{R}/\mathbb{Q})$. Thus $x \,\mathsf{Vit}\, y$ iff $x - y$ is rational. □

EXAMPLE 4.4.7. Come back to Banach spaces $\ell^\infty, \ell^p, \mathbf{c}, \mathbf{c}_0$ discussed in Section 4.1. Each of them can be considered to be a Polish group in the sense of component-wise addition in $\mathbb{R}^{\mathbb{N}}$. Each of them canonically acts on $\mathbb{R}^{\mathbb{N}}$ also by component-wise addition. For the sake of brevity, the orbit equivalence relations of these actions, *i.e.* $\mathsf{E}_{\ell^\infty}^{\mathbb{R}^{\mathbb{N}}}$, $\mathsf{E}_{\ell^p}^{\mathbb{R}^{\mathbb{N}}}$, $\mathsf{E}_{\mathbf{c}}^{\mathbb{R}^{\mathbb{N}}}$, $\mathsf{E}_{\mathbf{c}_0}^{\mathbb{R}^{\mathbb{N}}}$, are denoted by the same symbols, resp. $\ell^\infty, \ell^p, \mathbf{c}, \mathbf{c}_0$. □

EXAMPLE 4.4.8 (The group of permutations). The group S_∞ of all permutations of $\mathbb{N}$ (that is, all bijections $f : \mathbb{N} \xrightarrow{\text{onto}} \mathbb{N}$, with the superposition as the group operation) is a Polish group in the Polish product topology of $\mathbb{N}^{\mathbb{N}}$. It acts on any set of the form $X^{\mathbb{N}}$ as follows: for every $g \in S_\infty$ and $x \in X^{\mathbb{N}}$, $(g \cdot x)(k) = x(g^{-1}(k))$ for all k, or equivalently $(g \cdot x)(g(k)) = x(k)$ for all k. Formally, $g \cdot x = x \circ g^{-1}$, where $\circ$ is the superposition in the right-hand side.

Take $X = \mathbb{N}^{\mathbb{N}}$. Note that $(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$ with the product topology is a Polish space and the above action is Polish. Its orbit equivalence relation $\mathsf{E}_{S_\infty}^{(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}}$ is quite similar to T_2, but in fact not equal. Indeed if $x, y \in (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$ satisfy $x(0) = x(1) = y(0) = u$ and $x(k) = u(l) = v$ for all $k \geq 2$, $l \geq 1$, where $u \neq v \in \mathbb{N}^{\mathbb{N}}$, then $x \,\mathsf{T}_2\, y$ holds while $x \,\mathsf{E}_{S_\infty}^{(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}}\, y$ fails. Still Lemma 5.1.3 will prove that T_2 and $\mathsf{E}_{S_\infty}^{(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}}$ are equivalent in a certain well-defined sense. □

4.5. Probability measures

Studies of Borel equivalence relations, especially *countable* ones, reveal deep connections of this field with some branches of mathematics, generally speaking, outside of descriptive set theory, and, in particular, with ergodic theory.

Recall that a *Borel measure* on a Borel set X (as usual, in a Polish space) is any σ-additive function μ defined on the σ-algebra $\mathbf{Bor}(X)$ of all Borel sets $X' \subseteq X$, and with values in $[0, +\infty]$. In this case, a set $A \subseteq X$ (not necessarily Borel) is μ-*measurable* if there are Borel sets U, D such that $\mu(D) = 0$ and $A \,\Delta\, U \subseteq D$. In this case we can assign to A the measure value equal to $\mu(U)$, of course.

A Borel measure μ is:

σ-finite: iff $X = \bigcup_n X_n$, where all sets $X_n \subseteq X$ are Borel and $\mu(X_n) < +\infty$;

probability measure: iff $\mu(X) = 1$.

Note that probability measures are by definition σ-additive.

DEFINITION 4.5.1. $\boldsymbol{P}(X)$ is the set of all probability measures on X. □

EXAMPLE 4.5.2. Even finite sets carry continuum-many probability measures. Basically, a probability measure on $X = \{x_1, \ldots, x_n\}$ is the same as a partition $1 = p_1 + \cdots + p_n$ into n non-negative reals. Thus, for any $0 \leq p \leq 1$, there is a probability measure on the two-element set $2 = \{0, 1\}$ that assigns p to $\{0\}$ and $1 - p$ to $\{1\}$: it will be called *the $(p, 1-p)$-measure*. In particular, for $p = \frac{1}{2}$, *the $(\frac{1}{2}, \frac{1}{2})$-measure* assigns $\frac{1}{2}$ to both $\{0\}$ and $\{1\}$. □

EXAMPLE 4.5.3 (Product measures). Suppose that I is a countable set and $0 \le p \le 1$. Let λ_p denote the product measure of I-many copies of the $(p, 1-p)$-measure on $2 = \{0, 1\}$. In other words, λ_p is the unique Borel probability measure on 2^I such that for any pair of disjoint finite sets $u, v \subseteq I$, the set

$$Z_{uv} = \{a \in 2^I : \forall i \in u\,(a(i) = 0) \wedge \forall i \in v\,(a(i) = 1))\}$$

satisfies $\lambda_p(Z_{uv}) = p^n(1-p)^k$, where $n = \mathtt{card}\, u$ and $k = \mathtt{card}\, v$. In particular, for $p = \frac{1}{2}$, $\lambda_{1/2}$ is the only Borel probability measure on 2^I such that for any pair of disjoint finite sets $u, v \subseteq I$, the set Z_{uv} satisfies $\lambda_{1/2}(Z_{uv}) = p^m$, where $m = \mathtt{card}\, u + \mathtt{card}\, v$. □

EXERCISE 4.5.4. Suppose that $\mathbb{G}$ is a countable group, for instance, $\mathbb{Z}$ or F_2, acting on $2^{\mathbb{G}}$ by shift (see Example 4.4.4). Let $X = (2)^{\mathbb{G}}$ be the free domain: it consists of all points $x \in 2^{\mathbb{G}}$ satisfying $\forall g \in \mathbb{G} \smallsetminus \{1\}\,(g \cdot x \ne x)$, where 1 is the neutral element of $\mathbb{G}$. Then X is an invariant Borel set (that is, $g \cdot X = X$ for all $g \in \mathbb{G}$), and the shift action of $\mathbb{G}$ on X is free. Prove that $\lambda_{1/2}(X) = 1$. □

4.6. Invariant and ergodic measures

These are two important categories of measures. Suppose that E is a Borel equivalence relation on a Borel set X. A probability measure $\mu \in \boldsymbol{P}(X)$ is

E-*ergodic*: iff, for any Borel E-invariant set $A \subseteq X$, $\mu(A) = 0$ or $\mu(A) = 1$;

E-*non-atomic*: iff $\mu([x]_{\mathsf{E}}) = 0$ for all $x \in X$.

Note that if E is a countable equivalence relation, then for μ to be non-atomic it is sufficient (and also necessary) that $\mu(\{x\}) = 0$ for all points x.

DEFINITION 4.6.1. If E is an equivalence relation on a set X, then $[[\mathsf{E}]]$ is the set of all E-preserving partial Borel bijections $f : X \to X$. Thus $f \in [[\mathsf{E}]]$ iff there exist Borel sets $A, B \subseteq X$ such that f is a Borel bijection of A onto B and $x \mathrel{\mathsf{E}} f(x)$ holds for all $x \in A$. □

A probability measure $\mu \in \boldsymbol{P}(X)$ is

E-*invariant*: iff $\mu(A) = \mu(B)$ whenever there exists $f \in [[\mathsf{E}]]$, $f : A \xrightarrow{\text{onto}} B$.

DEFINITION 4.6.2. $\mathbf{EINV}_{\mathsf{E}}$ is the set of all E-invariant E-ergodic Borel probability measures (on the Borel set $X = \mathtt{dom}\, \mathsf{E}$). □

The notion of an E-invariant measure becomes more transparent in the case when E is induced by a Borel action of a Polish group $\mathbb{G}$. A measure μ is called $\mathbb{G}$-*invariant* iff simply $\mu(A) = \mu(g \cdot A)$ for all $g \in \mathbb{G}$ and Borel $A \subseteq X$.

EXERCISE 4.6.3. In this case, prove that if $\mathbb{G}$ *is countable* and μ is a Borel probability measure, then μ is $\mathbb{G}$-invariant iff it is E-invariant. *Hint.* Any map $f \in [[\mathsf{E}]]$ consists of countably many Borel pieces of maps of the form $x \mapsto g \cdot x$, $g \in \mathbb{G}$. Use Theorem 2.12.1 to show that the domains of those partial maps are Borel. □

EXAMPLE 4.6.4 (E_0-invariant ergodic measures). Recall that E_0 is defined on $2^{\mathbb{N}}$ so that $a \mathrel{\mathsf{E}_0} b$ iff there is $n \in \mathbb{N}$ such that $a \restriction n = b \restriction n$. There is a unique probability measure $\lambda = \lambda_{1/2}$ on $2^{\mathbb{N}}$ satisfying $\lambda(\mathscr{O}_s) = 2^{-\operatorname{lh} s}$ for every sequence $s \in 2^{<\omega}$, where $\mathscr{O}_s = \{a \in 2^{\mathbb{N}} : s \subset a\}$, a basic clopen set in $2^{\mathbb{N}}$.

Recall that E_0 is induced by the action $u, x \mapsto u \cdot x$ of the Δ-group $\mathscr{P}_{\mathtt{fin}}(\mathbb{N})$ defined in Example 4.4.2. The measure λ is obviously invariant with respect to this action, and hence E_0-invariant by Exercise 4.6.3. Prove that λ is E_0-ergodic.

Consider an E_0-invariant Borel set $X \subseteq 2^{\mathbb{N}}$, and show that $\lambda(X) = 0$ or $= 1$ (the 0-1 law, 17.1 in **[Kec95]**). Suppose toward the contrary that $0 < \lambda(X) < 1$ strictly. By the Lebesgue density theorem (17.9 in **[Kec95]**) there exist points $a \in X$ and $b \in Y = 2^{\mathbb{N}} \setminus X$ such that

$$\lim_{n\to\infty} \frac{\lambda(\mathscr{O}_{a\restriction n} \cap X)}{2^{-n}} = \lim_{n\to\infty} \frac{\lambda(\mathscr{O}_{b\restriction n} \cap Y)}{2^{-n}} = 1,$$

where $2^{-n} = \lambda(\mathscr{O}_{b\restriction n}) = \lambda(\mathscr{O}_{a\restriction n})$. Therefore, there is a number n such that both $\lambda(\mathscr{O}_{a\restriction n} \cap X)$ and $\lambda(\mathscr{O}_{b\restriction n} \cap Y) = \lambda(\mathscr{O}_{b\restriction n} \setminus X)$ are strictly bigger than 2^{-n-1}. Note that $u = \{i < n : a(i) \neq b(i)\} \in \mathscr{P}(\mathbb{N})$ and $u \cdot \mathscr{O}_{a\restriction n} = \mathscr{O}_{b\restriction n}$; therefore, by the invariance of X, $u \cdot (\mathscr{O}_{a\restriction n} \cap X) = \mathscr{O}_{b\restriction n} \cap X$, and $\lambda(\mathscr{O}_{b\restriction n} \cap X) = \lambda(\mathscr{O}_{a\restriction n} \cap X) > 2^{-n-1}$. But $\lambda(\mathscr{O}_{b\restriction n} \setminus X) > 2^{-n-1}$ as well, a contradiction. See 3.2 in **[KM04]** for a more general result. □

Recall that the Baire category of $2^{\mathbb{N}}$ is E_0-ergodic in the same sense: E_0-invariant Borel sets are either meager or comeager by Lemma 4.4.3.

EXERCISE 4.6.5 (Uniqueness). Prove that $\lambda = \lambda_{1/2}$ is the only E_0-invariant E_0-ergodic Borel probability measure on $2^{\mathbb{N}}$. *Hint.* Suppose that $\mu \in \mathbf{EINV}_{\mathsf{E}_0}$ is a Borel probability measure on $2^{\mathbb{N}}$. Then $\mu(\mathscr{O}_s) = \mu(\mathscr{O}_t)$ for any $s, t \in 2^{<\omega}$ of the same length $\mathtt{lh}\, s = \mathtt{lh}\, t = n$, because there is $u \in \mathscr{P}_{\mathtt{fin}}(\mathbb{N})$ such that $u \cdot \mathscr{O}_s = \mathscr{O}_t$. Therefore, $\mu(\mathscr{O}_s) = 2^{-\mathtt{lh}\, s} = \lambda(\mathscr{O}_s)$, $\forall\, s$, and it easily follows that $\mu = \lambda$. □

EXAMPLE 4.6.6 (Shift invariant measures). Recall that $\mathsf{E}(\mathbb{Z}, 2)$ is the shift equivalence relation defined on $2^{\mathbb{Z}}$ so that $x \;\mathsf{E}(\mathbb{Z}, 2)\; y$ iff there is $j \in \mathbb{Z}$ such that $y = j \cdot x$, that is, $y(k) = x(k - j)$, $\forall k \in \mathbb{Z}$. But in this case there exist continuum-many invariant ergodic measures for $\mathsf{E}(\mathbb{Z}, 2)$! Indeed, fix an arbitrary real p, $0 < p < 1$. Let λ_p be the probability measure on $2^{\mathbb{Z}}$ equal to the product of $\mathbb{Z}$-many copies of the $(p, 1-p)$-measure on $\{0, 1\}$. Thus, if $k_1 < \cdots < k_n$ are integers in $\mathbb{Z}$ and $i_1, \ldots, i_k = 0, 1$, then the set

$$X = \{x \in 2^{\mathbb{Z}} : x(k_1) = i_1 \wedge \cdots \wedge x(k_n) = i_n\}$$

satisfies $\lambda_p(X) = p^m (1-p)^{n-m}$, where m is the number of all indices ℓ, $1 \le \ell \le n$, with $i_\ell = 1$. The measure λ_p is shift-invariant, and hence $\mathsf{E}(\mathbb{Z}, 2)$-invariant. It takes some effort to prove that λ_p also is $\mathsf{E}(\mathbb{Z}, 2)$-ergodic, therefore $\lambda_p \in \mathbf{EINV}_{\mathsf{E}(\mathbb{Z},2)}$ for every real $0 < p < 1$; see, *e.g.*, 3.1 in **[KM04]**. □

We continue with yet another example of invariant measures.

THEOREM 4.6.7 (Haar measure). *If $\mathbb{G}$ is a Polish locally compact group, then there is a unique (up to a multiplicative constant) σ-finite Borel measure $\mu_{\mathbb{G}}$ on $\mathbb{G}$ such that:*

(i) $\mu_{\mathbb{G}}(K) < +\infty$ *for every compact* $K \subseteq \mathbb{G}$;

(ii) $\mu_{\mathbb{G}}(U) > 0$ *for every open* $\varnothing \neq U \subseteq \mathbb{G}$;

(iii) $\mu_{\mathbb{G}}(A) = \mu_{\mathbb{G}}(gA)$ *for every Borel* $A \subseteq \mathbb{G}$ *and every* $g \in \mathbb{G}$.

This measure is called the (left) Haar measure. □

Examples: the ordinary Lebesgue measure on $\mathbb{R}$ is the Haar measure for the additive group of the reals. The probability measure λ, as in Example 4.6.4, is the Haar measure for $2^{\mathbb{N}}$ identified with the compact group $(\mathbb{Z}_2)^{\mathbb{N}}$.

Note that (iii) means that $\mu_{\mathbb{G}}$ is invariant *w.r.t.* the left shift action of $\mathbb{G}$ on itself. The definition of a right Haar measure is similar, and the two are generally distinct. (But they coincide if $\mathbb{G}$ is abelian or compact.) If $\mathbb{G}$ is a compact group, then by (i) $\mu_{\mathbb{G}}(\mathbb{G})$ is finite positive, and by normalizing we make it a probability measure. See **[Hal74]** on Haar measures.

Yet in some cases invariant ergodic measures do not exist.

DEFINITION 4.6.8. A Borel equivalence relation E on X is *compressible* if there is a Borel map $f \in [[\mathsf{E}]]$, $f : X \to X$, such that the complement $Y = X \smallsetminus \operatorname{ran} f$ of the full image $\operatorname{ran} f = f[X]$ is a complete section for E, that is, $[Y]_{\mathsf{E}} = X$. □

EXAMPLE 4.6.9. The *tail equivalence relation* $\mathsf{E_t}$, defined on $2^{\mathbb{N}}$ by

$$a \,\mathsf{E_t}\, b \quad \text{iff} \quad \exists m\, \exists n\, \forall k\, (a(m+k) = b(n+k))$$

for $a, b \in 2^{\mathbb{N}}$, is a standard example of a compressible equivalence relation. To get a "compressing" map f choose any k and define, for $a \in 2^{\mathbb{N}}$, $f_k(a) = k^\frown a$, that is, $f_k(a)(0) = k$ and $f_k(a)(n+1) = a(n)$, $\forall n$. See § 2 in **[DJK94]** for more on compressible equivalence relations. □

THEOREM 4.6.10. *Let* E *be a countable Borel equivalence relation. Then* E *admits an invariant probability measure iff it is not compressible.* □

We are not going to prove here this theorem of NADKARNI, see 5.1 in **[DJK94]**. Yet the following exercise contains an element of the "iff" part.

EXERCISE 4.6.11. Prove that the tail equivalence relation $\mathsf{E_t}$ does not admit an invariant probability measure. Use the fact that the "compressing" maps f_k, defined as above, belong to $[[\mathsf{E_t}]]$ and have disjoint full images. □

EXERCISE 4.6.12. Prove that the Vitali equivalence relation Vit (see Example 4.4.6) does not admit an invariant probability measure. Use the fact that the only invariant Borel measure for Vit is the Lebesgue measure on $\mathbb{R}$, and it is not a probability measure. □

We proceed with a counterexample that shows that certain combinations of properties which cannot be achieved for Borel probability measures even in a rather elementary case. An equivalence relation E on a Borel set X is *smooth* if there is a Borel map $\vartheta : X \to 2^{\mathbb{N}}$ such that $x \,\mathsf{E}\, y \iff \vartheta(x) = \vartheta(y)$.

LEMMA 4.6.13. *If* E *is a smooth equivalence relation on a Borel set* X, *then there is no* E*-non-atomic* E*-ergodic Borel probability measure, and for an* E*-invariant* E*-ergodic Borel probability measure to exist, it is necessary and sufficient that* E *has at least one finite equivalence class.*

PROOF. Let a Borel map $\vartheta : X \to 2^{\mathbb{N}}$ witness the smoothness, so that $x \,\mathsf{E}\, y \iff \vartheta(x) = \vartheta(y)$ for $x, y \in X$. Let μ be a probability measure on X. Then $\nu(A) = \mu(\vartheta^{-1}[A])$ ($A \subseteq 2^{\mathbb{N}}$) is a probability measure on $2^{\mathbb{N}}$. If there is a Borel set $A \subseteq 2^{\mathbb{N}}$ with $0 < \nu(A) < 1$, then the preimage $\vartheta^{-1}[A]$ witnesses that μ is non-ergodic. If such a set A does not exist, then there is $a \in 2^{\mathbb{N}}$ such that $\vartheta(x) = a$ for μ-almost all x. In other words, if $x \in X$ is such that $\vartheta(x) = a$, then $\mu([x]_{\mathsf{E}}) = 1$, so μ is not non-atomic. In this case in order to be E-invariant, μ has to be invariant *w.r.t.* any Borel bijection $f : [x]_{\mathsf{E}} \xrightarrow{\text{onto}} [x]_{\mathsf{E}}$, which is possible only in the case when $[x]_{\mathsf{E}}$ is finite and μ is uniformly distributed on $[x]_{\mathsf{E}}$. □

CHAPTER 5

Borel reducibility of equivalence relations

There are several reasonable ways to compare equivalence relations in terms of existence of a *reduction*, that is, a map of certain kind which allows us to derive one of the equivalence relations from the other one. The Borel reducibility $\leq_{\mathrm{B}}$ is the key one. The plan of this chapter is to define $\leq_{\mathrm{B}}$ and to present a diagram that displays mutual $\leq_{\mathrm{B}}$-reducibility of the equivalence relations introduced in Section 4.1 (the key equivalence relations). The proof of related reducibility/irreducibility claims will be the main content of the remainder of the book.

5.1. Borel reducibility

Suppose that E and F are equivalence relations on Borel sets X, Y in some Polish spaces. We define the following:

$\mathsf{E} \leq_{\mathrm{B}} \mathsf{F}$ (*Borel reducibility* of E to F) iff there is a Borel map $\vartheta : X \to Y$ (called *reduction*) such that $x \mathrel{\mathsf{E}} y \iff \vartheta(x) \mathrel{\mathsf{F}} \vartheta(y)$ for all $x, y \in X$;

$\mathsf{E} \sim_{\mathrm{B}} \mathsf{F}$ iff $\mathsf{E} \leq_{\mathrm{B}} \mathsf{F}$ and $\mathsf{F} \leq_{\mathrm{B}} \mathsf{E}$ (*Borel bi-reducibility*, or *Borel equivalence*);

$\mathsf{E} <_{\mathrm{B}} \mathsf{F}$ iff $\mathsf{E} \leq_{\mathrm{B}} \mathsf{F}$ but not $\mathsf{F} \leq_{\mathrm{B}} \mathsf{E}$ (*strict Borel reducibility*).

If $\mathsf{E} \leq_{\mathrm{B}} \mathsf{F}$ (resp. $\mathsf{E} <_{\mathrm{B}} \mathsf{F}$, $\mathsf{E} \sim_{\mathrm{B}} \mathsf{F}$), then E is said to be *Borel reducible* (resp. *Borel strictly reducible*, *Borel equivalent* or *bi-reducible*) to F. Sometimes $\mathbb{X}/\mathsf{E} \leq_{\mathrm{B}} \mathbb{Y}/\mathsf{F}$ is used instead of $\mathsf{E} \leq_{\mathrm{B}} \mathsf{F}$.

REMARK 5.1.1. We shall occasionally consider analytic non-Borel equivalence relations, for instance, those of the form $\mathsf{E} \restriction X$, where X is a non-Borel Σ^1_1 subset of the domain of a Borel equivalence relation E. If E, F are Σ^1_1 equivalence relations on not necessarily Borel domains resp. X, Y, then $\mathsf{E} \leq_{\mathrm{B}} \mathsf{F}$ will be understood as the existence of a Borel map ϑ satisfying $X \subseteq \operatorname{dom} \vartheta$ and still $x \mathrel{\mathsf{E}} y \iff \vartheta(x) \mathrel{\mathsf{F}} \vartheta(y)$ for all $x, y \in X$. □

Borel isomorphism: $\mathsf{E} \cong_{\mathrm{B}} \mathsf{F}$ iff there is a Borel bijection $\vartheta : X \xrightarrow{\text{onto}} Y$ such that $x \mathrel{\mathsf{E}} x'$ iff $\vartheta(x) \mathrel{\mathsf{F}} \vartheta(x')$ for all $x, x' \in \mathbb{X}$.

Borel isomorphism $\mathsf{E} \cong_{\mathrm{B}} \mathsf{F}$ implies Borel bi-reducibility $\mathsf{E} \sim_{\mathrm{B}} \mathsf{F}$, of course, but not the other way around. A large family of pairwise $\sim_{\mathrm{B}}$-equivalent *hyperfinite* equivalence relations, considered in Chapter 8, contains infinitely many pairwise $\cong_{\mathrm{B}}$-inequivalent relations.

Borel reducibility of ideals: $\mathscr{I} \leq_{\mathrm{B}} \mathscr{J}$ iff $\mathsf{E}_{\mathscr{I}} \leq_{\mathrm{B}} \mathsf{E}_{\mathscr{J}}$. Thus it is required that there is a Borel map $\vartheta : \mathscr{P}(A) \to \mathscr{P}(B)$ such that $x \,\Delta\, y \in \mathscr{I}$ iff $\vartheta(x) \,\Delta\, \vartheta(y) \in \mathscr{J}$. (Here $\mathscr{I}, \mathscr{J}$ are ideals on countable sets A, B.)

In the domain of ideals, $\le_{\mathrm{B}}$ is weaker than all reducibilities of more special nature discussed in Section 3.2, in the sense that, for instance, each of $\mathscr{I} \le_{\mathrm{RB}} \mathscr{J}$ and $\mathscr{I} \le_{\mathrm{B}}^{\Delta} \mathscr{J}$ implies $\mathscr{I} \le_{\mathrm{B}} \mathscr{J}$. The only exception is the reducibility via inclusion $\le_{\mathrm{I}}$—it does not imply $\le_{\mathrm{B}}$. Indeed we have $\mathscr{S}_{\{1/n\}} \subseteq \mathscr{Z}_0$, but on the other hand the summable ideal $\mathscr{S}_{\{1/n\}}$ and the density-0 ideal $\mathscr{Z}_0$ are known to be $\le_{\mathrm{B}}$-incomparable; see below.

It would be interesting to figure out the exact relationship between $\le_{\mathrm{B}}$ and the Δ-reducibility $\le_{\mathrm{B}}^{\Delta}$. If the next question answers in the negative, then the whole theory of Borel reducibility for Borel ideals can be greatly simplified because reduction maps that are Δ-homomorphisms are much easier to deal with.

QUESTION 5.1.2. Is there a pair of Borel ideals $\mathscr{I}, \mathscr{J}$ such that $\mathscr{I} \le_{\mathrm{B}} \mathscr{J}$ but not $\mathscr{I} \le_{\mathrm{B}}^{\Delta} \mathscr{J}$? □

The remainder of the book will concentrate on the Borel reducibility/irreducibility theorems. The following rather elementary lemma gives some examples.

LEMMA 5.1.3. (i) $\Delta_{\mathbb{N}^{\mathbb{N}}} \sim_{\mathrm{B}} \Delta_{\mathbb{N}}^{+}$.

(ii) $\mathsf{T}_2 \sim_{\mathrm{B}} \mathsf{E}_{S_\infty}^{(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}}$ (see Example 4.4.8).

(iii) *If W satisfies $2 \le \operatorname{card} W \le \aleph_0$, then* $\mathsf{E}_0 \sim_{\mathrm{B}} \mathsf{E}_0(W)$ (see Definition 4.1.1).

PROOF. (i) By definition, $\Delta_{\mathbb{N}}^{+}$ is an equivalence relation on $\mathbb{N}^{\mathbb{N}}$, and $x \, \Delta_{\mathbb{N}}^{+} \, y$ holds iff $\operatorname{ran} x = \operatorname{ran} y$. Thus, the map $\vartheta(x) = \chi_{\operatorname{ran} x}$ (the characterictic function) witnesses that $\Delta_{\mathbb{N}}^{+} \le_{\mathrm{B}} \Delta_{\mathbb{N}^{\mathbb{N}}}$. To prove the converse put, for $x \in \mathbb{N}^{\mathbb{N}}$,

$$r(x) = \{x(0)\,,\ x(0)+x(1)+1\,,\ x(0)+x(1)+x(2)+2\,,\ \ldots\}\,;$$

then $\vartheta(x) = \chi_{r(x)}$ witnesses $\Delta_{\mathbb{N}^{\mathbb{N}}} \le_{\mathrm{B}} \Delta_{\mathbb{N}}^{+}$.

(ii) Suppose that $x, y \in (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$. Then $x \, \mathsf{T}_2 \, y$ means that

$$\forall k \, \exists l \, (x(k) = y(l)) \quad \text{and} \quad \forall l \, \exists k \, (x(k) = y(l)),$$

while $x \, \mathsf{E}_{S_\infty}^{(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}} \, y$ means that there is a bijection $f : \mathbb{N} \xrightarrow{\text{onto}} \mathbb{N}$ such that $x(k) = y(f(k))$ for all k. The latter condition is, generally speaking, stronger than the former one, but the two are equivalent provided that for every k there exist infinitely many indices l such that $x(k) = x(l)$ and the same for y. It follows that the map $\vartheta : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$, defined so that $\vartheta(x) = x'$ iff $x'(2^n(2k+1)-1) = x(k)$ for all n, k, is a Borel reduction of T_2 to $\mathsf{E}_{S_\infty}^{(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}}$.

A Borel reduction ϑ of $\mathsf{E}_{S_\infty}^{(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}}$ to T_2 can be defined as follows: $\vartheta(x) = x'$, where $x'(k) = n_x(k)^\frown x(k)$ for all k, $n_x(k)$ is the number of all l satisfying $x(l) = x(k)$ (including $l = k$) or 0 if there exist infinitely many of such l, and $n^\frown a$ for $a \in \mathbb{N}^{\mathbb{N}}$ is defined as the only element of $\mathbb{N}^{\mathbb{N}}$ such that $(n^\frown a)(0) = n$ and $(n^\frown a)(j+1) = a(j)$ for all j.

(iii) In the non-trivial direction, to prove, say, $\mathsf{E}_0(\mathbb{N}) \le_{\mathrm{B}} \mathsf{E}_0$, define $\tilde{x} \in 2^{\mathbb{N}}$ for every $x \in \mathbb{N}^{\mathbb{N}}$ so that $\tilde{x}(2^n(2k+1)-1) = 1$ whenever $x(n) = k$ — for all $n, k \in \mathbb{N}$. The map $x \mapsto \tilde{x}$ is a Borel reduction of $\mathsf{E}_0(\mathbb{N})$ to E_0. □

5.2. Injective Borel reducibility—embedding

A special type of Borel reductions consists of those via injective maps.

$\mathsf{E} \sqsubseteq_{\mathrm{B}} \mathsf{F}$ iff there is a Borel *embedding*, that is, a 1-to-1 reduction of E to F;

$\mathsf{E} \approx_{\mathrm{B}} \mathsf{F}$ iff $\mathsf{E} \sqsubseteq_{\mathrm{B}} \mathsf{F}$ and $\mathsf{F} \sqsubseteq_{\mathrm{B}} \mathsf{E}$ (a rare form, [**HKL98**, § 0]);

$\mathsf{E} \sqsubseteq^{\mathrm{i}}_{\mathrm{B}} \mathsf{F}$ iff there is a Borel *invariant* embedding, *i.e.*, an embedding ϑ such that $\operatorname{ran}\vartheta = \{\vartheta(x) : x \in \mathbb{X}\}$ is an F-*invariant* set (meaning that the F-*saturation* $[\operatorname{ran}\vartheta]_{\mathsf{F}} = \{y : \exists\, x\, (y \mathrel{\mathsf{F}} \vartheta(x))\}$ equals $\operatorname{ran}\vartheta$).

Thus, if E, F are equivalence relations on sets resp. X, Y, then a Borel embedding of E in F is a Borel injection (a 1-to-1 map) $\vartheta : X \to Y$ satisfying $x \mathrel{\mathsf{E}} y \iff \vartheta(x) \mathrel{\mathsf{F}} \vartheta(y)$ for all $x, y \in X$. Note that the set $Y' = \operatorname{ran}\vartheta$ is a Borel subset of Y, and ϑ is a Borel isomorphism between the relations E and $\mathsf{F} \restriction Y'$. It is not likely that $\le_{\mathrm{B}}$ implies $\sqsubseteq_{\mathrm{B}}$ in all cases, yet sometimes such a strengthening is possible.

PROPOSITION 5.2.1. *Suppose that* E *is one of the equivalence relations* E_1, E_2, E_3, *while* F *is a Borel equivalence relation, and* $\mathsf{F} \le_{\mathrm{B}} \mathsf{E}$. *Then* $\mathsf{F} \sqsubseteq_{\mathrm{B}} \mathsf{E}$.

PROOF. Consider E_1 to be an equivalence relation on $(2^{\mathbb{N}})^{\mathbb{N}}$ defined as $x \mathrel{\mathsf{E}_1} y$ iff $x(k) = y(k)$ for all but finite k. Let X be the domain of F. Suppose that $\vartheta : X \to (2^{\mathbb{N}})^{\mathbb{N}}$ is a Borel reduction of F to E_1. Define $\vartheta' : X \to (2^{\mathbb{N}})^{\mathbb{N}}$ so that $\vartheta'(x)(k+1) = \vartheta(x)(k)$ for all x and k, and $\vartheta'(x)(0) = f(x)$, where $f : X \to (2^{\mathbb{N}})^{\mathbb{N}}$ is an arbitrary Borel injection. Note that adding f does not change the property of being a reduction, but makes ϑ' an injection.

In the E_3 case, f has to be adjoined in different way: $\vartheta'(x)(k)(n+1) = \vartheta(x)(k)(n)$, but $\vartheta'(x)(k)(0) = f(k)$ for all x, k, n.

It remains to consider E_2. Let X be the domain of F. Suppose that $\vartheta : X \to \mathscr{P}(\mathbb{N})$ is a Borel reduction of F to E_2, so that $x \mathrel{\mathsf{F}} y$ iff $\sum_{k \in \vartheta(x) \Delta \vartheta(y),\, k \ge 1} \frac{1}{k} < +\infty$. We define $\vartheta'(x) = \{2n : n \in \vartheta(x)\}$; this is still a Borel reduction. Put $U = \{2^n + 1 : n \in \mathbb{N}\}$, obviously $\sum_{k \in U} \frac{1}{k} < +\infty$. Fix a Borel injection $f : X \to \mathscr{P}(U)$. The map $\vartheta''(x) = \vartheta'(x) \cup f(x)$ is the Borel embedding required. □

COROLLARY 5.2.2. *Suppose that* E *is one of the equivalence relations* E_1, E_2, E_3, *while* F *is a Borel equivalence relation, and* $\mathsf{F} \le_{\mathrm{B}} \mathsf{E}$. *Then there exists a Borel set* $X \subseteq \operatorname{dom}\mathsf{E}$ *such that* F *is Borel isomorphic to* $\mathsf{E} \restriction X$. □

5.3. Borel, continuous, and Baire measurable reductions

The Borel reducibility and related notions in Section 5.1 admit weaker Baire measurable (BM, for brevity) versions, which claims that the reduction postulated to exist is only a BM, not necessarily Borel, map. (Recall that a map is *Baire measurable* if the preimages of open sets are sets with the *Baire* property.) Those versions will be denoted with a subscript BM instead of B. Also there are stronger continuous versions that will be denoted with a subscript C. Thus

$\mathsf{E} \le_{\mathrm{BM}} \mathsf{F}$, $\mathsf{E} \sim_{\mathrm{BM}} \mathsf{F}$, $\mathsf{E} <_{\mathrm{BM}} \mathsf{F}$, $\mathsf{E} \sqsubseteq_{\mathrm{BM}} \mathsf{F}$: mean the reducibility, resp., bi-reducibility, strict reducibility, embedding by *Baire measurable* maps.

$\mathsf{E} \le_{\mathrm{C}} \mathsf{F}$, $\mathsf{E} \sim_{\mathrm{C}} \mathsf{F}$, $\mathsf{E} <_{\mathrm{C}} \mathsf{F}$, $\mathsf{E} \sqsubseteq_{\mathrm{C}} \mathbf{F}$: mean the reducibility, resp., bi-reducibility, strict reducibility, embedding by *continuous* maps.

It is known that a Baire measurable map defined on a Polish space is continuous on a comeager set. Thus BM reducibility is the same as a Borel or even continuous reducibility on a comeager set. On the other hand, according to the following result of JUST [**Jus90a**] and LOUVEAU [**Lou94**], continuous reducibility on the full domain can sometimes be derived from Borel reducibility.

LEMMA 5.3.1. *If* $\mathscr{I}$ *is a Borel ideal on a countable set* A, E *an equivalence relation on a Polish space* $\mathbb{X}$, *and* $\mathsf{E}_{\mathscr{I}} \le_{\mathrm{BM}} \mathsf{E}$ (via a Baire measurable reduction), *then* $\mathsf{E}_{\mathscr{I}} \le_{\mathrm{C}} \mathsf{E} \times \mathsf{E}$ (via a continuous reduction). *In addition, there is a set* $X \subseteq A$, $X \notin \mathscr{I}$ *such that* $\mathsf{E}_{\mathscr{I}\restriction X} \le_{\mathrm{C}} \mathsf{E}$, *where* $\mathscr{I} \restriction X = \mathscr{I} \cap \mathscr{P}(X)$.

Here $\mathsf{E} \times \mathsf{E}$ is an equivalence relation on $\mathbb{X} \times \mathbb{X}$ defined so that $\langle x, y\rangle$ and $\langle x', y'\rangle$ are equivalent iff both $x \mathrel{\mathsf{E}} x'$ and $y \mathrel{\mathsf{E}} y'$. Note that $\mathsf{E} \times \mathsf{E} \le_{\mathrm{C}} \mathsf{E}$ holds for various equivalence relations E, and in such a case the condition $\mathsf{E}_{\mathscr{I}} \le_{\mathrm{C}} \mathsf{E} \times \mathsf{E}$ in the lemma can be replaced by $\mathsf{E}_{\mathscr{I}} \le_{\mathrm{C}} \mathsf{E}$.

PROOF. We have to define continuous maps $\vartheta_0, \vartheta_1 : \mathscr{P}(A) \to \mathbb{X}$ such that, for every $x, y \in \mathscr{P}(\mathbb{N})$, $x \mathbin{\Delta} y \in \mathscr{I}$ iff both $\vartheta_0(x) \mathrel{\mathsf{E}} \vartheta_0(y)$ and $\vartheta_1(x) \mathrel{\mathsf{E}} \vartheta_1(y)$. Suppose *w.l.o.g.* that $A = \mathbb{N}$. Let $\vartheta : \mathscr{P}(\mathbb{N}) \to \mathbb{X}$ witness that $\mathsf{E}_{\mathscr{I}} \le_{\mathrm{BM}} \mathsf{E}$. Then ϑ is continuous on a dense $\mathbf{G}_\delta$ set $D = \bigcap_i D_i \subseteq \mathscr{P}(\mathbb{N})$, all D_i being dense open, and $D_{i+1} \subseteq D_i$. A sequence $0 = n_0 < n_1 < n_2 < \cdots$ and, for every i, a set $u_i \subseteq [n_i, n_{i+1})$ can be easily defined, by induction on i, so that $x \cap [n_i, n_{i+1}) = u_i \Longrightarrow x \in D_i$.[1] Let

$$N_1 = \textstyle\bigcup_i [n_{2i}, n_{2i+1}), \quad N_2 = \bigcup_i [n_{2i+1}, n_{2i+2}), \quad U_1 = \bigcup_i u_{2i}, \quad U_2 = \bigcup_i u_{2i+1}.$$

Now set $\vartheta_1(x) = \vartheta((x \cap N_1) \cup U_2)$ and $\vartheta_2(x) = \vartheta((x \cap N_2) \cup U_1)$ for $x \subseteq \mathbb{N}$.

To prove the second claim, let X be that one of the sets N_1, N_2 which does not belong to $\mathscr{I}$. (Or any one of them if neither belongs to $\mathscr{I}$.) Suppose that $X = N_1 \notin \mathscr{I}$. Then the map ϑ_1 proves $\mathsf{E}_{\mathscr{I}\restriction X} \le_{\mathrm{C}} \mathsf{E}$. □

The following question should perhaps be answered in the negative in general and be open for some particular cases.

QUESTION 5.3.2. Suppose that $\mathsf{E} \le_{\mathrm{B}} \mathsf{F}$ are Borel equivalence relations. Does there always exist a *continuous* reduction ? □

5.4. Additive reductions

There is a special useful type of continuous reducibility, actually a "clone" of the Rudin–Blass order of ideals considered in Section 3.2.

Suppose that $X = \prod_{k\in\mathbb{N}} X_k$ and $Y = \prod_{k\in\mathbb{N}} Y_k$, the sets X_i, Y_i are finite, $0 = n_0 < n_1 < n_2 < \cdots$, and $H_i : X_i \to \prod_{n_i \le k < n_{i+1}} Y_k$ for every i. Define

$$\Psi(x) = H_0(x(0)) \cup H_1(x(1)) \cup H_2(x(2)) \cup \cdots \in Y$$

for each $x \in X$. Maps Ψ of this kind are called *additive* (FARAH [**Far01b**]). More generally, if, in addition, $0 = m_0 < m_1 < m_2 < \cdots$, and $H_i : \prod_{m_i \le j < m_{i+1}} X_j \to \prod_{n_i \le k < n_{i+1}} Y_k$ for every i, then define

$$\Psi(x) = H_0(x \restriction [m_0, m_1)) \cup H_1(x \restriction [m_1, m_2)) \cup H_2(x \restriction [m_2, m_3)) \cup \cdots \in Y$$

for each $x \in X$. FARAH [**Far01b**] calls maps Ψ of this kind *asymptotically additive*. All of them are continuous functions $X \to Y$ in the sense of the product Polish topology. (Recall that X_i, Y_i are finite.)

Suppose now that E and F are equivalence relations on $X = \prod_k X_k$ and $Y = \prod_k Y_k$, respectively.

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[1] Sets such as u_i are called *stabilizers*. They are of much help in study of Borel ideals.

Additive reducibility: $\mathsf{E} \leq_{\mathrm{A}} \mathsf{F}$ if there is an additive reduction of E to F. As usual, $\mathsf{E} \sim_{\mathrm{A}} \mathsf{F}$ means that simultaneously $\mathsf{E} \leq_{\mathrm{A}} \mathsf{F}$ and $\mathsf{F} \leq_{\mathrm{A}} \mathsf{E}$, while $\mathsf{E} <_{\mathrm{A}} \mathsf{F}$ means that $\mathsf{E} \leq_{\mathrm{A}} \mathsf{F}$ but not $\mathsf{F} \leq_{\mathrm{A}} \mathsf{E}$.

A version: $\mathsf{E} \leq_{\mathrm{AA}} \mathsf{F}$ if there exists an asymptotically additive reduction of E to F.

The additive reducibility coincides with $\leq_{\mathrm{RB}}^{++}$ on the domain of Borel ideals:

LEMMA 5.4.1 (FARAH [**Far01b**]). *Assume that $\mathscr{I}$ and $\mathscr{J}$ are Borel ideals on $\mathbb{N}$. Then $\mathscr{I} \leq_{\mathrm{RB}}^{++} \mathscr{J}$ iff $\mathsf{E}_{\mathscr{I}} \leq_{\mathrm{A}} \mathsf{E}_{\mathscr{J}}$.*

By definition, $\mathsf{E}_{\mathscr{I}}$ and $\mathsf{E}_{\mathscr{J}}$ are equivalence relations on $\mathscr{P}(\mathbb{N})$. However, we can consider them to be equivalence relations on $2^{\mathbb{N}} = \prod_{k \in \mathbb{N}} \{0,1\}$, as usual, which yields the intended meaning for the relation $\mathsf{E}_{\mathscr{I}} \leq_{\mathrm{A}} \mathsf{E}_{\mathscr{J}}$.

PROOF. If $\mathscr{I} \leq_{\mathrm{RB}}^{++} \mathscr{J}$ via a sequence of finite sets w_i with $\mathtt{max}\, w_i < \mathtt{min}\, w_{i+1}$, then we put $n_0 = 0$ and $n_i = \mathtt{min}\, w_i$ for $k \geq 1$, so that $w_i \subseteq [n_i, n_{i+1})$, and, for every i, put $H_i(0) = [n_i, n_{i+1}) \times \{0\}$ and let $H_i(1)$ be the characteristic function of w_i within $[n_i, n_{i+1})$. Conversely, if $\mathsf{E}_{\mathscr{I}} \leq_{\mathrm{A}} \mathsf{E}_{\mathscr{J}}$ via a sequence $0 = n_0 < n_1 < n_2 < \cdots$ and a family of maps $H_i : \{0,1\} \to 2^{[n_i, n_{i+1})}$, then $\mathscr{I} \leq_{\mathrm{RB}}^{++} \mathscr{J}$ via the sequence of sets $w_i = \{k \in [n_i, n_{i+1}) : H_i(0)(k) \neq H_i(1)(k)\}$. □

5.5. Diagram of Borel reducibility of key equivalence relations

The diagram in Figure 1 (page 68) begins, at the low end, with cardinals $1 \leq n \in \mathbb{N}$, $\aleph_0$, $\mathfrak{c}$, naturally identified with the equivalence relation of equality on resp. finite (of a certain number n of elements), countable, uncountable Polish spaces. As all uncountable Polish spaces are Borel isomorphic, the equivalence relations $\Delta_{\mathbb{X}}$, $\mathbb{X}$ a Polish space, are characterized, modulo $\leq_{\mathrm{B}}$, or even modulo Borel isomorphism between the domains, by the cardinality of the domain, which can be any finite $1 \leq n < \omega$, or $\aleph_0$, or $\mathfrak{c} = 2^{\aleph_0}$.

The linearity breaks above E_0: each one of the four equivalence relations E_1, E_2, E_3, E_∞ of the next level is strictly $<_{\mathrm{B}}$-bigger than E_0, and they are pairwise $\leq_{\mathrm{B}}$-incomparable with each other.

The framebox [?] points on an interesting open problem (Question 5.7.5 below). The framebox [$\mathbf{c}_0$-eqs] denotes $\mathbf{c}_0$-*equalities*, a family of Borel equivalence relations introduced by FARAH [**Far01b**]: all of them are $\leq_{\mathrm{B}}$-between E_3 and $\mathbf{c}_0 \sim_{\mathrm{B}} \mathsf{Z}_0$, and there is continuum-many $\leq_{\mathrm{B}}$-incomparable among them.

The "non-P domain" denotes the family of all Borel equivalence relations that cannot be induced by a Polish action. E_1 belongs to this family, and it is conjectured that E_1 is a $\leq_{\mathrm{B}}$-least ER in this family. SOLECKI [**Sol96, Sol99**] proved this conjecture for equivalence relations generated by Borel ideals: for a Borel ideal $\mathscr{I}$ to be not a P-ideal, it is necessary and sufficient that $\mathsf{E}_1 \leq_{\mathrm{B}} \mathsf{E}_{\mathscr{I}}$; see Corollary 11.8.3.

Finally, the framebox [ctble] denotes the family of all Borel countable equivalence relations (meaning that equivalence classes are at most countable); all of them are Borel reducible to E_∞ and are induced by Borel actions of countable groups by the FELDMAN–MOORE theorem [**FM77**, Thm. 1] (Theorem 7.4.1 below). The following theorem of ADAMS–KECHRIS [**AK00**] shows that this is quite a rich family with its own complicated $\leq_{\mathrm{B}}$-structure. See a brief survey of results from recent years on Borel countable equivalence relations in Chapter 9.

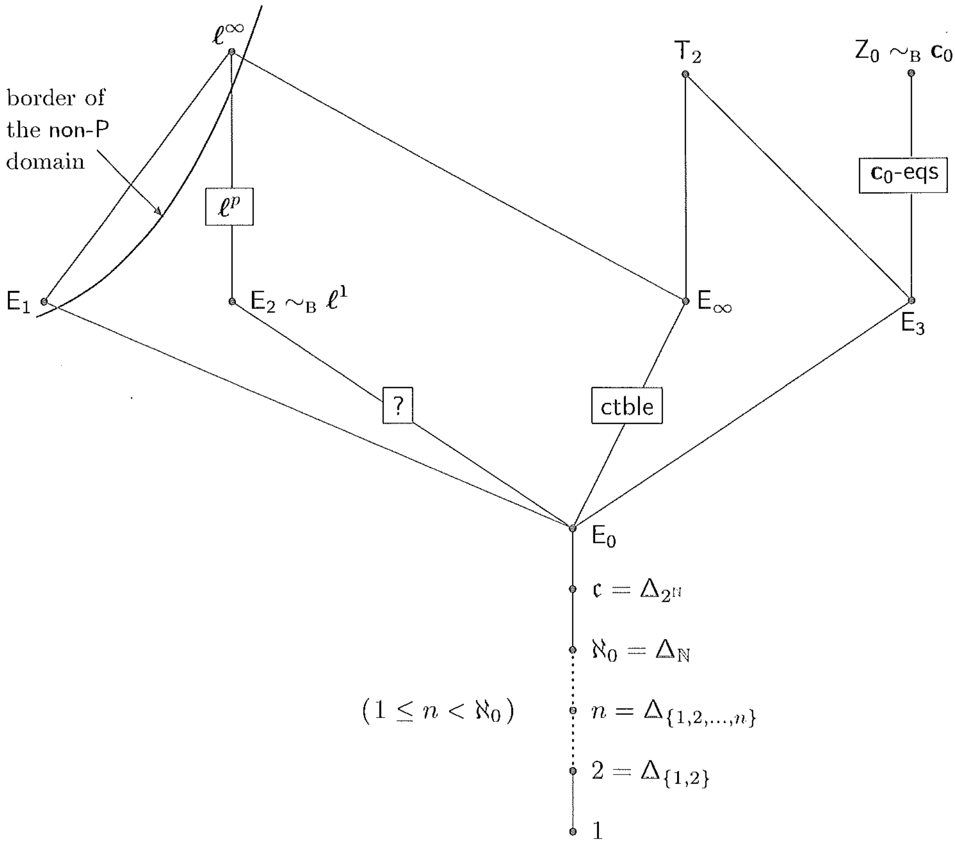

FIGURE 1. Reducibility between the key equivalence relations. Connecting lines here indicate Borel reducibility of lower equivalence relations to upper ones.

THEOREM 5.5.1 (not to be proved here). *There is a map* $A \mapsto \mathsf{E}_A$ *assigning a countable Borel equivalence relation* E_A *to each Borel subset* $A \subseteq 2^{\mathbb{N}}$ *such that* $A \subseteq B \iff \mathsf{E}_A \le_{\mathrm{B}} \mathsf{E}_B$. *It follows that there exist continuum many pairwise* $\le_{\mathrm{B}}$-*incomparable countable Borel equivalence relations.* □

A somewhat weaker result that implies the existence of continuum-many pairwise $\le_{\mathrm{B}}$-incomparable (not necessarily countable) Borel equivalence relations will be established by Theorem 16.6.3.

5.6. Reducibility and irreducibility on the diagram

Here we discuss, without going into technicalities, the structure of the diagram in Figure 1 and related theorems.

Recall that straight line connections on the diagram indicate Borel reducibility of the equivalence relation at its lower end to the equivalence relation at its upper end. Some of these reducibility claims are witnessed by simple and obvious reductions. Slightly less obvious are reductions of E_∞ and E_3 to T_2 and E_3 to $\mathbf{c}_0$; see Lemmas 6.1.2 and 6.1.3. Finally, to prove that $\mathsf{E}_1, \mathsf{E}_\infty$, and all of ℓ^p (including

$\ell^1 \sim_B \mathsf{E}_2$), are Borel reducible to ℓ^∞, we apply ROSENDAL's theorem [**Ros05**] saying that ℓ^∞ is a $\le_B$-largest $\mathbf{F}_\sigma$.

Lemmas 6.2.3 and 6.2.4 below prove $\mathsf{E}_2 \sim_B \ell^1$ and $\mathbf{c}_0 \sim_B \mathsf{Z}_0$.

See Theorem 6.5.1 on the equivalence $\ell^p \le_B \ell^q \Longleftrightarrow p \le q$.

It is a most interesting question whether the diagram in Figure 1 is complete in the sense that there is no Borel reducibility interrelations between the ERs mentioned in the diagram except for those explicitly indicated by straight lines. Some of these irreducibility claims are trivial by a simple cardinality argument: clearly, an equivalence relation E having strictly more equivalence classes than F is not Borel reducible to F.

However, this argument is not applicable in more complicated cases, beginning with the irreducibility claim $\mathsf{E}_0 \not\le_B \Delta_{2^{\mathbb{N}}}$: each of the two relations has exactly continuum-many classes. Here we have to employ the Borel-ness. Suppose toward the contrary that $\vartheta : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is a Borel reduction of E_0 to $\Delta_{2^{\mathbb{N}}}$. Then the preimage $\vartheta^{-1}[y] = \{x : \vartheta(x) = y\}$ of any $y \in 2^{\mathbb{N}}$ is countable (or empty); in other words, $P = \{\langle y, x\rangle : \vartheta(x) = y\}$ is a Borel set with countable cross-sections. It can be uniformized by a Borel uniform set $Q \subseteq P$ by Theorem 2.12.5. The set $T = \{x \in 2^{\mathbb{N}} : \langle \vartheta(x), x\rangle \in Q\}$ is Borel by Corollary 2.12.2 as a Borel preimage of Q. However, T is a *transversal*, that is, T has exactly one common element with each E_0-class. But this contradicts the Borel-ness of T; see a short argument after Example 4.4.2.[2]

As for the rest of the diagram, to establish its completeness, one has to prove the following irreducibility claims:

(1) $\mathsf{E}_1 \not\le_B : \mathsf{E}_2, \mathsf{T}_2, \mathbf{c}_0$;

(2) $\ell^\infty \not\le_B : \mathsf{E}_1, \mathsf{E}_2, \mathsf{T}_2, \mathbf{c}_0$;

(3) $\mathsf{E}_2 \not\le_B : \mathsf{E}_1, \mathsf{T}_2, \mathbf{c}_0$;

(4) $\mathsf{E}_\infty \not\le_B : \mathsf{E}_1, \mathsf{E}_2, \mathbf{c}_0$ (this group contains open problems);

(5) $\mathsf{E}_3 \not\le_B : \ell^\infty$;

(6) $\mathsf{T}_2 \not\le_B : \ell^\infty, \mathbf{c}_0$;

(7) $\mathbf{c}_0 \not\le_B : \ell^\infty, \mathsf{T}_2$.

Beginning with (1), we note that E_1 is not Borel reducible to equivalence relations induced by Polish actions by Theorem 11.8.1 (KECHRIS–LOUVEAU). On the other hand, $\mathsf{E}_2, \mathsf{T}_2, \mathbf{c}_0$ obviously belong to this category of ERs.

(2) follows from (1) and (3) since $\mathsf{E}_1 \le_B \ell^\infty$ and $\mathsf{E}_2 \le_B \ell^\infty$.

The result $\mathsf{E}_2 \not\le_B \mathbf{c}_0$ in (3) is HJORTH's Theorem 6.3.1(ii). The result $\mathsf{E}_2 \not\le_B \mathsf{E}_1$ (Corollary 11.1.4) will be established by reference to KECHRIS's Theorem 11.1.1 on the structure of ideals Borel reducible to E_1.

The results $\mathsf{E}_2 \not\le_B \mathsf{T}_2$ and $\mathbf{c}_0 \not\le_B \mathsf{T}_2$ in (3) and (7) are proved in Chapter 13 (Corollary 13.9.2); this will involve turbulence theory by HJORTH and KECHRIS.

The result of (5) is Lemma 6.1.1. It also implies $\mathbf{c}_0 \not\le_B \ell^\infty$ in (7).

(6) was obtained by HJORTH; see Chapter 17.

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[2] Alternatively, one can derive $\mathsf{E}_0 \not\le_B \Delta_{2^N}$ from an old result of SIERPIŃSKI [**Sie18**]: any linear ordering of all E_0-classes yields a Lebesgue non-measurable set of the same descriptive complexity as the given ordering.

This leaves us with (4). We do not know how to prove $\mathsf{E}_\infty \not\leq_{\mathrm{B}} \mathsf{E}_1$ easily and directly. There are two indirect ways. The first one is to apply some results in the theory of countable and hyperfinite equivalence relations; see Corollary 11.2.2. The second one is based on Theorems 5.7.3 (3rd dichotomy) and 11.8.1; see Corollary 11.8.5.

QUESTION 5.6.1. Is E_∞ Borel reducible to $\mathbf{c}_0$? to ℓ^1? to any other ℓ^p? □

A related question, whether E_∞ is Borel reducible to E_3, answers in the negative on the base of the 6th dichotomy theorem by Corollary 14.0.1.

The irreducibility results in (1)–(7) can be partitioned into two rather distinct categories. The first group consists of those having proofs that involve only common methods of descriptive set theory, such as the proof of $\mathsf{E}_0 \not\leq_{\mathrm{B}} \Delta_{2^{\mathbb{N}}}$ outlined above. This includes such results as $\mathsf{E}_2 \not\leq_{\mathrm{B}} \mathbf{c}_0$, $\ell^\infty \not\leq_{\mathrm{B}} \mathbf{c}_0$, $\mathsf{E}_3 \not\leq_{\mathrm{B}} \ell^\infty$, $\mathbf{c}_0 \not\leq_{\mathrm{B}} \ell^\infty$, and also $\mathsf{E}_2 \not\leq_{\mathrm{B}} \mathsf{E}_1$ as a transitional claim between the first and second group: it refers to Theorem 11.1.1, a special result on the $\leq_{\mathrm{B}}$-structure of ideals below $\mathscr{I}_1$, rather complicated but still based on classics of descriptive set theory.

Note that some results in this group belong to the earliest of this type. For instance, JUST proved that E_2 is mutually $\leq_{\mathrm{B}}$-irreducible with Z_0 [**Jus90b**] and with $\mathsf{E}_{\mathtt{Fin}\otimes\mathtt{Fin}}$ [**Jus90a**]. According to [**KL97**, 1.4], the irreducibility claim $\mathsf{E}_1 \not\leq_{\mathrm{B}} \mathsf{E}_\infty$ goes back to an even earlier paper [**FR85**].

The other group consists of irreducibility results that involve (as far as we know) methods that definitely go beyond common tools of descriptive set theory. This includes such resuts as $\mathsf{E}_1 \not\leq_{\mathrm{B}} \mathsf{E}_2$, $\mathsf{E}_1 \not\leq_{\mathrm{B}} \mathsf{T}_2$, $\mathsf{E}_1 \not\leq_{\mathrm{B}} \mathbf{c}_0$, based on the fact that E_1 is not reducible to a Polish action (Theorem 11.8.1), $\mathsf{E}_2 \not\leq_{\mathrm{B}} \mathsf{T}_2$ and $\mathbf{c}_0 \not\leq_{\mathrm{B}} \mathsf{T}_2$ based on the turbulence theory, $\mathsf{E}_\infty \not\leq_{\mathrm{B}} \mathsf{E}_1$ and $\mathsf{E}_\infty \not\leq_{\mathrm{B}} \mathsf{E}_3$ based on the 3th and 6th dichotomy theorems, respectively (see the next section), and finally $\mathsf{T}_2 \not\leq_{\mathrm{B}} \ell^\infty$ and $\mathsf{T}_2 \not\leq_{\mathrm{B}} \mathbf{c}_0$ based on the theory of *pinned* equivalence relations (see Chapter 17).

5.7. Dichotomy theorems

Another general problem related to Figure 1 is the $\leq_{\mathrm{B}}$-structure of certain domains, for instance, $\leq_{\mathrm{B}}$-intervals between adjacent equivalence relations. Some results in this direction are known as dichotomy theorems because of their distinguished dichotomical form. The following two theorems will be proved in Chapter 10.

THEOREM 5.7.1 (**1st dichotomy**, SILVER [**Sil80**] and HARRINGTON). *Every Borel, even every* $\mathbf{\Pi}^1_1$ *equivalence relation* E **either** *has at most countably many equivalence classes, formally,* $\mathsf{E} \leq_{\mathrm{B}} \Delta_{\mathbb{N}}$, **or** *satisfies* $\Delta_{2^{\mathbb{N}}} \leq_{\mathrm{B}} \mathsf{E}$.

THEOREM 5.7.2 (**2nd dichotomy**, HARRINGTON, KECHRIS, and LOUVEAU [**HKL90**]). *Every Borel equivalence relation* E *satisfies* **either** $\mathsf{E} \leq_{\mathrm{B}} \mathfrak{c}$ **or** $\mathsf{E}_0 \leq_{\mathrm{B}} \mathsf{E}$.

Thus not only the strict $<_{\mathrm{B}}$-interval between the equivalence relations $\aleph_0 = \Delta_{\mathbb{N}}$ and $\mathfrak{c} = \Delta_{2^{\mathbb{N}}}$ is empty, but the union of the lower $\leq_{\mathrm{B}}$-cone of the former and the upper $\leq_{\mathrm{B}}$-cone of the latter cover the whole family of Borel equivalence relations! The same is true for the $<_{\mathrm{B}}$-interval between the equivalence relations $\mathfrak{c}$ and E_0.

What is going on in the $<_{\mathrm{B}}$-intervals between E_0 and the equivalence relations E_1, E_2, E_3? The following dichotomy theorems provide some answers.

THEOREM 5.7.3 (**3rd dichotomy**, KECHRIS and LOUVEAU [**KL97**]). *Every equivalence relation* $\mathsf{E} \leq_{\mathrm{B}} \mathsf{E}_1$ *satisfies* **either** $\mathsf{E} \leq_{\mathrm{B}} \mathsf{E}_0$ **or** $\mathsf{E} \sim_{\mathrm{B}} \mathsf{E}_1$.

THEOREM 5.7.4 (**4th dichotomy**, HJORTH [**Hjo00a**]). *Every equivalence relation* $\mathsf{E} \le_{\mathrm{B}} \mathsf{E}_2$ **either** *is essentially countable* **or** *satisfies* $\mathsf{E} \sim_{\mathrm{B}} \mathsf{E}_2$.

An equivalence relation E is *essentially countable* iff it is Borel reducible to a Borel countable equivalence relation. (Recall that *countable* means that all equivalence classes are at most countable.) The "either" case in 4th dichotomy remains not entirely clear. This is marked by the framebox $\boxed{?}$ on the diagram.

QUESTION 5.7.5. In Theorem 5.7.4, can the "either" case be strengthened to the condition $\mathsf{E} \le_{\mathrm{B}} \mathsf{E}_0$? □

The fifth dichotomy theorem is a bit more special, and it will not be considered in this book. See the end of Section 13.1 for more detail.

THEOREM 5.7.6 (**6th dichotomy**, HJORTH and KECHRIS [**HK97, HK01**]). *Any equivalence relation* $\mathsf{E} \le_{\mathrm{B}} \mathsf{E}_3$ *satisfies* **either** $\mathsf{E} \le_{\mathrm{B}} \mathsf{E}_0$ **or** $\mathsf{E} \sim_{\mathrm{B}} \mathsf{E}_3$.

Theorems 5.7.3, 5.7.4, and 5.7.6 will be proved in Chapters 11, 15, and 14, respectively.

On the other hand, the interval between E_0 and E_∞ contains all countable Borel equivalence relations and among them plenty of pairwise $\sim_{\mathrm{B}}$-inequivalent equivalence relations by Theorem 5.5.1. See Chapters 7, 8, and 9 on countable Borel equivalence relations.

It was once considered [**HK97**] to be a plausible hypothesis that any Borel equivalence relation which is not $\le_{\mathrm{B}} \mathsf{E}_\infty$, *i.e.*, not essentially countable, satisfies $\mathsf{E}_i \le_{\mathrm{B}} \mathsf{E}$ for at least one $i = 1, 2, 3$. This turns out to be not the case: FARAH [**Far01a, Far99**] and VELICKOVIC [**Vel99**] found an independent family of Borel equivalence relations, not countable and also $\le_{\mathrm{B}}$-incomparable with the equivalence relations $\mathsf{E}_1, \mathsf{E}_2, \mathsf{E}_3$.

QUESTION 5.7.7. It there any reasonable "basis" in the family of Borel non-countable equivalence relations above E_0? □

5.8. Borel ideals in the structure of Borel reducibility

Some of the equivalence relations in Figure 1 are obviously generated by Borel ideals; for some other ones this is not clear. This leads to the question, what is the place of Borel ideals in the whole structure of Borel equivalence relations? The answer obtained in the studies in recent years can be formulated as follows: Borel ideals are $\le_{\mathrm{B}}$-cofinal, but rather rare, in the $\le_{\mathrm{B}}$-structure of Borel equivalence relations. We prove the following theorem, the cofinality claim of which is due to ROSENTAL [**Ros05**] (Theorem 18.4.1 in Chapter 18). The other claim is contained in Corollary 13.9.4.

THEOREM 5.8.1. *For any Borel equivalence relation* E *there exists a Borel ideal* $\mathscr{I} \subseteq \mathscr{P}(\mathbb{N})$ *such that* $\mathsf{E} \le_{\mathrm{B}} \mathsf{E}_{\mathscr{I}}$. *On the other hand there is no Borel ideal* $\mathscr{I}$ *such that* $\mathsf{T}_2 \sim_{\mathrm{B}} \mathsf{E}_{\mathscr{I}}$.

CHAPTER 6

"Elementary" results

This chapter is devoted to the reducibility/irreducibility results in Figure 1 on page 68, elementary in the sense that they do not involve any special concepts or technical methods beyond usual methods of modern descriptive set theory (including a rather elementary application of forcing). Some of them are really simple, as *e.g.* some lemmas on E_3 and T_2 in Section 6.1 or the equivalence relations $\mathbf{c}_0 \sim_{\mathrm{B}} \mathsf{Z}_0$ and $\mathsf{E}_2 \sim_{\mathrm{B}} \ell^1$ in Section 6.2, while some others are quite tricky. The latter category includes HJORTH's theorem on the irreducibility of non-trivial summable ideals to $\mathbf{c}_0$ in Section 6.3, interrelations in the family of equivalence relations ℓ^p in Section 6.5, and the $\leq_{\mathrm{B}}$-universality of ℓ^∞ in the class of all $\mathbf{F}_\sigma$ equivalence relations in Section 6.6.

6.1. Equivalence relations E_3 and T_2

Equivalence relations E_3 and T_2, together with $\mathbf{c}_0 \sim_{\mathrm{B}} \mathsf{Z}_0$, are the only non-$\mathbf{\Sigma}^0_2$ equivalence relations explicitly mentioned in Figure 1 on page 68.

LEMMA 6.1.1. *E_3 is Borel irreducible to ℓ^∞.*

PROOF. Let us consider E_3 to be an equivalence relation on $(2^{\mathbb{N}})^{\mathbb{N}}$ defined so that $x \mathrel{\mathsf{E}_3} y$ iff $x(n) \mathrel{\mathsf{E}_0} y(n)$ for all n. Recall that, for $a, b \in 2^{\mathbb{N}}$, $a \mathrel{\mathsf{E}_0} b$ means that the set $a \bigtriangleup b = \{m : a(m) \neq b(m)\}$ is finite.

Suppose toward the contrary that $\vartheta : (2^{\mathbb{N}})^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ is a Borel reduction of E_3 to ℓ^∞. Since obviously $\ell^\infty \sim_{\mathrm{B}} \ell^\infty \times \ell^\infty$, Lemma 5.3.1 reduces the general case to the case of continuous ϑ. Define $\mathbf{0}, \mathbf{1} \in 2^{\mathbb{N}}$ by $\mathbf{0}(n) = 0$, $\mathbf{1}(n) = 1$, $\forall n$. Define $\mathbb{0} \in (2^{\mathbb{N}})^{\mathbb{N}}$ so that $\mathbb{0}(k) = \mathbf{0}$, $\forall k$. Finally, for every k define $\mathbf{a}_k \in 2^{\mathbb{N}}$ by $\mathbf{a}_k(n) = 1$ for $n < k$ and $\mathbf{a}_k(n) = 0$ for $n \geq k$.

We claim that there are increasing sequences of natural numbers $\{k_m\}$ and $\{j_m\}$ such that $|\vartheta(x)(j_m) - \vartheta(\mathbb{0})(j_m)| > m$ for all m and all $x \in (2^{\mathbb{N}})^{\mathbb{N}}$ satisfying

$$x(k) = \begin{cases} \mathbf{a}_{k_i} & \text{whenever } i < m \text{ and } k = k_i, \\ \mathbf{0} & \text{for all } k < k_m \text{ not of the form } k_i. \end{cases}$$

To see that this implies a contradiction define $x \in (2^{\mathbb{N}})^{\mathbb{N}}$ so that $x(k_i) = \mathbf{a}_{k_i}$, $\forall i$ and $x(k) = \mathbf{0}$ for all k not of the form k_i. Then obviously $x \mathrel{\mathsf{E}_3} \mathbb{0}$, but $|\vartheta(x)(j_m) - \vartheta(\mathbb{0})(j_m)| > m$ for all m; hence, $\vartheta(x) \mathrel{\ell^\infty} \vartheta(\mathbb{0})$ fails, as required.

We put $k_0 = 0$. To define j_0 and k_1, consider $x_0 \in (2^{\mathbb{N}})^{\mathbb{N}}$ defined by $x_0(0) = \mathbf{1}$ but $x_0(k) = \mathbf{0}$ for all $k \geq 1$. Then $x_0 \mathrel{\mathsf{E}_3} \mathbb{0}$ fails, and hence $\vartheta(x_0) \mathrel{\ell^\infty} \vartheta(\mathbb{0})$ fails as well. Take an arbitrary j_0 with $|\vartheta(x_0)(j_0) - \vartheta(\mathbb{0})(j_0)| > 0$. As ϑ is continuous, there is a number $k_1 > 0$ such that $|\vartheta(x)(j_0) - \vartheta(\mathbb{0})(j_0)| > 0$ holds for every $x \in (2^{\mathbb{N}})^{\mathbb{N}}$ with $x(0) = \mathbf{a}_{k_1}$ and $x(k) = \mathbf{0}$ for all $0 < k < k_1$.

To define j_1 and k_2, consider $x_1 \in (2^{\mathbb{N}})^{\mathbb{N}}$ defined so that $x_1(0) = \mathbf{a}_{k_1}$, $x_1(k) = \mathbf{0}$ whenever $0 < k < k_1$, and $x_1(k_1) = \mathbf{1}$. Once again there is a number j_1 with

$|\vartheta(x_1)(j_1) - \vartheta(\mathbb{O})(j_1)| > 1$, and a number $k_2 > k_1$ such that $|\vartheta(x)(j_1) - \vartheta(\mathbb{O})(j_1)| > 1$ for every $x \in (2^{\mathbb{N}})^{\mathbb{N}}$ with $x(0) = \mathbf{a}_{k_1}$, $x(k_1) = \mathbf{a}_{k_1}$, and $x(k) = \mathbf{0}$ for all $0 < k < k_1$ and $k_1 < k < k_2$.

Et cetera. □

LEMMA 6.1.2. E_3 *is Borel reducible to both* T_2 *and* $\mathbf{c}_0$.

PROOF. If $a \in 2^{\mathbb{N}}$ and $s \in 2^{<\omega}$, then define $s \cdot a \in 2^{\mathbb{N}}$ by $(s \cdot a)(k) = a(k) +_2 s(k)$ for $k < \mathtt{lh}\, s$ and $(s \cdot a)(k) = a(k)$ for $k \geq \mathtt{lh}\, s$. If $m \in \mathbb{N}$, then $m^\frown a \in 2^{\mathbb{N}}$ denotes the concatenation. In these terms, if $x, y \in (2^{\mathbb{N}})^{\mathbb{N}}$, then obviously

$$x \mathrel{\mathsf{E}_3} y \iff \{m^\frown(s \cdot x(m)) : s \in 2^{<\omega},\ m \in \mathbb{N}\} = \{m^\frown(s \cdot y(m)) : s \in 2^{<\omega},\ m \in \mathbb{N}\}.$$

Now any bijection $2^{<\omega} \times \mathbb{N} \xrightarrow{\text{onto}} \mathbb{N}$ yields a Borel reduction of E_3 to T_2.

To reduce E_3 to $\mathbf{c}_0$ we make use of a Borel map $\vartheta : (2^{\mathbb{N}})^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ such that $\vartheta(x)(2^n(2k+1) - 1) = n^{-1} x(n)(k)$. □

LEMMA 6.1.3. *If* E *is a countable Borel equivalence relation, then it is Borel reducible to* T_2.

PROOF. Let E be a Borel countable equivalence relation on $2^{\mathbb{N}}$. It follows from the Countable-to-1 Enumeration (Theorem 2.12.3) that there is a Borel map $f : 2^{\mathbb{N}} \times \mathbb{N} \to 2^{\mathbb{N}}$ such that $[a]_{\mathsf{E}} = \{f(a, n) : n \in \mathbb{N}\}$ for all $a \in 2^{\mathbb{N}}$. The map ϑ sending every $a \in 2^{\mathbb{N}}$ to $x = \vartheta(a) \in (2^{\mathbb{N}})^{\mathbb{N}}$ such that $x(n) = f(a, n)$, $\forall n$, is a required reduction. □

See further study on T_2 in Chapter 17, where it will be shown that T_2 is not Borel reducible to a big family of equivalence relations that includes $\mathbf{c}_0, \boldsymbol{\ell}^p, \boldsymbol{\ell}^\infty$, $\mathsf{E}_1, \mathsf{E}_2, \mathsf{E}_3, \mathsf{E}_\infty$. On the other hand, the equivalence relations in this list, with the exception of $\mathsf{E}_3, \mathsf{E}_\infty$, are not Borel reducible to T_2; this follows from the turbulence theory presented in Chapter 13.

6.2. Discretization and generation by ideals

Some equivalence relations in Figure 1 on page 68 are explicitly generated by ideals, like E_i, $i = 0, 1, 2, 3$. Some other equivalence relations are defined differently. It will be shown in Chapter 18 that *every* Borel equivalence relation E is Borel reducible to an equivalence relation of the form $\mathsf{E}_{\mathscr{I}}$, $\mathscr{I}$ being a Borel ideal. On the other hand, the equivalence relations $\mathbf{c}_0, \boldsymbol{\ell}^1, \boldsymbol{\ell}^\infty$ turn out to be Borel equivalent to some meaningful Borel ideals. Moreover, these equivalence relations admit "discretization" by means of restriction to certain subsets of $\mathbb{R}^{\mathbb{N}}$.

DEFINITION 6.2.1. We define $\mathbb{X} = \prod_{n \in \mathbb{N}} X_n = \{x \in \mathbb{R}^{\mathbb{N}} : \forall n\, (x(n) \in X_n)\}$, where $X_n = \{\frac{0}{2^n}, \frac{1}{2^n}, \ldots, \frac{2^n}{2^n}\}$. □

LEMMA 6.2.2. $\mathbf{c}_0 \leq_{\mathrm{B}} \mathbf{c}_0 \restriction \mathbb{X}$ *and* $\boldsymbol{\ell}^p \leq_{\mathrm{B}} \boldsymbol{\ell}^p \restriction \mathbb{X}$ *for every* $1 \leq p < \infty$. *On the other hand,* $\boldsymbol{\ell}^\infty \leq_{\mathrm{B}} \boldsymbol{\ell}^\infty \restriction \mathbb{Z}^{\mathbb{N}}$.

PROOF. We first show that $\mathbf{c}_0 \leq_{\mathrm{B}} \mathbf{c}_0 \restriction [0,1]^{\mathbb{N}}$. Let π be any bijection of $\mathbb{N} \times \mathbb{Z}$ onto $\mathbb{N}$. For $x \in \mathbb{R}^{\mathbb{N}}$, define $\vartheta(x) \in [0,1]^{\mathbb{N}}$ as follows. Suppose that $k = \pi(n, \eta)$ $(\eta \in \mathbb{Z})$. If $\eta \leq x(n) < \eta + 1$, then let $\vartheta(x)(k) = x(n)$. If $x(n) \geq \eta + 1$, then put $\vartheta(x)(k) = 1$. If $x(n) < \eta$, then put $\vartheta(x)(k) = 0$. Then ϑ is a Borel reduction of $\mathbf{c}_0$ to $\mathbf{c}_0 \restriction [0,1]^{\mathbb{N}}$. Now we prove that $\mathbf{c}_0 \restriction [0,1]^{\mathbb{N}} \leq_{\mathrm{B}} \mathbf{c}_0 \restriction \mathbb{X}$. For $x \in [0,1]^{\mathbb{N}}$ define $\psi(x) \in \mathbb{X}$ so that $\psi(x)(n)$ is the largest number of the form $\frac{i}{2^n}$, $0 \leq i \leq 2^n$ smaller than $x(n)$. Then obviously $x \ \mathbf{c}_0 \ \psi(x)$ holds for every $x \in [0,1]^{\mathbb{N}}$, and hence ψ is a Borel reduction of $\mathbf{c}_0 \restriction [0,1]^{\mathbb{N}}$ to $\mathbf{c}_0 \restriction \mathbb{X}$.

Thus $\mathbf{c}_0 \le_{\mathrm{B}} \mathbf{c}_0 \restriction \mathbb{X}$, and hence in fact $\mathbf{c}_0 \sim_{\mathrm{B}} \mathbf{c}_0 \restriction \mathbb{X}$.

The argument for $\boldsymbol{\ell}^1$ is pretty similar. The result for $\boldsymbol{\ell}^\infty$ is obvious: given $x \in \mathbb{R}^{\mathbb{N}}$, replace every $x(n)$ by the largest integer value $\le x(n)$.

The version for $\boldsymbol{\ell}^p, 1 < p < \infty$, needs some comment in the first part (reduction to $[0,1]^{\mathbb{N}}$). Note that if $\eta \in \mathbb{Z}$ and $\eta - 1 \le x(n) < \eta < \zeta \le y(n) < \zeta + 1$, then the value $(y(n) - x(n))^p$ in the distance $\|y - x\|_p = (\sum_n |y(n) - x(n)|^p)^{\frac{1}{p}}$ is replaced by $(\zeta - \eta) + (\eta - x(n))^p + (y(n) - \zeta)^p$ in $\|\vartheta(y) - \vartheta(x)\|_p$. Thus if this happens infinitely many times, then both distances are infinite, while otherwise this case can be neglected. Further, if $\eta - 1 \le x(n) < \eta \le y(n) < \eta + 1$, then $(y(n) - x(n))^p$ in $\|y - x\|_p$ is replaced by $(\eta - x(n))^p + (y(n) - \eta)^p$ in $\|\vartheta(y) - \vartheta(x)\|_p$. However $(\eta - x(n))^p + (y(n) - \eta)^p \le (y(n) - x(n))^p \le 2^{p-1}((\eta - x(n))^p + (y(n) - \eta)^p)$, and hence these parts of the sums in $\|y - x\|_p$ and $\|\vartheta(y) - \vartheta(x)\|_p$ differ from each other by a factor between 1 and 2^{p-1}. Finally, if $\eta \le x(n), y(n) < \eta + 1$ for one and the same $\eta \in \mathbb{Z}$, then the term $(y(n) - x(n))^p$ in $\|y - x\|_p$ appears unchanged in $\|\vartheta(y) - \vartheta(x)\|_p$. Thus totally $\|y - x\|_p$ is finite iff $\|\vartheta(y) - \vartheta(x)\|_p$ is also. □

LEMMA 6.2.3 (OLIVER [**Oli03**]). $\mathbf{c}_0 \sim_{\mathrm{B}} \mathsf{Z}_0$.

Recall that $\mathsf{Z}_0 = \mathsf{E}_{\mathscr{Z}_0}$, where $\mathscr{Z}_0$ is the null-density ideal (Section 3.1).

PROOF. Prove that $\mathbf{c}_0 \le_{\mathrm{B}} \mathsf{Z}_0$. It suffices, by Lemma 6.2.2, to define a Borel reduction $\mathbf{c}_0 \restriction \mathbb{X} \to \mathsf{Z}_0$, *i.e.*, a Borel map $\vartheta : \mathbb{X} \to \mathscr{P}(\mathbb{N})$ such that $x \ \mathbf{c}_0 \ y \iff \vartheta(x) \bigtriangleup \vartheta(y) \in \mathscr{Z}_0$ for all $x, y \in \mathbb{X}$. Let $x \in \mathbb{X}$. Then, for every n, we have $x(n) = \frac{k(n)}{2^n}$ for some natural $k(n) \le 2^n$. The value of $k(n)$ determines the intersection $\vartheta(x) \cap [2^n, 2^{n+1})$: for each $j < 2^n$, we define $2^n + j \in \vartheta(x)$ iff $j < k(n)$. Then $x(n) = \frac{\mathtt{card}\,(\vartheta(x) \cap [2^n, 2^{n+1}))}{2^n}$ for every n, and moreover

$$|y(n) - x(n)| = \frac{\mathtt{card}\,([\vartheta(x) \bigtriangleup \vartheta(y)] \cap [2^n, 2^{n+1}))}{2^n}$$

for all $x, y \in \mathbb{X}$ and n. This easily implies that ϑ is as required.

To prove $\mathsf{Z}_0 \le_{\mathrm{B}} \mathbf{c}_0$, we have to define a Borel map $\vartheta : \mathscr{P}(\mathbb{N}) \to \mathbb{R}^{\mathbb{N}}$ such that $x \bigtriangleup x \in \mathscr{Z}_0 \iff \vartheta(x) \ \mathbf{c}_0 \ \vartheta(x)$. Most elementary ideas like $\vartheta(x)(n) = \frac{\mathtt{card}\,(x \cap [0,n))}{n}$ do not work. The right way is based on the following observation: for any sets $s, t \subseteq [0, n)$ to satisfy $\mathtt{card}\,(s \bigtriangleup t) \le k$ it is necessary and sufficient that $|\,\mathtt{card}\,(s \bigtriangleup z) - \mathtt{card}\,(t \bigtriangleup z)| \le k$ for all $z \subseteq [0, n)$. To make use of this fact, let us fix an enumeration (with repetitions) $\{z_j\}_{j \in \mathbb{N}}$ of all finite subsets of $\mathbb{N}$ such that

$$\{z_j : 2^n \le j < 2^{n+1}\} \quad = \quad \text{all subsets of } [0, n)$$

for every n. Put, for every $x \in \mathscr{P}(\mathbb{N})$ and $2^n \le j < 2^{n+1}$, $\vartheta(x)(j) = \frac{\mathtt{card}\,(x \cap z_j)}{n}$. Then $\vartheta : \mathscr{P}(\mathbb{N}) \to [0,1]^{\mathbb{N}}$ is a required reduction. □

Recall for every sequence of reals $r_n \ge 0$, that $\mathsf{E}_{\{r_n\}}$ is an equivalence relation on $\mathscr{P}(\mathbb{N})$ generated by the ideal $\mathscr{S}_{\{r_n\}} = \{x \subseteq \mathbb{N} : \sum_{n \in x} r_n < +\infty\}$. It follows from the next lemma, attributed to KECHRIS in [**Hjo00a**, 2.4], that in the non-trivial case all of $\mathsf{E}_{\{r_n\}}$ are $\sim_{\mathrm{B}}$-equivalent to each other.

LEMMA 6.2.4. *If* $r_n \ge 0$, $r_n \to 0$, $\sum_n r_n = +\infty$, *then* $\mathsf{E}_{\{r_n\}} \sim_{\mathrm{B}} \boldsymbol{\ell}^1$. *In particular, the equivalence relation* $\mathsf{E}_2 = \mathsf{E}_{\{1/n\}}$ *satisfies* $\mathsf{E}_2 \sim_{\mathrm{B}} \boldsymbol{\ell}^1$.

PROOF. To prove $\mathsf{E}_{\{r_n\}} \le_{\mathrm{B}} \ell^1$, define $\vartheta(x) \in \mathbb{R}^{\mathbb{N}}$ for any $x \in \mathscr{P}(\mathbb{N})$ as follows: $\vartheta(x)(n) = r_n$ for every $n \in x$, and $\vartheta(x)(n) = 0$ for any other n. Then $x \mathbin{\triangle} y \in \mathscr{S}_{\{r_n\}} \iff \vartheta(x)\, \ell^1\, \vartheta(y)$, as required.

To prove the other direction, it suffices to define a Borel reduction of $\ell^1 \restriction \mathbb{X}$ to $\mathsf{E}_{\{r_n\}}$. We can associate a (generally, infinite) set $s_{nk} \subseteq \mathbb{N}$ with any pair of n and $k < 2^n$, so that the sets s_{nk} are pairwise disjoint and $\sum_{j \in s_{nk}} r_j = 2^{-n}$. The map $\vartheta(x) = \bigcup_n \bigcup_{k<2^n x(n)} s_{nk}$, $x \in \mathbb{X}$, is the reduction required. □

A few words on other cases. If $\sum_n r_n < +\infty$, then obviously $\mathscr{S}_{\{r_n\}} = \mathscr{P}(\mathbb{N})$, and hence $\mathsf{E}_{\{r_n\}}$ makes all sets $x \subseteq \mathbb{N}$ equivalent. The case $r_n \not\to 0$ will be considered in detail in Section 15.1.

6.3. Summables irreducible to density-0

The $\le_{\mathrm{B}}$-independence of ℓ^1 and $\mathbf{c}_0$, the two best known "Banach" equivalence relations, is quite important. In one direction it is provided by (ii) of the next theorem. As for the other direction, Lemma 6.1.1 contains an even stronger irreducibility claim. (Indeed $\mathsf{E}_3 \le_{\mathrm{B}} \mathbf{c}_0$ by Lemma 6.1.2, and $\ell^1 \le_{\mathrm{B}} \ell^\infty$ by Theorem 6.6.1.)

Is there any example of Borel ideals $\mathscr{I} \le_{\mathrm{B}} \mathscr{J}$ that do not satisfy $\mathscr{I} \le_{\mathrm{B}}^{\Delta} \mathscr{J}$? Typically, the reductions found to witness $\mathscr{I} \le_{\mathrm{B}} \mathscr{J}$ are Δ-homomorphisms and even better maps. The next theorem proves that Borel reducibility yields $\le_{\mathrm{RB}}^{++}$-reduction in quite a representative case.

Suppose that $\mathscr{I}, \mathscr{J}$ are ideals over $\mathbb{N}$. Let us say that $\mathscr{I} \le_{\mathrm{RB}}^{++} \mathscr{J}$ *holds exponentially* if there exist a sequence of natural numbers k_i with and $k_{i+1} \ge 2k_i$ and a sequence of sets $w_i \subseteq [k_i, k_{i+1})$ that withesses $\mathscr{I} \le_{\mathrm{RB}}^{++} \mathscr{J}$; in other words, the equivalence $A \in \mathscr{I} \iff w_A = \bigcup_{k \in A} w_k \in \mathscr{J}$ holds for all $A \subseteq \mathbb{N}$.

THEOREM 6.3.1. *Suppose that* $r_n \ge 0$, $r_n \to 0$, $\sum_n r_n = +\infty$. *Then*

(i) (FARAH [**Far99**, 2.1]) *If* $\mathscr{J}$ *is a Borel P-ideal and* $\mathscr{S}_{\{r_n\}} \le_{\mathrm{B}} \mathscr{J}$, *then we have* $\mathscr{S}_{\{r_n\}} \le_{\mathrm{RB}}^{++} \mathscr{J}$ *exponentially*;

(ii) (HJORTH [**Hjo00a**]) $\mathscr{S}_{\{r_n\}}$ *is not Borel-reducible to* $\mathscr{Z}_0$.

PROOF. (i) Let a Borel map $\vartheta : \mathscr{P}(\mathbb{N}) \to \mathscr{P}(\mathbb{N})$ witness $\mathscr{S}_{\{r_n\}} \le_{\mathrm{B}} \mathscr{J}$. Let, according to Theorem 3.5.1, ν be an LSC submeasure on $\mathbb{N}$ with $\mathscr{J} = \mathtt{Exh}_\nu$. Thus $x \in \mathscr{J}$ iff $\mathtt{lim}_{n\to\infty} \nu(x \cap [n,\infty)) = 0$. The map $\nu : \mathscr{P}(\mathbb{N}) \to \mathbb{R}$ is Borel as well. It follows by Corollary 2.3.4 that there exists a parameter $p \in \mathbb{N}^{\mathbb{N}}$ such that both ϑ and ν, as well as the set $\mathscr{J}$, are $\Delta^1_1(p)$.[1]

The proof can be carried by a pedestrian topological argument based on the fact that Borel maps are continuous on comeager sets, but we prefer to utilize a forcing-style proof. The topological argument will be outlined in Section 6.4 for the convenience of the reader.

[1] It is worth noting that none of the objects $\vartheta, \nu, \mathscr{J}$ belongs directly to the domain of the effective hierarchy as defined in Section 1.6. However, ϑ is naturally identified with a map $2^{\mathbb{N}} \to 2^{\mathbb{N}}$, hence, it is a subset of the product space $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$. And if the latter belongs to $\Delta^1_1(p)$ or any other effective class, then we say that the former does also. As for ν, every true real $x \in \mathbb{R}$ can be identified with its Dedekind cut $Q_x = \{q \in \mathbb{Q} : q < x\}$ in the rationals, and hence also with a subset of $\mathbb{N}$ via a fixed recursive bijection $\mathbb{N} \xrightarrow{\text{onto}} \mathbb{Q}$. Thus ν is identified with a map $\mathscr{P}(\mathbb{N}) \to \mathscr{P}(\mathbb{N})$, and hence also with a subset of $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$, as above. And $\mathscr{J}$ as a subset of $\mathscr{P}(\mathbb{N})$ can be identified with a subset of $\mathbb{N}^{\mathbb{N}}$.

DEFINITION 6.3.2. Let us fix a countable transitive model $\mathfrak{M}$ of $\mathbf{ZFC}^-$ containing both p and the sequence $\{r_n\}_{n\in\mathbb{N}}$ and being an elementary submodel of the universe *w.r.t.* all analytic formulas with parameters in $\mathfrak{M}$. Such a model exists by Corollary A.1.5. □

See Appendix A.1 on $\mathbf{ZFC}^-$, its models, and Section A.4 on Cohen forcing.

In the course of the proof of Theorem 6.3.1(i), the model $\mathfrak{M}$ will be the ground model for Cohen forcing. In the remainder of the proof, "generic" means Cohen-generic over $\mathfrak{M}$, and $\mathbb{C} = \{\langle m, w\rangle : m \in \mathbb{N} \wedge w \subseteq [0, m)\}$ is the Cohen forcing notion for $\mathscr{P}(\mathbb{N})$ as defined in Remark A.4.6.

Let us fix an enumeration $\{D_n\}_{n\in\mathbb{N}}$ of all open dense sets $D \subseteq \mathbb{C}$, $D \in \mathfrak{M}$. We let $D'_i = \bigcap_{n\le i} D_n$; these sets are still in the same list and $D'_{i+1} \subseteq D'_i$ for all i.

We are going to define an increasing sequence of natural numbers $0 = k_0 = \alpha_0 < \gamma_0 < k_1 < \alpha_1 < \gamma_1 < k_2 < \cdots$ and, for every i, a set $s_i \subseteq [\gamma_i, \alpha_{i+1})$ and also sets $w_i^u \subseteq [0, k_{i+1})$ for all $u \subseteq [0, \gamma_i)$, such that, for all generic $x \subseteq [\alpha_{i+1}, \infty)$ and all $u, v \subseteq [0, \gamma_i)$, we have

(1) $\nu((\vartheta(u \cup s_i \cup x) \,\triangle\, \vartheta(v \cup s_i \cup x)) \cap [k_{i+1}, \infty)) < 2^{-i}$;

(2) $\vartheta(u \cup s_i \cup x) \cap [0, k_{i+1}) = w_i^u$;

and in addition, for all i,

(3) if $u \subseteq [0, \gamma_i)$, then the condition $\langle \alpha_{i+1}, u \cup s_i\rangle$ belongs to D'_i;

(4) $k_{i+1} > 2k_i$;

(5) there is a set $g_i \subseteq [\alpha_i, \gamma_i)$ such that $|r_i - \sum_{n\in g_i} r_n| < 2^{-i}$.

DEFINITION 6.3.3 (The construction). The construction of $k_i, \alpha_i, \gamma_i, s_i, w_i^u$ goes on by induction on i. As $k_0 = \alpha_0 = 0$ are defined, we assume (the inductive hypothesis) that $i \in \mathbb{N}$ and $\alpha_i \in \mathbb{N}$ have been defined. The goal is to define γ_i, k_{i+1}, and other relevant sets, up to the number α_{i+1}.

Step 1. We begin with γ_i. Recall that $r_n \ge 0$, $r_n \to 0$, $\sum_n r_n = +\infty$ in Theorem 6.3.1. It follows that there is a finite set $g_i \subseteq \mathbb{N}$ such that $|r_i - \sum_{n\in g_i} r_n| < 2^{-i}$ and $\mathtt{min}\, g_i > \alpha_i$. Let γ_i be the least number $> \alpha_i$ satisfying $g_i \subseteq [\alpha_i, \gamma_i)$. We have fixed (5), by the way.

Step 2. Here we define numbers $k_{i+1} < \alpha_{i+1}$ (and $k_{i+1} > \gamma_i$). This is based on the next lemma:

LEMMA 6.3.4. *If $n \in \mathbb{N}$ and $i \in \mathbb{N}$, then there are numbers $n' > k > n$ and a set $s \subseteq [n, n')$ such that*

$$\nu((\vartheta(u \cup s \cup b) \,\triangle\, \vartheta(v \cup s \cup b)) \cap [k, \infty)) \;<\; 2^{-i}$$

holds for all $u, v \subseteq [0, n)$ and all generic $b \subseteq [n', \infty)$.

PROOF. Let $a \subseteq [n, \infty)$ be a generic set (that is, Cohen-generic over $\mathfrak{M}$). If u, $v \subseteq [0, n)$, then $(u \cup a) \,\triangle\, (v \cup a) = u \,\triangle\, v \in \mathscr{S}_{\{r_n\}}$; hence $\vartheta(u \cup a) \,\triangle\, \vartheta(v \cup a) \in \mathscr{J}$. Therefore, the union $U_n(a) = \bigcup_{u,v\subseteq[0,n)} \vartheta(u \cup a) \,\triangle\, \vartheta(v \cup a)$ belongs to $\mathscr{J}$. It follows, by the choice of ν, that there is a number $k > n$ such that

$$(*) \qquad \nu((U_n(a) \cap [k, \infty)) \;<\; 2^{-i}.$$

Let us denote the displayed formula by $\Phi_{ikn}(a, p)$. (The parameter $p \in \mathfrak{M} \cap \mathbb{N}^{\mathbb{N}}$ participates implicitly via occurrences of ϑ and ν.) Yet Φ_{ikn} is essentially a

Σ^1_1 formula with p as the only parameter. (It is an easy exercise to get a Σ^1_1 representation of Φ from Σ^1_1 and Π^1_1 definitions of ϑ and ν.) Therefore Φ_{ikn} is absolute for every transitive model of $\mathbf{ZFC}^-$ containing a and p by Theorem A.1.6, in particular, for $\mathfrak{M}[a]$. It follows that $\Phi_{ikn}(a,p)$ is true in the model $\mathfrak{M}[a] = \mathfrak{M}[G(a)]$, where $G(a) = \{p = \langle m, w\rangle \in \mathbb{C} : a \cap [0,m) = w\}$ is the Cohen-generic set associated with a; see Remark A.4.6.

Therefore, by Theorem A.3.3, there is a condition $p = \langle n', s\rangle \in G(a)$ which Cohen-forces $\Phi_{ikn}(a_{\underset{\sim}{G}}, \mathring{p})$ in the sense that $\Phi_{ikn}(b,p)$ is true in $\mathfrak{M}[b]$ whenewer $b \in \mathscr{P}(\mathbb{N})$ is a set Cohen-generic over $\mathfrak{M}$ and $b \cap [0, n') = s$. Now note that $a \cap [0,n') = s$ because $p \in G(a)$. On the other hand $a \subseteq [n, \infty)$. Therefore $s \subseteq [n,\infty)$. And now we can assume *w.l.o.g.* that $n' > k$ and $s \subseteq [n, n')$. (Indeed otherwise we can replace n' by $k+1$ anyway.)

We assert that the numbers $k < n'$ are as required. Indeed, consider an arbitrary generic $b \subseteq [n', \infty)$ and a pair of sets $u, v \subseteq [0,n)$. The set $b' = b \cup s$ is Cohen-generic as well, and by definition $b' \cap [0,n') = s$, so that $p \in G(b')$, and hence $\Phi_{ikn}(b', p)$ is true in the model $\mathfrak{M}[b] = \mathfrak{M}[b']$ and then in the universe of all sets as well by the absoluteness (see above). Thus

$$\nu((U_n(b') \cap [k,\infty)) < 2^{-i};$$

therefore, $\nu((\vartheta(u \cup b') \vartriangle \vartheta(v \cup b')) \cap [k,\infty)) < 2^{-i}$, as required. □ (Lemma)

Apply Lemma 6.3.4 with $n = \gamma_i$ to get numbers $k_{i+1} < \alpha_{i+1}$ (and $k_{i+1} > \gamma_i$) and a set $s_i \subseteq [\gamma_i, \alpha_{i+1})$ such that (1) holds for all $u, v \subseteq [0,\gamma_i)$, and all generic $x \subseteq [\alpha_{i+1}, \infty)$. Note that these initial values of k_{i+1}, α_{i+1}, s_i will be increased at the following Steps 3, 4, and 5.

Step 3. Increase k_{i+1} and α_{i+1} if necessary to fulfill (4).

Step 4. To fulfill (2) we need another auxiliary lemma.

LEMMA 6.3.5. *If $n < k < n'$ and $s \subseteq [n, n')$, then there exist a number $n'' > n'$, a set $s' \subseteq [n', n'')$, and for every $u \subseteq [0,n)$ a set $w_u \subseteq [0,k)$ such that $\vartheta(u \cup s \cup s' \cup b) \cap [0,k) = w_u$ for all $u \subseteq [0,n)$ and all generic $b \subseteq [n'', \infty)$.*

PROOF (Sketch). Consider an arbitrary generic set $a \subseteq [n', \infty)$. Let

$$w_u = \vartheta(u \cup s \cup a) \cap [0,k).$$

The formula $\bigwedge_{u \subseteq [0,n)} \big(w_u = \vartheta(u \cup s \cup a) \cap [0,k)\big)$ (a finite conjunction) is forced by a Cohen condition $p = \langle n'', s'\rangle \in G(a)$. Thus $s' = a \cap [0, n'')$, and hence we can assume that $n'' > n'$ and $s' \subseteq [n, n')$. In addition if $b' \subseteq [n', \infty)$ is a Cohen-generic set and $s' = b' \cap [0, n'')$, then $\bigwedge_{u \subseteq [0,n)} \big(w_u = \vartheta(u \cup s \cup b') \cap [0,k)\big)$. □ (Lemma)

To accomplish Step 4, apply the lemma with $n = \gamma_i$, $n' = \alpha_{i+1}$, $s = s_i$, and let the "new" α_{i+1} and s_i to be n'' and s', respectively.

Step 5. To fulfill (3), fix an enumeration $\mathscr{P}([0,\gamma_i)) = \{u_j : j < k\}$ where $k = 2^{\gamma_i}$. As the set D'_i is dense, there is a condition $\langle n_1, w_1\rangle \in D'_i$ stronger than $\langle \alpha_{i+1}, u_0 \cup s_i\rangle$. Clearly, $n_1 \geq n_0 = \alpha_{i+1}$ and $u_0 \cup s_i = w_1 \cap [0, \alpha_{i+1})$. Let $t_1 = w_1 \cap [\alpha_{i+1}, \infty)$. Thus, $t_1 \subseteq [\alpha_{i+1}, n_1)$ and the condition $\langle n_1, u_0 \cup s_i \cup t_1\rangle$ belongs to D'_i. Similarly, there is a number $n_2 \geq n_1$ and a set $t_2 \subseteq [n_1, n_2)$ such that the condition $\langle n_2, u_1 \cup s_i \cup t_1 \cup t_2\rangle$ belongs to D'_i. And so on. We obtain a sequence of numbers $\alpha_{i+1} = n_0 \leq n_1 \leq n_2 \leq \cdots \leq n_k$ and sets $t_\ell \subseteq [n_{\ell-1}, n_\ell)$

such that all conditions of the form $\langle n_\ell, u_\ell \cup s_i \cup t_1 \cup \cdots \cup t_\ell\rangle$ belong to D'_i. Let n_k be the "new" α_{i+1} and $s_i \cup t_1 \cup \cdots \cup t_k$ be the "new" s_i.

This ends the inductive step of the construction. □ (Definition 6.3.3)

The following is an immediate corollary of (3):

(3′) If $x \subseteq \mathbb{N}$ and $x \cap [\gamma_i, \alpha_{i+1}) = s_i$ for infinitely many i, then x is generic.

Indeed suppose that $x \in \mathscr{P}(\mathbb{N})$ and $x \cap [\gamma_i, \alpha_{i+1}) = s_i$ for infinitely many i. To see that x is generic, let $D \subseteq \mathbb{C}$, $D \in \mathfrak{M}$ be an arbitrary open dense set. Then $D = D_n$ for some n, and there is a number $i > n$ such that $x \cap [\gamma_i, \alpha_{i+1}) = s_i$. Then the set $w = x \cap [0, \alpha_{i+1})$ satisfies $w = u \cup s_i$, where $u = x \cap [0, \gamma_i)$; therefore, by definition the condition $p = \langle \alpha_{i+1}, w\rangle$ belongs to D'_i, and hence to $D = D_n$. And $x \cap [0, \alpha_{i+1}) = w$, as required.

Let us complete the proof of Theorem 6.3.1(i). It follows from (5) that the map $a \mapsto g_a = \bigcup_{i\in a} g_i$ $(a \subseteq \mathbb{N})$ is a reduction of $\mathscr{S}_{\{r_n\}}$ to $\mathscr{S}_{\{r_n\}} \restriction N$, where $N = \bigcup_i [\alpha_i, \gamma_i)$. Let $S = \bigcup_i s_i$; note that $S \cap N = \varnothing$.

Put $\xi(z) = \vartheta(z \cup S) \,\triangle\, \vartheta(S)$ for every $z \subseteq N$. Then, for all sets $x, y \subseteq N$,

$$x \,\triangle\, y \in \mathscr{S}_{\{r_n\}} \iff \vartheta(x \cup S) \,\triangle\, \vartheta(y \cup S) \in \mathscr{J} \iff \xi(x) \,\triangle\, \xi(y) \in \mathscr{J},$$

thus, ξ reduces $\mathscr{S}_{\{r_n\}} \restriction N$ to $\mathscr{J}$. Now put $w_i = \xi(g_i) \cap [k_i, k_{i+1})$ and $w_a = \bigcup_{i\in a} w_i$ for $a \in \mathscr{P}(\mathbb{N})$. We assert that the map $i \mapsto w_i$ proves $\mathscr{S}_{\{r_n\}} \leq^{++}_{\mathrm{RB}} \mathscr{J}$. In view of the above, it remains to show that $\xi(g_a) \,\triangle\, w_a \in \mathscr{J}$ for every $a \in \mathscr{P}(\mathbb{N})$.

As $\mathscr{J} = \mathtt{Exh}_\nu$, it suffices to demonstrate that $\nu(w_i \triangle (\xi(g_a) \cap [k_i, k_{i+1}))) < 2^{-i}$ for all $i \in a$, but $\nu(\xi(g_a) \cap [k_i, k_{i+1})) < 2^{-i}$ for $i \notin a$. After dropping the common term $\vartheta(S)$, it suffices to check that

(a) $\nu((\vartheta(g_i \cup S) \,\triangle\, \vartheta(g_a \cup S)) \cap [k_i, k_{i+1})) < 2^{-i}$ for all $i \in a$, but

(b) $\nu((\vartheta(S) \,\triangle\, \vartheta(g_a \cup S)) \cap [k_i, k_{i+1})) < 2^{-i}$ for $i \notin a$.

Note that every set of the form $x \cup S$, where $x \subseteq N$, is generic by (3′). It follows by (2) that we can assume, in (a) and (b), that $a \subseteq [0, i]$, *i.e.*, resp. $\max a = i$ and $\max a < i$. We can finally apply (1), with $u = a \cup \bigcup_{j<i} s_j$, $x = \bigcup_{j>i} s_j$, and $v = u_i \cup \bigcup_{j<i} s_j$ if $i \in a$ while $v = \bigcup_{j<i} s_j$ if $i \notin a$.

(ii) Otherwise, $\mathscr{S}_{\{r_n\}} \leq^{++}_{\mathrm{RB}} \mathscr{Z}_0$ exponentially by (i). Let this be witnessed by $i \mapsto w_i$ and a sequence of numbers k_i, so that $k_{i+1} \geq 2k_i$ and $w_i \subseteq [k_i, k_{i+1})$. If $d_i = \frac{\mathtt{card}\,(w_i)}{k_{i+1}} \to 0$, then easily $\bigcup_i w_i \in \mathscr{Z}_0$ by the choice of $\{k_i\}$. Otherwise, there is a set $a \in \mathscr{S}_{\{r_n\}}$ such that $d_i > \varepsilon$ for all $i \in a$ and one and the same $\varepsilon > 0$, so that $w_a = \bigcup_{i\in a} w_i \notin \mathscr{Z}_0$. In both cases we have a contradiction with the assumption that the map $i \mapsto w_i$ witnesses $\mathscr{S}_{\{r_n\}} \leq^{++}_{\mathrm{RB}} \mathscr{Z}_0$. □ (Theorem 6.3.1)

QUESTION 6.3.6. FARAH [**Far99**] points out that Theorem 6.3.1(i) also holds for the ideal $\mathscr{I}_3$ (instead of $\mathscr{S}_{\{r_n\}}$) and asks for which other ideals it is true. □

6.4. How to eliminate forcing

The goal of this section is to explain how forcing can be avoided in proofs such as the proof of Theorem 6.3.1. The forcing proofs of this kind become fully available after such a correction for those who are new to forcing, but accordingly somewhat more awkward for the practitioners of forcing.

The use of forcing in the proof of Theorem 6.3.1 amounts roughly to the following two principles:

1) There exists a dense $\mathbf{G}_\delta$ set $G \subseteq \mathscr{P}(\mathbb{N})$ (or $G \subseteq 2^{\mathbb{N}}$ if we identify $\mathscr{P}(\mathbb{N})$ with $2^{\mathbb{N}}$) whose elements are called generic;

2) If $x \in \mathscr{P}(\mathbb{N})$ is generic, $\varphi(x)$ a formula of some sort, and the sentence $\varphi(x)$ is true, then it is "forced" in the sense that there exists a number n such that $\varphi(x')$ is true for all generic x' such that $x' \cap [0,n) = x \cap [0,n)$.

In topological terms, 2) means that a certain function (2-valued in the example considered) defined on $\mathscr{P}(\mathbb{N})$ is continuous on G, a suitably dense $\mathbf{G}_\delta$ set. This gives us a clue as how to eliminate forcing. Namely we have to define a dense $\mathbf{G}_\delta$ set $G \subseteq \mathscr{P}(\mathbb{N})$ on which all maps relevant to the arguments are continuous. Since the number of maps considered is at most countable in such cases, the existence of a set required is a consequence of the following classical result:

PROPOSITION 6.4.1. *If $\mathbb{X}, \mathbb{Y}$ are Polish spaces and $F : \mathbb{X} \to \mathbb{Y}$ a Borel map, then there is a dense $\mathbf{G}_\delta$ set $G \subseteq \mathbb{X}$ such that $F \restriction D$ is continuous.* □

In the proof of Theorem 6.3.1(i), the family of Borel maps to be made continuous contains the following maps:

(1) all maps $f_{nk} : \mathscr{P}(\mathbb{N}) \to \mathbb{R}$ of the form $f_{nk}(a) = \nu(U_n(a) \cap [k,\infty))$ (see the proof of Lemma 6.3.4);

(2) all maps $f_k : \mathscr{P}(\mathbb{N}) \to \mathbb{R}$ of the form $f_{uk}(a) = \vartheta(u \cup a) \cap [0,k)$ (where $k \in \mathbb{N}$ and $u \subseteq \mathbb{N}$ is finite, see the proof of Lemma 6.3.5).

After the choice of a dense $\mathbf{G}_\delta$ set $G \subseteq \mathbb{X}$ on which all those maps are continuous, the proof of Theorem 6.3.1(i) goes on with the understanding of "generic" as an "element of D". There are some easy and rather clear changes, of course. For instance, to define the sequence of sets $D_n \subseteq \mathscr{P}(\mathbb{N})$ for (3′), we let $G = \bigcap_n G_n$, where all G_n are (topologically) open dense in $\mathscr{P}(\mathbb{N})$, and then we put

$$D_n = \{p = \langle n, u \rangle \in \mathbb{C} : \forall a \in \mathscr{P}(\mathbb{N})\, (a \cap n = u \Longrightarrow a \in G_n)\}.$$

The awkwardness of such a modification can be seen in the fact that the family of functions to be made continuous is determined *a posteriori*, that is, after the proof is essentially outlined. The forcing setup defines this family *a priori*, as all maps are coded in a chosen model of $\mathbf{ZFC}^-$.

6.5. The family ℓ^p

The next theorem of DOUGHERTY and HJORTH [**DH99**] shows that Borel reducibility between equivalence relations ℓ^p is fully determined by the value of p.

THEOREM 6.5.1. *If $1 \le p < q < \infty$, then $\ell^p <_{\mathrm{B}} \ell^q$.*

PROOF. **Part 1**: show that $\ell^q \not\le_{\mathrm{B}} \ell^p$.

By Lemma 6.2.2, it suffices to prove that $\ell^q \restriction \mathbb{X} \not\le_{\mathrm{B}} \ell^p \restriction \mathbb{X}$. Suppose, on the contrary, that $\vartheta : \mathbb{X} \to \mathbb{X}$ is a Borel reduction of $\ell^q \restriction \mathbb{X}$ to $\ell^p \restriction \mathbb{X}$. Arguing as in the proof of Theorem 6.3.1, we can reduce the general case to the case when there exist increasing sequences of numbers $0 = j(0) < j(1) < j(2) < \cdots$ and $0 = a_0 < a_1 < a_2 < \cdots$ and a map $\tau : \mathbb{Y} \to \mathbb{X}$, where $\mathbb{Y} = \prod_{n=0}^{\infty} X_{j(n)}$, which reduces $\ell^q \restriction \mathbb{Y}$ to $\ell^p \restriction \mathbb{X}$ and has the form $\tau(x) = \bigcup_{n\in\mathbb{N}} t_n^{x(n)}$, where $t_n^r \in \prod_{k=a_n}^{a_{n+1}-1} X_k$ for every $r \in X_{j_n}$ (see Definition 6.2.1).

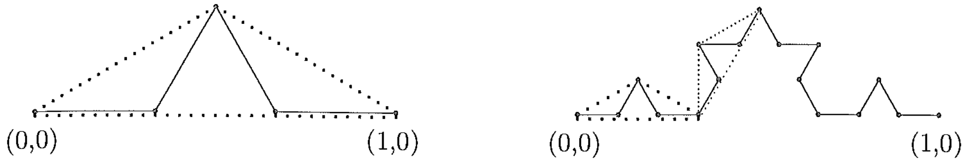

FIGURE 2. $r = \frac{1}{3}$: Left, step 1; Right, step 2.

Case 1. There are $c > 0$ and a number N such that $\|\tau_n^1 - \tau_n^0\|_p \geq c$ for all $n \geq N$. Since $p < q$, there is a non-decreasing sequence of natural numbers $i_n \leq j_n$, $n = 0, 1, 2, \ldots$, such that $\sum_n 2^{p(i_n - j_n)}$ diverges but $\sum_n 2^{q(i_n - j_n)}$ converges. *Hint*: $i_n \approx j_n - p^{-1} \log_2 n$.

Now consider an arbitrary $n \geq N$. As $\|\tau_n^1 - \tau_n^0\|_p \geq c$ and because $\|\ldots\|_p$ is a norm, there exists a pair of rationals $u(n) < v(n)$ in X_{j_n} with $v(n) - u(n) = 2^{i_n - j_n}$ and $\|\tau_n^{v(n)} - \tau_n^{u(n)}\|_p \geq c\, 2^{i_n - j_n}$. In addition, put $u(n) = v(n) = 0$ for $n < N$. Then the ℓ^q-distance between the infinite sequences $u = \{u(n)\}_{n \in \mathbb{N}}$ and $v = \{v(n)\}_{n \in \mathbb{N}}$ is equal to $\sum_{n=N}^{\infty} 2^{q(i_n - j_n)} < +\infty$, while the ℓ^p-distance between $\tau(u)$ and $\tau(v)$ is non-smaller than $\sum_{n=N}^{\infty} c^p\, 2^{p(i_n - j_n)} = \infty$. But this contradicts the assumption that τ is a reduction.

Case 2. Otherwise. Then there is a strictly increasing sequence $n_0 < n_1 < n_2 < \cdots$ with $\|\tau_{n_k}^1 - \tau_{n_k}^0\|_p \leq 2^{-k}$ for all k. Now let $x \in \mathbb{Y}$ be the constant 0 while $y \in \mathbb{Y}$ is defined by $y(n_k) = 1, \forall k$ and $y(n) = 0$ for all other n. Then $x\ \ell^q\ y$ fails ($|y(n) - x(n)| \not\to 0$) but $\tau(x)\ \ell^p\ \tau(y)$ holds; a contradiction.

Part 2. Show that $\ell^p \leq_{\mathrm{B}} \ell^q$.

It suffices to prove that $\ell^p \restriction [0,1]^{\mathbb{N}} \leq_{\mathrm{B}} \ell^q$ (Lemma 6.2.2). We assume *w.l.o.g.* that $q < 2p$: any bigger q can be approached in several steps. For $\vec{x} = \langle x, y \rangle \in \mathbb{R}^2$, let $\|\vec{x}\|_h = (x^h + y^h)^{1/h}$.

LEMMA 6.5.2. *For every* $\frac{1}{2} < \alpha < 1$ *there is a continuous map* $K = K_\alpha : [0,1] \to [0,1]^2$ *and positive real numbers* $m < M$ *such that for all* $x < y$ *in* $[0,1]$ *we have* $m(y - x)^\alpha \leq \|K_\alpha(y) - K_\alpha(x)\|_2 \leq M(y - x)^\alpha$.

PROOF (Lemma). The construction of such a map K can be easier described in terms of fractal geometry rather than by an analytic expression. Let $r = 4^{-\alpha}$, so that $\frac{1}{4} < r < \frac{1}{2}$ and $\alpha = -\log_4 r$. Starting with the segment $[(0,0), (1,0)]$ of the horizontal axis of the cartesian plane, we replace it by four smaller segments each of length r (thin lines on Figure 2, left). Each of these we replace by four segments of length r^2 (thin lines on Figure 2, right). And so on, infinitely many steps. The resulting curve K is parametrized by giving the vertices of the polygons values equal to multiples of 4^{-n}, n being the number of the polygon. For instance, the vertices of the left polygon on Figure 2 are given values $0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$.

Note that the curve $K : [0,1] \to [0,1]^2$, approximated by the polygons, is bounded by certain triangles built on the sides of the polygons. For instance, the whole curve lies inside the triangle bounded by dotted lines in Figure 2, left. (The dotted line that follows the basic side $[(0,0), (1,0)]$ of the triangle is drawn slightly below its true position.) Further, the parts $0 \leq t \leq \frac{1}{4}$ and $\frac{1}{4} \leq t \leq \frac{1}{2}$ of the curve lie inside the triangles bounded by (slightly different) dotted lines in Figure 2, right. And so on. Let us call those triangles *bounding triangles*.

To prove the inequality of the lemma, consider any pair of reals $x < y \in [0,1]$. Let n be the least number such that x, y belong to non-adjacent intervals, resp., $[\frac{i-1}{4^n}, \frac{i}{4^n}]$ and $[\frac{j-1}{4^n}, \frac{j}{4^n}]$, with $j > i+1$. Then $4^{-n} \le |y-x| \le 8 \cdot 4^{-n}$.

The points $K(x)$ and $K(y)$ then belong to one and the same side or adjacent sides of the $n-1$-th polygon. Let C be a common vertice of these sides. It is quite clear geometrically that the euclidean distances from $K(x)$ and $K(y)$ to C do not exceed r^{n-1} (the length of the side), thus $\|K(x) - K(y)\|_2 \le 2\, r^{n-1}$.

Estimation from below needs more work. The points $K(x)$, $K(y)$ belong to the bounding triangles built on the segments, resp., $[K(\frac{i-1}{4^n}), K(\frac{i}{4^n})]$ and $[K(\frac{j-1}{4^n}), K(\frac{j}{4^n})]$, and obviously $i+1 < j \le i+8$, so that there exist at most six bounding triangles between these two. Note that adjacent bounding triangles meet each other at only two possible angles (that depend on r but not on n). Taking it as geometrically evident that non-adjacent bounding triangles are disjoint, we conclude that there is a constant $c > 0$ (that depends on r but not on n) such that the distance between two non-adjacent bounding triangles of rank n, having at most six bounding triangles of rank n between them, does not exceed $c \cdot r^n$. In particular, $\|K(x) - K(y)\|_2 \ge c \cdot r^n$. Combining this with the inequalities above, we conclude that $m(y-x)^\alpha \le \|K(y) - K(x)\|_2 \le M(y-x)^\alpha$, where $m = \frac{c}{8^\alpha}$ and $M = \frac{2}{r}$ (and $\alpha = -\log_4 r$). □ (Lemma)

Coming back to the theorem, let $\alpha = p/q$, and let K_α be as in the lemma. Let $x = \langle x_0, x_1, x_2, \dots \rangle \in [0,1]^{\mathbb{N}}$. Then $K_\alpha(x_i) = \langle x_i', x_i'' \rangle \in [0,1]^2$. We put $\vartheta(x) = \langle x_0', x_0'', x_1', x_1'', x_2', x_2'', \dots \rangle$. Prove that ϑ reduces $\ell^p \restriction [0,1]^{\mathbb{N}}$ to ℓ^q.

Let $x = \{x_i\}_{i\in\mathbb{N}}$ and $y = \{y_i\}_{i\in\mathbb{N}}$ belong to $[0,1]^{\mathbb{N}}$. We have to prove that $x \,\ell^p\, y$ iff $\vartheta(x) \,\ell^q\, \vartheta(y)$. To simplify the picture, note the following:

$$2^{-1/2}\|w\|_2 \le \max\{w', w''\} \le \|w\|_q \le \|w\|_1 \le 2\|w\|_2$$

for every $w = \langle w', w'' \rangle \in \mathbb{R}^2$. The task takes the following form:

$$\sum_i (x_i - y_i)^p < \infty \iff \sum_i \|K_\alpha(x_i) - K_\alpha(y_i)\|_2{}^q < \infty\,.$$

Furthermore, by the choice of K_α, this converts to

$$\sum_i (x_i - y_i)^p < \infty \iff \sum_i (x_i - y_i)^{\alpha q} < \infty\,,$$

which holds because $\alpha q = p$. □ (Theorem 6.5.1)

6.6. ℓ^∞: maximal $\mathbf{K}_\sigma$

Recall that $\mathbf{K}_\sigma$ denotes the class of all σ-compact sets in Polish spaces. Easy computations show that this class contains, among others, the equivalence relations E_1, E_∞, ℓ^p, $1 \le p \le \infty$, considered to be sets of pairs in the corresponding Polish spaces. Note that if E is a $\mathbf{K}_\sigma$ equivalence relation on a Polish space $\mathbb{X}$, then $\mathbb{X}$ is $\mathbf{K}_\sigma$ as well, since projections of compact sets are compact. Thus $\mathbf{\Sigma}^0_2$ equivalence relations on $\mathbf{K}_\sigma$ sets in Polish spaces are $\mathbf{K}_\sigma$ equivalence relations.

THEOREM 6.6.1. *Every $\mathbf{K}_\sigma$ equivalence relation on a Polish space, in particular, E_1, E_∞, ℓ^p, is Borel reducible to ℓ^∞.*[2]

PROOF (from ROSENDAL [**Ros05**]). Let $\mathbb{A}$ be the set of all $\subseteq$-increasing sequences $a = \{a_n\}_{n\in\mathbb{N}}$ of subsets $a_n \subseteq \mathbb{N}$—a closed subset of the Polish space $\mathscr{P}(\mathbb{N})^{\mathbb{N}}$. Define an equivalence relation H on $\mathbb{A}$ by

$$\{a_n\}\,\mathsf{H}\,\{b_n\} \quad \text{iff} \quad \exists N\,\forall m\,(a_m \subseteq b_{N+m} \wedge b_m \subseteq a_{N+m}).$$

The theorem is a consequence of the following two claims:

CLAIM 6.6.2. $\mathsf{H} \le_{\mathrm{B}} \ell^\infty$.

PROOF. This is easy. Given a sequence $a = \{a_n\}_{n\in\mathbb{N}} \in \mathscr{P}(\mathbb{N})^{\mathbb{N}}$, define $\vartheta(a) \in \mathbb{N}^{\mathbb{N}\times\mathbb{N}}$ by $\vartheta(a)(n,k)$ to be the least $j \le k$ such that $n \in a_j$, or $\vartheta(a)(n,k) = k$ whenever $n \notin a_k$. Then $\{a_n\}\,\mathsf{H}\,\{b_n\}$ iff there is N such that $|\vartheta(a)(n,k) - \vartheta(b)(n,k)| \le N$ for all n, k. □

CLAIM 6.6.3. *Every $\mathbf{K}_\sigma$ equivalence relation E on a Polish space $\mathbb{X}$ is Borel reducible to H.*

PROOF. We have $\mathsf{E} = \bigcup_n E_n$, where each E_n is a compact subset of $\mathbb{X}\times\mathbb{X}$ (not necessarily an equivalence relation) and $E_n \subseteq E_{n+1}$. We can assume *w.l.o.g.* that each E_n is reflexive and symmetric on its domain $D_n = \operatorname{dom} E_n = \operatorname{ran} E_n$ (a compact set), in particular, $x \in D_n \Longrightarrow \langle x, x\rangle \in E_n$. Define $P_0 = E_0$ and

$$P_{n+1} = P_n \cup E_{n+1} \cup P_n^{(2)}, \quad \text{where } P_n^{(2)} = \{\langle x,y\rangle : \exists z\,(\langle x,z\rangle \in P_n \wedge \langle z,y\rangle \in P_n)\},$$

by induction. Thus all P_n are still compact subsets of $\mathbb{X}\times\mathbb{X}$, moreover, of E since E is an equivalence relation, and $E_n \subseteq P_n \subseteq P_{n+1}$; therefore $\mathsf{E} = \bigcup_n P_n$.

Let $\{U_k : k \in \mathbb{N}\}$ be a basis for the topology of $\mathbb{X}$. Put, for any $x \in \mathbb{X}$, $\vartheta_n(x) = \{k : U_k \cap R_n(x) \ne \varnothing\}$, where $R_n(x) = \{y : \langle x,y\rangle \in R_n\}$. Then obviously $\vartheta_n(x) \subseteq \vartheta_{n+1}(x)$, and hence $\vartheta(x) = \{\vartheta_n(x)\}_{n\in\mathbb{N}} \in \mathbb{A}$. Then ϑ reduces E to H.

Indeed if $x\,\mathsf{E}\,y$, then $\langle y,x\rangle \in P_n$ for some n, and for all m and $z \in \mathbb{X}$ we have $\langle x,z\rangle \in R_m \Longrightarrow \langle y,z\rangle \in R_{1+\max\{m,n\}}$. In other words, $R_m(x) \subseteq R_{1+\max\{m,n\}}(y)$ and hence $\vartheta_m(x) \subseteq \vartheta_{1+\max\{m,n\}}(y)$ hold for all m. Similarly, for some n' we have $\vartheta_m(y) \subseteq \vartheta_{1+\max\{m,n'\}}(y)$, $\forall m$. Thus $\vartheta(x)\,\mathsf{H}\,\vartheta(y)$.

Conversely, suppose that $\vartheta(x)\,\mathsf{H}\,\vartheta(y)$. Thus, for some N, we have $R_m(x) \subseteq R_{N+m}(y)$ and $R_m(y) \subseteq R_{N+m}(x)$ for all m and y. Taking m big enough for P_m to contain $\langle x,x\rangle$, we obtain $x \in R_{N+m}(y)$, so that immediately $x\,\mathsf{E}\,y$. □

□ (Theorem 6.6.1)

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[2] The result for ℓ^p is due to SU GAO [**Gao98**]. He defines

$$d_p(x,s) = \left(\sum_{k=0}^{\operatorname{lh} s-1} |x(k) - s(k)|^p\right)^{\frac{1}{p}}$$

for all $x \in \mathbb{R}^{\mathbb{N}}$ and $s \in \mathbb{Q}^{<\omega}$ (a finite sequence of rationals). One easily proves that the ℓ^p-distance $(\sum_{k=0}^{\infty} |x(k) - y(k)|^p)^{\frac{1}{p}}$ between any pair of $x, y \in \mathbb{R}^{\mathbb{N}}$ is finite iff there is a constant C such that $|d_p(x,s) - d_p(y,s)| < C$ for all $s \in \mathbb{Q}^{<\omega}$. This yields the reduction required.

CHAPTER 7

Introduction to countable equivalence relations

This class of Borel equivalence relations was discovered in the 1970s in the framework of ergodic theory. In spite of significant progress, especially in problems related to interconnections between countable equivalence relations, group actions, and probability measures, the Borel reducibility structure within this class remained obscure until the very end of the 1990s. Since then, many remarkable results related to Borel countable equivalence relations have been obtained, and we give a brief review of them in Chapter 9.

This chapter is devoted to several basic results. After notational remarks in Section 7.1, we present a few rather simple results on smooth and hyperfinite equivalence relations in Section 7.2. We then prove (Theorem 7.3.1) that, given a countable equivalence relation F satisfying $\mathbb{N}\mathsf{F} \le_{\mathrm{B}} \mathsf{F}$, the property of "being Borel reducible to F" is σ-additive as a property of Borel domains; this will simplify several reducibility/irreducibility proofs below. In Section 7.4 we prove that every countable Borel equivalence relation is induced by a Borel action of a countable group and is Borel reducible to E_∞; hence the whole domain $\le_{\mathrm{B}} \mathsf{E}_\infty$ is equal to the class of essentially countable Borel equivalence relations. The equivalence relation E_∞ turns out to be (countable but) non-hyperfinite by Theorem 7.5.1. (Hyperfinite equivalence relations are those E satisfying $\mathsf{E} \le_{\mathrm{B}} \mathsf{E}_0$; we consider this subclass in Chapter 8.) It follows that $\mathsf{E}_0 <_{\mathrm{B}} \mathsf{E}_\infty$ strictly. We finish with a sufficient condition of essential countability in Lemma 7.6.1.

7.1. Several types of equivalence relations

The following types of equivalence relations are relevant to the domain of countable equivalence relations. A Borel equivalence relation E on a (Borel) set X is:

countable: if every E-class $[x]_{\mathsf{E}} = \{y \in X : x \mathrel{\mathsf{E}} y\}$, $x \in X$, is at most countable;

essentially countable: if $\mathsf{E} \le_{\mathrm{B}} \mathsf{F}$, where F is a countable Borel equivalence relation;

finite: if every E-class $[x]_{\mathsf{E}} = \{y \in X : x \mathrel{\mathsf{E}} y\}$, $x \in X$, is finite;

of type n: if every E-class $[x]_{\mathsf{E}}$, $x \in X$, contains at most n elements;

hyperfinite: if $\mathsf{E} = \bigcup_n \mathsf{F}_n$ for an increasing sequence of Borel finite equivalence relations F_n;

smooth: if $\mathsf{E} \le_{\mathrm{B}} \Delta_{2^{\mathbb{N}}}$;

hypersmooth: if $\mathsf{E} = \bigcup_n \mathsf{F}_n$ for an increasing sequence of smooth equivalence relations F_n.

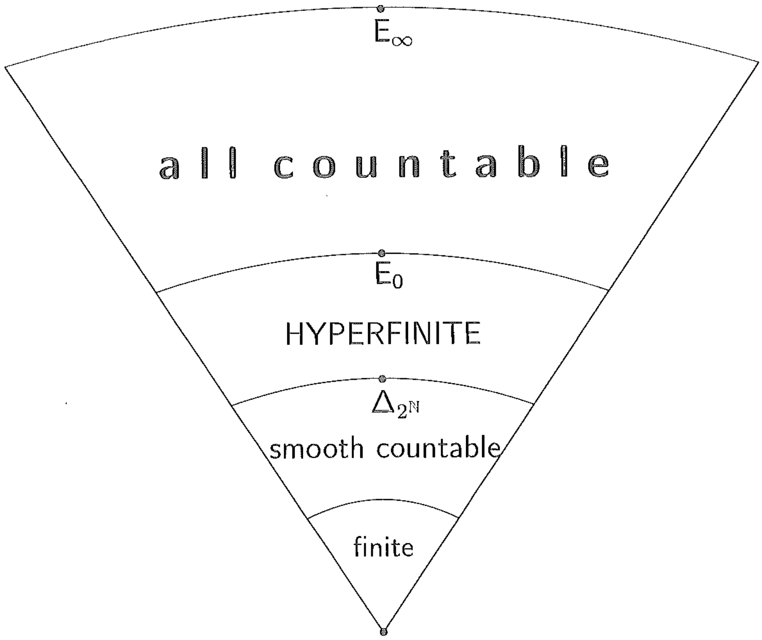

FIGURE 3. finite $\subsetneqq$ smooth countable $\subsetneqq$ hyperfinite $\subsetneqq$ countable

By quite obvious reasons we have

$$\text{type } n \subseteq \text{finite} \subseteq \text{hyperfinite} \subseteq \text{countable} \subseteq \text{essentially countable.}$$

On the other hand, smooth $\subseteq$ hypersmooth, and we will see that finite $\subseteq$ smooth, and hence hyperfinite $\subseteq$ hypersmooth. Every smooth equivalence relation E is "essentially hyperfinite", in the sense that $\mathsf{E} \le_{\mathrm{B}} \mathsf{F}$ for some hyperfinite equivalence relation F (we can take $\mathsf{F} = \Delta_{2^{\mathbb{N}}}$ here), and if E is in addition countable, then it is really hyperfinite.

Figure 3 displays these relationships. E_∞ is the $\le_{\mathrm{B}}$-largest among all Borel countable equivalence relations by Theorem 7.4.1, while E_0 is the $\le_{\mathrm{B}}$-largest among all hyperfinite equivalence relations by Theorem 8.1.1. The strictness of the inclusions displayed follows from Theorem 7.5.1 (E_∞ is countable non-hyperfinite) and Proposition 7.2.1 (E_0 is hyperfinite but non-smooth), and to get a Borel countable smooth equivalence relation with infinite classes, define E on $\mathbb{N}^{\mathbb{N}}$ so that $a \mathrel{\mathsf{E}} b$ iff $a(k) = b(k)$ for all $k \ge 1$ (but $a(0)$ may not be equal to $b(0)$).

Equivalence relations that are simultaneously countable and hypersmooth also belong to the hyperfinite class by Theorem 8.1.1 below. See Chapter 8 for more on hyperfinite equivalence relations.

7.2. Smooth and below

By definition (see Section 7.1) an equivalence relation E is smooth iff there is a Borel map $\vartheta : X \to 2^{\mathbb{N}}$ such that the equivalence $x \mathrel{\mathsf{E}} y \iff \vartheta(x) = \vartheta(y)$ holds for all $x, y \in X = \mathtt{dom}\,\mathsf{E}$. In other words, it is required that the equivalence classes can be counted by reals (here, elements of $2^{\mathbb{N}}$) in a Borel way. An important subspecies of smooth equivalence relations consists of those having a Borel *transversal*: a set with exactly one element in every equivalence class.

PROPOSITION 7.2.1. *Assume that* E *is a Borel equivalence relation on a Borel set* X *in a Polish space. Then*:

(i) *if* E *has a Borel transversal, then it is smooth*;

(ii) *if* E *is finite (i.e., with finite classes), then it admits a Borel transversal*;

(iii) *if* E *is countable and smooth, then it admits a Borel transversal*;

(iv) *if every* E*-class is a closed set and the saturation* $[\mathscr{O}]_{\mathsf{E}}$ *of every open set* $\mathscr{O} \subseteq \mathbb{X}$ *is Borel, then* E *admits a Borel transversal, hence, it is smooth.*[1]

In addition,

(v) *the equivalence relation* E_0 *is not smooth*;

(vi) *we have* $\Delta_{2^{\mathbb{N}}} <_{\mathrm{B}} \mathsf{E}_0$ *strictly, thus by Theorem* 3.2.1(i) $\Delta_{2^{\mathbb{N}}}$ *is not* $\sim_{\mathrm{B}}$*-equivalent to an equivalence relation of the form* $\mathsf{E}_{\mathscr{I}}$, *where* $\mathscr{I}$ *is a Borel ideal on* $\mathbb{N}$;

(vii) *there exists a smooth equivalence relation* E *without a Borel transversal.*

PROOF. (i) Let T be a Borel transversal for E. The map $\vartheta(x) =$ "the only element of T E-equivalent to x" reduces E to Δ_T. To see that ϑ is Borel, note that $\vartheta(x) = y \iff x \in X \wedge y \in T \wedge x \mathrel{\mathsf{E}} y$.

(ii) Consider the set

$$T = \{x \in X : \forall\, y \in X\, (x \mathrel{\mathsf{E}} y \Longrightarrow x \le y)\}$$

of the $\le$-least elements of E-classes, where $\le$ is a fixed Borel linear order on the domain of E. That T is Borel follows from Corollary 2.12.2. Indeed let $P = \{\langle x, y\rangle : x \mathrel{\mathsf{E}} y \wedge y < x\}$. This is a Borel set: $y < x$ is equivalent to $y \le x \wedge \neg\, x \le y$. And all its cross-sections are finite since so are the E-classes. Therefore, its projection

$$Y = \{x \in X : \exists\, y\, (x \mathrel{\mathsf{E}} y \wedge y < x)\}$$

is Borel by Corollary 2.12.2. And finally $T = X \setminus Y$.

(iii) Use Theorem 2.12.5 (Countable-to-1 Uniformization).

(iv) Since every Borel set is a continuous image of $\mathbb{N}^{\mathbb{N}}$, we can assume that E is an equivalence relation on $\mathbb{N}^{\mathbb{N}}$. Then, for every $x \in \mathbb{N}^{\mathbb{N}}$, the equivalence class $[x]_{\mathsf{E}}$ is a closed subset of $\mathbb{N}^{\mathbb{N}}$, naturally identified with a tree, say, $T_x \subseteq \mathbb{N}^{<\omega}$. Let $\vartheta(x)$ denote the left-most branch of T_x. Then $x \mathrel{\mathsf{E}} \vartheta(x)$ and $x \mathrel{\mathsf{E}} y \Longrightarrow \vartheta(x) = \vartheta(y)$, so that it remains to show that $Z = \{\vartheta(x) : x \in \mathbb{N}^{\mathbb{N}}\}$ is Borel. Note that

$$z \in Z \iff \forall\, m\, \forall\, s, t \in \mathbb{N}^m\, (s <_{\mathsf{lex}} t \wedge z \in \mathscr{O}_t \Longrightarrow [z]_{\mathsf{E}} \cap \mathscr{O}_t = \varnothing),$$

where $<_{\mathsf{lex}}$ is the lexicographical order on $\mathbb{N}^m$ and $\mathscr{O}_s = \{x \in \mathbb{N}^{\mathbb{N}} : s \subset x\}$. However, $[x]_{\mathsf{E}} \cap \mathscr{O}_t = \varnothing$ iff $x \notin [\mathscr{O}_t]_{\mathsf{E}}$ and $[\mathscr{O}_t]_{\mathsf{E}}$ is Borel for every t.

(v) Otherwise, E_0 has a Borel transversal T by (iii), which is a contradiction by Lemma 4.4.3.

[1] SRIVASTAVA [**Sri81**] proved the result for equivalence relations with $\mathbf{G}_\delta$ classes. This is the best possible as E_0 is a Borel equivalence relation, whose classes are $\mathbf{F}_\sigma$ and saturations of open sets are even open, but have no Borel transversal. See also [**Kec95**, 18.20 iv].

(vi) The fact that $\Delta_{2^{\mathbb{N}}} \le_{\mathrm{B}} \mathsf{E}_0$ is witnessed by any perfect set $X \subseteq 2^{\mathbb{N}}$ which is a *partial* transversal for E_0 (*i.e.*, every $x \ne y$ in X are E_0-inequivalent). On the other hand, $\Delta_{2^{\mathbb{N}}}$ is smooth but E_0 is non-smooth by (v).

(vii) By Corollary 2.5.6 there is closed set $P \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ with $\mathtt{dom}\, P = \mathbb{N}^{\mathbb{N}}$, not uniformizable by a Borel set. Define $\langle x, y\rangle \mathrel{\mathsf{E}} \langle x', y'\rangle$ iff both $\langle x, y\rangle$ and $\langle x', y'\rangle$ belong to P and $x = x'$. □

7.3. Assembling countable equivalence relations

Here we establish a result that will be used in the proof of Theorem 8.1.1 and some other theorems below. It shows that in certain cases the notion of being Borel reducible to a given countable Borel equivalence relation is σ-additive.

THEOREM 7.3.1. *Let* F *be a countable Borel equivalence relation satisfying* $\mathbb{N}\mathsf{F} \le_{\mathrm{B}} \mathsf{F}$, *and let* E *be a Borel equivalence relation on a Borel set* $X = \bigcup_k X_k$ *with all* X_k *also Borel. Suppose that* $\mathsf{E} \restriction X_k \le_{\mathrm{B}} \mathsf{F}$ *for each* k. *Then* $\mathsf{E} \le_{\mathrm{B}} \mathsf{F}$.

The product $\mathbb{N}\mathsf{F}$ means the union of countably many Borel isomorphic copies of F defined on pairwise disjoint and F-disconnected Borel sets (in one and the same Polish space). For instance if $\mathtt{dom}\, \mathsf{F} = X$, then $\mathbb{N}\mathsf{F}$ can be defined on the cartesian product $\mathbb{N} \times X$ so that $\langle k, x\rangle \mathrel{\mathbb{N}\mathsf{F}} \langle n, y\rangle$ iff $k = n$ and $x \mathrel{\mathsf{F}} y$.

PROOF. It obviously suffices to prove that if E is a Borel equivalence relation defined on the union $X \cup Y$ of disjoint Borel sets X and Y, F is a countable Borel equivalence relation defined on the union $P \cup Q$ of disjoint Borel sets P and Q, F-disconnected in the sense that $p \not\mathrel{\mathsf{F}} q$ for all $p \in P$, $q \in Q$, and f, g are Borel reductions of resp. $\mathsf{E} \restriction X$, $\mathsf{E} \restriction Y$ to resp. $\mathsf{F} \restriction P$, $\mathsf{F} \restriction Q$, then there is a Borel reduction h of E to F such that $h \restriction X = f$.

As X, Y are *not* assumed to be necessarily E-disconnected, the key problem is to define $h(y)$ in the case when $y \in Y$ satisfies $g(y) \in \mathtt{ran}\, U$, where

$$U = \{\langle p, q\rangle \in P \times Q : \exists x \in X\, \exists y \in Y\, (x \mathrel{\mathsf{E}} y \wedge f(x) = p \wedge f(y) = q)\}$$

is a Σ^1_1 set. As f, g are reductions to F, U is a subset of the $\boldsymbol{\Pi}^1_1$ set

$$W = \{\langle p, q\rangle \in P \times Q : \forall \langle p', q'\rangle \in U\, (p \mathrel{\mathsf{F}} p' \iff q \mathrel{\mathsf{F}} q')\}.$$

Therefore by Separation (Theorem 2.3.3) there exists an intermediate Borel set V such that $U \subseteq V \subseteq W$.

The set U is 1-to-1 modulo F in the sense that the equivalence $p \mathrel{\mathsf{F}} p' \iff q \mathrel{\mathsf{F}} q'$ holds for any two pairs $\langle p, q\rangle$ and $\langle p', q'\rangle$ in U. The set V does not necessarily have this property. To obtain a Borel subset of V and a still superset of U, 1-to-1 modulo F, note that U is a subset of the $\boldsymbol{\Pi}^1_1$ set

$$R = \{\langle p', q'\rangle \in V : \forall \langle p, q\rangle \in V\, (p \mathrel{\mathsf{F}} p' \iff q \mathrel{\mathsf{F}} q')\}.$$

It follows that there exists a Borel set S with $U \subseteq S \subseteq R$. Clearly, S is 1-to-1 modulo F together with R. Since F is a countable equivalence relation, it follows by Theorems 2.12.1 and 2.12.3 (Countable-to-1 Projection and Countable-to-1 Enumeration) that the set $Z = \mathtt{ran}\, S$ is Borel and there is a Borel map $\vartheta : Z \to P$ such that $\langle \vartheta(q), q\rangle \in S$ for every $q \in Z$.

In particular, we have $\mathtt{ran}\, U \subseteq Z$ and $p \mathrel{\mathsf{F}} \vartheta(q)$ for all pairs $\langle p, q\rangle \in U$. In addition, it can be assumed *w.l.o.g.* that Z is F-invariant, *i.e.* $q \in Z \wedge q' \mathrel{\mathsf{F}} q \Longrightarrow q' \in Z$. (Indeed, consider the set $Z' = [Z]_{\mathsf{F}} = \{q' : \exists q \in Z\, (q \mathrel{\mathsf{F}} q')\}$. Note that F is the orbit equivalence relation of a Polish action of a countable group by Theorem 7.4.1 below. It follows that there exists a countable system $\{\beta_n\}_{n\in\mathbb{N}}$ of Borel isomorphisms of the set $P \cup Q = \operatorname{dom} \mathsf{F}$ such that $Z' = \bigcup_n \{\beta_n(q) : q \in Z\}$. Thus, Z' is Borel by the Countable-to-1 Projection. Then by the Countable-to-1 Enumeration there is a Borel map $\zeta : Z' \to Z$ such that $\zeta(q') \mathrel{\mathsf{F}} q'$ for all $q' \in Z'$. It remains to replace Z, ϑ by Z' and the map $\vartheta'(q') = \vartheta(\zeta(q'))$.)

Now we define a Borel reduction of E to F as follows. Naturally, put $h(x) = f(x)$ for $x \in X$. If $y \in Y$ and $g(y) \notin Z$, then put $h(y) = g(y)$, while in the case $g(y) \in Z$, we define $h(y) = \vartheta(g(y))$. □

The condition $\mathbb{N}\mathsf{F} \le_{\mathrm{B}} \mathsf{F}$ in the theorem holds for many naturally arising equivalence relations F.[2] In particular it holds for the equivalence relation $\mathsf{F} = \mathsf{E}_0$ and for the equalities $\mathsf{F} = \Delta_X$. This allows us to obtain the following corollary.

COROLLARY 7.3.2. *Suppose that* E *is a Borel equivalence relation on a Borel set* $X = \bigcup_k X_k$, *with all* X_k *also Borel. If the restriction* $\mathsf{E} \restriction X_k$ *is smooth (resp.* $\mathsf{E} \restriction X_k \le_{\mathrm{B}} \mathsf{E}_0$*) for all* k, *then* E *itself is smooth (resp.* $\mathsf{E} \le_{\mathrm{B}} \mathsf{E}_0$*).* □

7.4. Countable equivalence relations and group actions

This class of equivalence relations is a subject of ongoing intense study. We present here the following important theorem, leaving [**JKL02, Gab00, KM04**] as sources of further information regarding countable equivalence relations. Part (i) of the theorem is due to FELDMAN–MOORE [**FM77**, Thm 1]. Part (ii) see *e.g.* in [**DJK94**, 1.8].

THEOREM 7.4.1. *Every Borel countable equivalence relation* E *on a Borel set* X *in a Polish space:*

(i) *is induced by a Polish action of a countable group* $\mathbb{G}$ *on* X;

(ii) *satisfies* $\mathsf{E} \le_{\mathrm{B}} \mathsf{E}_\infty = \mathbb{E}(F_2, 2)$, *where* F_2 *is the free group with two generators and* $\mathbb{E}(F_2, 2)$ *is the equivalence relation induced by the shift action of* F_2 *on* 2^{F_2}; *see Example* 4.4.5.

PROOF. (i) We assume *w.l.o.g.* that $X = 2^{\mathbb{N}}$. According to Theorem 2.12.3 (Countable-to-1 Enumeration) there is a sequence of Borel maps $f_n : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ such that $[a]_{\mathsf{E}} = \{f_n(a) : n \in \mathbb{N}\}$ for each $a \in 2^{\mathbb{N}}$. Put $\Gamma'_n = \{\langle a, f_n(a)\rangle : a \in \mathbb{N}\}$ (the graph of f_n) and $\Gamma_n = \Gamma'_n \smallsetminus \bigcup_{k<n} \Gamma'_k$. The sets $P_{nk} = \Gamma_n \cap {\Gamma_k}^{-1}$ form a partition of (the graph of) E onto countably many Borel injective sets. Further, define $\Delta = \{\langle a, a\rangle : a \in 2^{\mathbb{N}}\}$ and let $\{D_m\}_{m\in\mathbb{N}}$ be an enumeration of all non-empty sets of the form $P_{nk} \smallsetminus \Delta$. Intersecting the sets D_m with the rectangles of the form

$$R_s = \{\langle a, b\rangle \in 2^{\mathbb{N}} \times 2^{\mathbb{N}} : s^\frown 0 \subset a \wedge s^\frown 1 \subset b\} \quad \text{and} \quad {R_s}^{-1},$$

we reduce the general case to the case when $\operatorname{dom} D_m \cap \operatorname{ran} D_m = \varnothing$, $\forall m$.

Now, for every m define $h_m(a) = b$ whenever either $\langle a, b\rangle \in D_m$ or $\langle a, b\rangle \in {D_m}^{-1}$, or $a = b \notin \operatorname{dom} D_m \cup \operatorname{ran} D_m$. Clearly, h_m is a Borel bijection $2^{\mathbb{N}} \xrightarrow{\text{onto}} 2^{\mathbb{N}}$. Thus $\{h_m\}_{m\in\mathbb{N}}$ is a family of Borel automorphisms of $2^{\mathbb{N}}$ such that $[a]_{\mathsf{E}} = \{h_m(a) : m \in \mathbb{N}\}$. It does not take much effort to expand this system to a Borel action of

[2] In fact it is rather difficult to cook up a Borel equivalence relation, with infinitely many equivalence classes, not satisfying this reduction. See several complicated examples in THOMAS [**Tho02**] and in the recent monograph of HJORTH and KECHRIS [**HK05**].

F_ω, the free group with countably many generators $a_1, a_2, a_3, \ldots$, on $2^{\mathbb{N}}$, whose induced equivalence relation is E.

(ii) First of all, by (i), $\mathsf{E} \le_{\mathrm{B}} \mathsf{R}$, where R is induced by a Borel action $\cdot$ of F_ω on $2^{\mathbb{N}}$. The map $\vartheta(a) = \{g^{-1} \cdot a\}_{g \in F_\omega}$, $a \in 2^{\mathbb{N}}$ is a Borel reduction of R to $\mathbb{E}(F_\omega, 2^{\mathbb{N}})$. If now F_ω is a subgroup of a countable group H, then $\mathbb{E}(F_\omega, 2^{\mathbb{N}}) \le_{\mathrm{B}} \mathbb{E}(H, 2^{\mathbb{N}})$ by means of the map sending every $\{a_g\}_{g \in F_\omega}$ to $\{b_h\}_{h \in H}$, where $b_g = a_g$ for $g \in F_\omega$ and b_h equal to any fixed $b' \in 2^{\mathbb{N}}$ for $h \in H \smallsetminus F_\omega$. Note that F_ω admits an injective homomorphism into F_2. (Indeed, let F be the subgroup of F_2 generated by all elements of the form $\alpha_n = a^n b^n$ and $\alpha_n^{-1} = b^{-n} a^{-n}$. The map sending every a_n to α_n and accordingly a_n^{-1} to α_n^{-1} is an isomorphism of F_ω onto F.) It follows that $\mathsf{E} \le_{\mathrm{B}} \mathbb{E}(F_2, 2^{\mathbb{N}})$.

It remains to define a Borel reduction of $\mathbb{E}(F_2, 2^{\mathbb{N}})$ to $\mathbb{E}(F_2, 2)$. The inequality $\mathbb{E}(F_2, 2^{\mathbb{N}}) \le_{\mathrm{B}} \mathbb{E}(F_2, 2^{\mathbb{Z} \smallsetminus \{0\}})$ is clear. Further $\mathbb{E}(F_2, 2^{\mathbb{Z} \smallsetminus \{0\}}) \le_{\mathrm{B}} \mathbb{E}(F_2 \times \mathbb{Z}, 3)$ by means of the map sending every $\{a_g\}_{g \in F_2}$ $(a_g \in 2^{\mathbb{Z} \smallsetminus \{0\}})$ to $\{b_{gj}\}_{g \in F_2,\, j \in \mathbb{Z}}$, where $b_{gj} = a_g(j)$ for $j \ne 0$ and $b_{g0} = 2$. In addition, for every group G it holds that $\mathbb{E}(G, 3) \le_{\mathrm{B}} \mathbb{E}(G \times \mathbb{Z}_2, 2)$ by means of the map sending every element $\{a_g\}_{g \in G}$ $(a_g = 0, 1, 2)$ to $\{b_{gi}\}_{g \in G,\, i \in \mathbb{Z}_2}$, where

$$b_{gi} = \begin{cases} 0, & \text{if } \ a_g = 0 \quad \textit{or} \quad a_g = 1 \text{ and } i = 0, \\ 1, & \text{if } \ a_g = 2 \quad \textit{or} \quad a_g = 1 \text{ and } i = 1. \end{cases}$$

Thus, $\mathbb{E}(F_2, 2^{\mathbb{N}}) \le_{\mathrm{B}} \mathbb{E}(F_2 \times \mathbb{Z} \times \mathbb{Z}_2, 2)$. However, $F_2 \times \mathbb{Z} \times \mathbb{Z}_2$ admits a homomorphism into the group F_ω, and then into F_2 by the above, so that $\mathbb{E}(F_2, 2^{\mathbb{N}}) \le_{\mathrm{B}} \mathbb{E}(F_2, 2)$, as required. □

7.5. Non-hyperfinite countable equivalence relations

It will be proved below (Theorem 8.1.1(i),(ii)) that hyperfinite equivalence relations form an initial segment, in the sense of $\le_{\mathrm{B}}$, within the collection of all Borel countable equivalence relations. Let us show that this is a proper initial segment; that is, not all Borel countable equivalence relations are hyperfinite.

THEOREM 7.5.1 (SLAMAN–STEEL [**SS88**]). *The equivalence relation E_∞ (Borel and countable) is not hyperfinite.*

PROOF. We present the original proof of this result given in [**SS88**]. There is another, more complicated proof, based on the fact that a certain property called *amenability* holds for all hyperfinite equivalence relations and associated groups like $\langle \mathbb{Z}; + \rangle$, but fails for E_∞ and the group F_2; see Corollary 9.3.3(i).

Given a pair of bijections $f, g : 2^{\mathbb{N}} \xrightarrow{\text{onto}} 2^{\mathbb{N}}$, we define an action $\mathfrak{a}_{fg}$ of the free group F_2 with two generators a, b on $2^{\mathbb{N}}$ as follows: if $w = a_1 a_2 \cdots a_n \in F_2$ then $\mathfrak{a}_{fg}(w, x) = w \cdot x = h_{a_1}(h_{a_2}(\cdots(h_{a_n}(x))\cdots))$, where $h_a = f$, $h_{a^{-1}} = f^{-1}$, $h_b = g$, $h_{b^{-1}} = g^{-1}$. Separately $\Lambda \cdot x = x$, where Λ, the empty word, is the neutral element of F_2. The maps f, g are *independent*, iff the action is free; that is, for all x, $w \cdot x = x$ implies $w = \Lambda$.

To prove the theorem, we define a free action of F_2 on $2^{\mathbb{N}}$ by *Lipschitz homeomorphisms*, i.e. those homeomorphisms $f : 2^{\mathbb{N}} \xrightarrow{\text{onto}} 2^{\mathbb{N}}$ satisfying

$$x \restriction n = y \restriction n \iff f(x) \restriction n = f(y) \restriction n$$

for all n and $y \in 2^{\mathbb{N}}$. Such an action can be extended to any set of the form $2^n = \{s \in 2^{<\omega} : \mathtt{lh}\, s = n\}$ so that $w \cdot (x \restriction n) = (w \cdot x) \restriction n$ for all $x \in 2^{\mathbb{N}}$.

LEMMA 7.5.2. *There exists an independent pair of Lipschitz homeomorphisms* $f, g : 2^{\mathbb{N}} \xrightarrow{\text{onto}} 2^{\mathbb{N}}$.

PROOF. Define $f \restriction 2^n$ and $g \restriction 2^n$ by induction on n. We will take care that

$$(*) \qquad \mathtt{lh}\, f(s) = \mathtt{lh}\, g(s) = \mathtt{lh}\, s, \quad f(s) \subset f(s^\frown i), \quad \text{and} \quad g(s) \subset g(s^\frown i)$$

for all $s \in 2^{<\omega}$ and $i = 0, 1$. Fix a linear ordering of length ω, of the set of all pairs $\langle w, s \rangle \in F_2 \times 2^{<\omega}$ such that $w \neq \Lambda$.

Put $f(\Lambda) = g(\Lambda) = \Lambda$ $(n = 0)$ and $f(\langle i \rangle) = g(\langle i \rangle) = \langle 1 - i \rangle$, $i = 0, 1$.

To carry out the step $n \to n + 1$, suppose that the values $f(s), g(s)$, and subsequently $w \cdot s$ for all $w \in F_2$, have been defined for all $w \in F_2$ and $s \in 2^{<\omega}$ with $\mathtt{lh}\, s \le n$. Let $\langle w_n, s_n \rangle$ be the least pair (in the sense of the ordering mentioned above) such that for $k = \mathtt{lh}\, s_n \le n$, there is $t \in 2^n$ with $s_n \subseteq t$ and $w_n \cdot t = t$, and $u \cdot s_n \neq v \cdot s_n$ for all initial subwords[3] $u \neq v$ of w_n *except* for the case when $u = \Lambda$ and $v = w_n$ or vice versa. (Pairs $\langle w, s \rangle$ of this kind do exist: as 2^n is finite, for every $s \in 2^n$ there is $w \in F_2 \smallsetminus \{\Lambda\}$ such that $w \cdot s = s$.)

We put $T_n = \{t \in 2^n : s_n \subseteq t \wedge w_n \cdot t = t\}$. The sets

$$C_t = \{u \cdot t : u \text{ is an initial subword of } w_n\}, \quad t \in T_n,$$

are pairwise disjoint. Indeed if $u \cdot t_1 = v \cdot t_2 = t'$, where u, v are initial subwords of w_n, then $u \neq v$ as otherwise $t_1 = u^{-1} \cdot t' = v^{-1} \cdot t' = t_2$. But then $u \cdot s_n = v \cdot s_n$ (as t_1, t_2 extend s_n), which contradicts the choice of s_n.

Consider any $t \in T_n$. The word w_n has the form $a_0 a_1 \cdots a_{m-1}$ for some $m \ge 1$, where all a_ℓ belong to $\{a, b, a^{-1}, b^{-1}\}$. Then $C_t = \{t_0, t_1, \dots, t_m\}$, where $t_0 = t$ and $t_{\ell+1} = a_\ell \cdot t_\ell$, $\forall \ell$. Easily $t_m = w_n \cdot t = t = t_0$, but $t_\ell \neq t_{\ell'}$ whenever $\ell < \ell' < m$. We define $a_0 \cdot (t_0{}^\frown i) = t_1{}^\frown(1 - i)$ for $i = 0, 1$, but $a_\ell \cdot (t_\ell{}^\frown i) = t_{\ell+1}{}^\frown i$ whenever $1 \le \ell < m$. Then easily $w_n \cdot (t^\frown i) = t^\frown(1 - i) \neq t$.

Note that this definition of *some* of the values of $a \cdot r, b \cdot r, a^{-1} \cdot r, b^{-1} \cdot r, r \in 2^{n+1}$, is self-consistent.[4] Thus it remains consistent on the union of all "cycles" C_t, $t \in T_n$. It follows that the action of f and g can be defined on 2^{n+1} so that $(*)$ holds, while the values of $a_\ell \cdot (t_\ell{}^\frown i)$ coincide with the above-defined ones within each cycle C_t, $t \in T_n$. Then $w_n \cdot (t^\frown i) \neq t^\frown i$ for all $t \in T_n$, $i = 0, 1$. It follows that there can be no pair $\langle w_{n'}, s_{n'} \rangle$, $n' > n$, equal to $\langle w_n, s_n \rangle$.

This definition results in a pair of Lipschitz homeomorphisms f, g of $2^{\mathbb{N}}$. To check the independence, suppose toward the contrary that $x \in 2^{\mathbb{N}}$, $w \in F_2$, $w \neq \Lambda$, and $w \cdot x = x$, and there is no shorter word w of this sort. Then there exists $k \in \mathbb{N}$ such that $s = x \restriction k$ satisfies $u \cdot s \neq v \cdot s$ for all initial subwords $u \neq v$ of w except for the case $u = \Lambda$ and $v = w$ (or vice versa). The pair $\langle w, s \rangle$ is equal to $\langle w_n, s_n \rangle$ for some $n \ge k$. Then the set T_n contains the element $t = x \restriction n$. Put $i = x(n)$. Then by definition $w \cdot (t^\frown i) = (w \cdot t)^\frown(1 - i) = t^\frown(1 - i) \neq t^\frown i$, contary to the assumption $w \cdot x = x$. □ (Lemma)

Fix a pair of independent Lipschitz homeomorphisms $f, g : 2^{\mathbb{N}} \xrightarrow{\text{onto}} 2^{\mathbb{N}}$. Define the action $\mathfrak{a}(w, x) = w \cdot x$ as above. This Polish (even "Lipschitz") action of F_2 on $2^{\mathbb{N}}$ induces a Borel countable equivalence relation $x \mathsf{E} y$ iff $\exists w \in F_2\, (y = w \cdot x)$. Let us show that E is not hyperfinite.

[3] Λ and w themselves are considered to be initial subwords of every word $w \in F_2$.

[4] The inconsistency would have appeared in the case $a_{m-1}^{-1} = a_0$. Then $a_0 \cdot (t_0{}^\frown i) = t_1{}^\frown(1 - i)$ while $a_{m-1}^{-1} \cdot (t_m{}^\frown i) = t_{m-1}{}^\frown i$, and $t_0 = t_m$. However, $a_{m-1}^{-1} \neq a_0$, since otherwise $a_0^{-1} s_n = (a_0 \dots a_{m-2}) \cdot s_n$, contrary to the choice of s_n.

Suppose toward the contrary that $\mathsf{E} = \bigcup_n \mathsf{F}_n$ where $\{\mathsf{F}_n\}_{n\in\mathbb{N}}$ is a $\subseteq$-increasing sequence of finite Borel equivalence relations. For every x let n_x be the least n such that $\{f(x), g(x), f^{-1}(x), g^{-1}(x)\}$ is a subset of $[x]_{\mathsf{F}_n}$. Then there exist a number n and a closed $X \subseteq 2^{\mathbb{N}}$ such that $n_x \le n$ for all $x \in X$, and $\mu(X) \ge 3/4$, where μ is the uniform probability measure on $2^{\mathbb{N}}$.

Define the subtree $T = \{x \restriction m : x \in X \wedge m \in \mathbb{N}\}$ of the tree $2^{<\omega}$. We claim that the set U of all pairs $\langle w, s\rangle \in F_2 \times 2^{<\omega}$, such that $\mathtt{lh}\, w = \mathtt{lh}\, s$ and $u \cdot s \in T$ for all initial subwords u of w (including Λ and w), is infinite.

To prove this fact, fix $\ell \in \mathbb{N}$ and find $\langle w, s\rangle \in U$ such that $\mathtt{lh}\, s = \mathtt{lh}\, w \ge \ell$. By the independence of f, g, we have $w \cdot x \ne x$ for all $w \in W = \{a, b, a^{-1}, b^{-1}\}$ and $x \in 2^{\mathbb{N}}$, in addition $w \cdot x \ne w' \cdot x$ for all $w \ne w'$ in W. Then by König's lemma, there is a number $m \ge \ell$ such that $w \cdot s \ne s$ and $w \cdot s \ne w' \cdot s$ for all $w \ne w'$ in W and all $s \in 2^m$. Note that the graph

$$\Gamma = \{\{s, t\} : s, t \in 2^m \wedge \exists w \in W\ (w \cdot s = t)\}$$

on 2^m has exactly $2 \cdot 2^m$ edges: indeed, by the choice of m for every $s \in 2^m$ there exist exactly four different nodes $t \in 2^m$ such that $\{s, t\} \in \Gamma$.

Consider the subgraph $G = \{\{s, t\} \in \Gamma : s, t \in T\}$. The intersection $T \cap 2^m$ contains at least $\frac{3}{4} \cdot 2^m$ elements (as X is a set of measure $\ge 3/4$). Accordingly, the difference $2^m \smallsetminus T$ contains at most $\frac{1}{4} \cdot 2^m$ elements. Thus, comparably to Γ, the subgraph G loses at most $4 \cdot \frac{1}{4} \cdot 2^m = 2^m$ edges. In other words, G, a graph with $\le 2^m$ nodes, has at least $2 \cdot 2^m - 2^m = 2^m$ edges.

Now we apply the following combinatorial fact.

LEMMA 7.5.3. *Every graph G on a finite set Y, containing not more nodes than edges, has a cycle with at least three nodes.*

PROOF (Sketch). Otherwise, Y contains an endpoint; that is, an element $y \in Y$ such that $\{y, y'\} \in G$ holds for at most one $y' \in Y \smallsetminus \{y\}$. This allows us to use induction on the number of nodes. □

Thus G contains a cycle $s_0, s_1, \dots, s_{k_1}, s_k = s_0$. Here $k \ge 3$, all s_k belong to $T \cap 2^m$, s_i, $i < k$, are pairwise different, and for every $i < k$ there exists $a_i \in W = \{a, b, a^{-1}, b^{-1}\}$ such that $a_i \cdot s_i = s_{i+1}$. The word $u = a_0 a_1 \cdots a_{k-1}$ is irreducible as otherwise $s_{i-1} = s_{i+1}$ for some $0 < i < k$. Moreover, the word uu (the concatenation of two copies of u) is irreducible also, as otherwise $s_1 = s_{k-1}$. Therefore, u^m (the concatenation of m copies of u) is irreducible as well, and so is its initial subword $w = u^m \restriction m$. It follows that $\langle w, s_0\rangle \in U$, as required.

As U is infinite, by König it contains an infinite branch, *i.e.* there is an (irreducible) word $w \in \{a, b, a^{-1}, b^{-1}\}^{\mathbb{N}}$ and $x \in 2^{\mathbb{N}}$ such that $\langle w \restriction m, x \restriction m\rangle \in U$ for all m. Then clearly $(w \restriction m) \cdot x \in X$ for all m, and hence $x \mathrel{\mathsf{F}_n} ((w \restriction m) \cdot x)$ by induction on m. Finally, $(w \restriction m) \cdot x \ne (w \restriction m') \cdot x$ holds whenever $m \ne m'$ by the independence of f, g. Thus, the equivalence class $[x]_{\mathsf{F}_n}$ is infinite; a contradiction.

Thus, E is a countable non-hyperfinite equivalence relation. Recall that $\mathsf{E} \le_{\mathrm{B}} \mathsf{E}_\infty$ by Theorem 7.4.1. Thus, E_∞ itself is non-hyperfinite as well by the equivalence (i) $\Longleftrightarrow$ (ii) of Theorem 8.1.1. □ (Theorem)

It is worth noting that despite Theorem 7.5.1, any countable Borel equivalence relation *is* hyperfinite modulo restriction to a large set by the next theorem (6.2 in [**HK96**] or 1.13 in [**JKL02**] or 12.1 in [**KM04**]):

THEOREM 7.5.4. *If* E *is a Borel countable equivalence relation on a Polish space* $\mathbb{X}$ *then* E *is* **generically hyperfinite**, *that is, there is an* E*-invariant co-meager set* $X \subseteq \mathbb{X}$ *such that* $\mathsf{E} \restriction X$ *is hyperfinite.* □

In particular the equivalence relation $\mathsf{E}_\infty = \mathbb{E}(F_2, 2)$ is hyperfinite on an E_∞-invariant co-meager set $X \subseteq 2^{F_2}$. Remarkably, this result fails in the measure-theoretic context, see 9.3.3(iv) below.

7.6. A sufficient condition of essential countability

We finish with a technical lemma, attributed to KECHRIS in [**Hjo00a**], that will be used in Chapter 15. Recall that equivalence relations which are Borel reducible to Borel countable ones are called *essentially countable*. The following lemma shows that maps much weaker than reductions lead to the same class.

LEMMA 7.6.1. *Suppose that* A, X *are Borel sets,* E *is a Borel equivalence relation on* A, *and* $\rho : A \to X$ *is a Borel map satisfying the following: First, the* ρ*-image of every* E*-class is at most countable. Second,* ρ*-images of different* E*-classes are pairwise disjoint. Then* E *is an essentially countable equivalence relation.*

PROOF. The relation $x \mathrel{\mathsf{R}} y$ iff $x, y \in Y$ belong to the ρ-image of one and the same E-class in A is a Σ^1_1-equivalence relation on the set $Y = \operatorname{ran} \vartheta$. Moreover,

$$\mathsf{R} \subseteq P = \{\langle x, y\rangle : \neg\, \exists\, a, b \in A\ (a \not\mathrel{\mathsf{E}} b \wedge x = \rho(a) \wedge y = \rho(b))\}\,,$$

where P is $\boldsymbol{\Pi}^1_1$. Thus, there is a Borel set U with $\mathsf{R} \subseteq U \subseteq P$. In particular, $U \cap (Y \times Y) = \mathsf{R}$. As all R-equivalence classes are at most countable, we can assume that all cross-sections of U are at most countable too.

To prove the lemma it suffices to find a Borel equivalence relation F with $\mathsf{R} \subseteq \mathsf{F} \subseteq U$. Say that a set $Z \subseteq X$ is *stable* if $U \cap (Z \times Z)$ is an equivalence relation. For example, Y is stable. We observe that the set $D_0 = \{y : Y \cup \{y\} \text{ is stable}\}$ is $\boldsymbol{\Pi}^1_1$ and satisfies $Y \subseteq D_0$; hence, there is a Borel set Z_1 with $Y \subseteq Z_1 \subseteq D_0$. Similarly,

$$D_1 = \{y' \in Z_1 : Y \cup \{y, y'\} \text{ is stable for all } y \in Z_1\}$$

is $\boldsymbol{\Pi}^1_1$ and satisfies $Y \subseteq D_1$ by the definition of Z_1, so there is a Borel set Z_2 with $Y \subseteq Z_2 \subseteq D_1$. Generally, we define

$$D_n = \{y' \in Z_n : Y \cup \{y_1, \dots, y_n, y'\} \text{ is stable for all } y_1, \dots, y_n \in Z_n\}$$

and find that $Y \subseteq D_n$, and choose a Borel set Z_n with $Y \subseteq Z_n \subseteq D_n$. Then, by the construction, $Y \subseteq Z = \bigcap_n Z_n$, and, for every finite $Z' \subseteq Z$, the set $Y \cup Z'$ is stable, so that Z itself is stable, and we can take $\mathsf{F} = U \cap (Z \times Z)$. □

CHAPTER 8

Hyperfinite equivalence relations

This is the most elementary class of equivalence relations above the equalities on Polish spaces, symbolized (but not entirely exhausted in all aspects) by the equivalence relation E_0, and a very interesting subclass of countable Borel equivalence relations. Theorem 8.1.1 is the main result of this chapter. It establishes the equivalence of several different characterizations of hyperfinite equivalence relations. Some additional results on hyperfinite equivalence relations are added at the end of the chapter. In particular, we prove the hyperfiniteless of the tail equivalence relation E_t and discuss the classification of hyperfinite equivalence relations up to Borel isomorphism.

8.1. Hyperfinite equivalence relations: The characterization theorem

The class of Borel hyperfinite equivalence relations has been a topic of intense study since the 1970s. Papers [**DJK94, JKL02**] and the book [**KM04**] give a comprehensive account of the results obtained regarding hyperfinite relations, with further references. Different parts of the following characterization theorem were established by different authors in the beginning of the 1990s or even earlier; see Theorems 5.1 and, partially, 7.1 in [**DJK94**] and 12.1(ii) in [**JKL02**], where links to original proofs are given.

THEOREM 8.1.1. *The following are equivalent for a Borel equivalence relation* E *on a Polish space* $\mathbb{X}$:

(i) $\mathsf{E} \le_{\mathrm{B}} \mathsf{E}_0$ *and* E *is countable*;

(ii) E *is hyperfinite*;

(iii) E *is hypersmooth and countable*;

(iv) *there exists a Borel set* $X \subseteq (2^{\mathbb{N}})^{\mathbb{N}}$ *such that the restricted equivalence relation* $\mathsf{E}_1 \restriction X$ *is countable and* E *is isomorphic, via a Borel bijection of* $\mathbb{X}$ *onto* X, *to* $\mathsf{E}_1 \restriction X$;[1]

(v) E *is induced by a Borel action of* $\mathbb{Z}$, *the additive group of the integers*;

(vi) *there exists a pair of Borel equivalence relations* F, G *of type* 2 *such that* $\mathsf{E} = \mathsf{F} \vee \mathsf{G}$;[2]

(vii) *there is a Borel partial order* $\preceq$ *on the domain of* E *such that every* E-*class is* $\preceq$-*ordered similarly to a subset of* $\mathbb{Z}$.

[1] This transitional condition refers to E_1, here considered to be an equivalence relation on $(2^{\mathbb{N}})^{\mathbb{N}}$ defined so that $x \mathsf{E}_1 y$ iff $x(n) = y(n)$ for all but finite n.

[2] $\mathsf{F} \vee \mathsf{G}$, the *join* of F, G, denotes the $\subseteq$-least equivalence relation which includes $\mathsf{F} \cup \mathsf{G}$.

Note that all Borel finite equivalence relations are smooth by Proposition 7.2.1. Accordingly, all hyperfinite equivalence relations are hypersmooth. On the other hand, all finite and hyperfinite equivalence relations are countable, of course. It follows from the theorem that, conversely, every hypersmooth countable equivalence relation is hyperfinite.

The theorem also shows that E_0 is a universal hyperfinite equivalence relation. (To see that E_0 is hyperfinite, define $x \mathsf{F}_n y$ iff $x \bigtriangleup y \subseteq [0, n)$ for $x, y \subseteq \mathbb{N}$.)

Regarding (vi), the result seems to be close to the best possible of its kind. Indeed by 1.21 in [**JKL02**] for *every countable* Borel equivalence relation E there exist Borel equivalence relations F, G of types $2, 3$, respectively, such that $\mathsf{E} \sim_{\mathrm{B}} \mathsf{F} \vee \mathsf{G}$.[3] For instance, this is true for $\mathsf{E} = \mathsf{E}_\infty$, a countable non-hyperfinite Borel equivalence relation. But then the corresponding *join* $\mathsf{F} \vee \mathsf{G}$ cannot be hyperfinite since the class of all hyperfinite equivalence relations is $\sim_{\mathrm{B}}$-invariant by (i), (ii) of the theorem.

Regarding (vii), it must be said that this characterization belongs to a wide class of definitions of different subclasses, especially among Borel *countable* equivalence relations, by induction of invariant structures on equivalence classes. The invariance means that the order $\le_x = \le \restriction [x]_{\mathsf{E}}$ depends on the class $[x]_{\mathsf{E}}$ but not on the specific choice of x inside $[x]_{\mathsf{E}}$. In other words, if $x \mathsf{E} y$, then $\le_x$ and $\le_y$ coincide.

Before the proof starts, let us mention a corollary first established (in different terms and by different methods) by Mycielski.

Exercise 8.1.2. Using (iii) $\Longrightarrow$ (ii) of the theorem, prove that the Vitali equivalence relation Vit (see Example 4.4.6) is hyperfinite. To show the hypersmoothness, note that $\mathsf{Vit} = \bigcup_n \mathsf{F}_n$ where $x \mathsf{F}_n y$ iff $|x - y| = k \cdot 2^{-n}$ for some $k \in \mathbb{N}$. □

Remark 8.1.3. It is not clear at all how to induce Vit by a Borel action of $\mathbb{Z}$, or, which is the same, how to define a mathematically meaningful Borel automorphism $f : \mathbb{R} \xrightarrow{\text{onto}} \mathbb{R}$ whose orbits $[x]_f = \{f^j(x) : j \in \mathbb{Z}\}$ are precisely the Vitali classes $[x]_{\mathsf{Vit}} = x + \mathbb{Q}$. As for E_0 (considered on $2^{\mathbb{N}}$), such an automorphism $f : 2^{\mathbb{N}} \xrightarrow{\text{onto}} 2^{\mathbb{N}}$ can be defined as follows. Suppose that $a \in 2^{\mathbb{N}}$. If $a(k) = 1, \forall k$, then let $f(a) = b$, where $b(k) = 0, \forall k$. Otherwise let $n = \mathtt{min}\{k : a(k) = 0\}$, and $f(a) = b$, where $b(k) = 0$ for $k < n$, $b(n) = 1$, and $b(k) = a(k)$ for $k > n$. Easily f-orbits $= \mathsf{E}_0$-classes, except for the fact that the E_0-classes of the constant-0 and constant-1 are joined into a single f-orbit. □

8.2. Proof of the characterization theorem

It does not seem possible to prove Theorem 8.1.1 by a simple cyclic argument. The structure of the proof is presented in Figure 4 on page 97.

The implications (ii) $\Longrightarrow$ (iii) and (i) $\Longrightarrow$ (iii) are quite elementary.

(iii) $\Longrightarrow$ (iv). Let $\mathsf{E} = \bigcup_n \mathsf{F}_n$ be a countable and hypersmooth equivalence relation on a space $\mathbb{X}$, let all F_n be smooth (and countable), and $\mathsf{F}_n \subseteq \mathsf{F}_{n+1}, \forall n$. We may assume that $\mathbb{X} = 2^{\mathbb{N}}$ and $\mathsf{F}_0 = \Delta_{2^{\mathbb{N}}}$. Let $T_n \subseteq \mathbb{X}$ be a Borel transversal for F_n (recall Proposition 7.2.1(iii)). Now let $\vartheta_n(x)$ be the only element of T_n

[3] But strengthening this to $\mathsf{E} = \mathsf{F} \vee \mathsf{G}$ fails. Indeed there exists a countable Borel equivalence relation E that cannot be presented in the form $\mathsf{E} = \mathsf{F}_1 \vee \cdots \vee \mathsf{F}_n$, where F_i are Borel *finite* equivalence relations; see a remark in [**JKL02**] after the proof of 1.21 with a reference to [**Gab00**].

$$\begin{array}{ccccccccccc}
 & & \text{(ii)} & \Longleftarrow & \Longleftarrow & \Longleftarrow & \Longleftarrow & \Longleftarrow & \text{(ii)} & & \\
 & & \Downarrow & & & & & & \Uparrow & & \\
\text{(i)} & \Longrightarrow & \text{(iii)} & \Longrightarrow & \text{(iv)} & \Longrightarrow & \text{(vii)} & \Longrightarrow & \text{(v)} & \Longrightarrow & \text{(i)} \\
 & & & & & & \Uparrow & & \Downarrow & & \\
 & & & & & & \text{(vi)} & \Longleftarrow & \text{(vi)} & &
\end{array}$$

FIGURE 4. The structure of the proof of Theorem 8.1.1

with $x \mathsf{F}_n \vartheta_n(x)$. Then $x \mapsto \{\vartheta_n(x)\}_{n\in\mathbb{N}}$ is a 1-to-1 Borel map $\mathbb{X} \to (2^{\mathbb{N}})^{\mathbb{N}}$ and $x \mathsf{E} y \iff \vartheta(x) \mathsf{E}_1 \vartheta(y)$. Take X to be the image of $\mathbb{X}$ under this map.

(iv) $\Longrightarrow$ (vii). Let X be as indicated. For every $\mathbb{N}$-sequence x and $n \in \mathbb{N}$, let $x \restriction_{>n} = x \restriction (n, \infty)$. It follows from Theorems 2.12.1 and 2.12.3 (Countable-to-1 Projection and Countable-to-1 Enumeration) that for every n the set $X \restriction_{>n} = \{x \restriction_{>n} : x \in X\}$ is Borel and there is a countable family of Borel functions $g_i^n : X \restriction_{>n} \to X$, $i \in \mathbb{N}$, such that the set $X_\xi = \{x \in X : x \restriction_{>n} = \xi\}$ is equal to $\{g_i^n(\xi) : i \in \mathbb{N}\}$ for all $\xi \in X \restriction_{>n}$. In other words,

$$\{g_i^n(\xi)(n) : i \in \mathbb{N}\} = \{x(n) : x \in X_\xi\}.$$

For every $x \in (2^{\mathbb{N}})^{\mathbb{N}}$, let $\varphi(x) = \{\varphi_n(x)\}_{n\in\mathbb{N}}$, where $\varphi_n(x)$ is the least number i such that $x(n) = f_i^n(x)(n)$; thus, $\varphi(x) \in \mathbb{N}^{\mathbb{N}}$. Let $\mu(x)$ be the sequence

$$\varphi_0(x)\,,\ \varphi_0'(x)\,,\ \varphi_1(x)+1\,,\ \varphi_1'(x)+1\,,\ \dots\,,\ \varphi_n(x)+n\,,\ \varphi_n'(x)+n\,,\ \dots\,,$$

where $\varphi_n'(x) = \max_{k\le n} \varphi_k(x)$. Easily if $x \neq y \in X$ satisfy $x \mathsf{E}_1 y$, *i.e.*, $x \restriction_{>n} = y \restriction_{>n}$ for some n, then $\varphi(x) \restriction_{>n} = \varphi(y) \restriction_{>n}$, but $\varphi(x) \neq \varphi(y)$, $\mu(x) \neq \mu(y)$, and $\mu(x) \restriction_{>m} = \mu(y) \restriction_{>m}$ for some $m \geq n$.

Let $<_{\texttt{alex}}$ be the anti-lexicographical partial order on $\mathbb{N}^{\mathbb{N}}$, *i.e.*, $a <_{\texttt{alex}} b$ iff there is n such that $a \restriction_{>n} = b \restriction_{>n}$ and $a(n) < b(n)$. For $x, y \in X$ define $x <_0 y$ iff $\mu(x) <_{\texttt{alex}} \mu(y)$. It follows from the above that $<_0$ linearly orders every E_1-class $[x]_{\mathsf{E}_1} \cap X$ of $x \in X$. Moreover, it follows from the definition of $\mu(x)$ that every $<_{\texttt{alex}}$-interval between some $\mu(x) <_{\texttt{alex}} \mu(y)$ contains only finitely many elements of the form $\mu(z)$. (For φ this would not be true.) We conclude that every class $[x]_{\mathsf{E}_1} \cap X$, $x \in X$, is linearly ordered by $<_0$ similarly to a subset of $\mathbb{Z}$, the integers.

(vii) $\Longrightarrow$ (v). Suppose that $\leq$ is an order as in (vii). That $\leq$ can be converted to a required Borel action of $\mathbb{Z}$ on X is rather elementary. Indeed, if a E-class $[x]_{\mathsf{E}}$ is $\leq$-ordered similarly to $\mathbb{Z}$ itself (the main case),

$$[x]_{\mathsf{E}} = \{\cdots < x_2 < x_1 < x_0 < x_1 < x_2 < \cdots\},$$

then we let $1 \cdot x_j = x_{j+1}$ for all $j \in \mathbb{Z}$. If $[x]_{\mathsf{E}}$ is $\leq$-ordered similarly to $\mathbb{N}$, that is,

$$[x]_{\mathsf{E}} = \{x_0 < x_1 < x_2 < x_3 < x_3 < \cdots\},$$

then re-order it similarly to $\mathbb{Z}$ as

$$[x]_{\mathsf{E}} = \{\cdots < x_3 < x_1 < x_0 < x_2 < x_4 < \cdots\},$$

and come to the main case. If the order is similar to $-\mathbb{N}$, then reverse it and come to the $\mathbb{N}$-case. If finally the set

$$[x]_{\mathsf{E}} = \{x_0 < x_1 < x_2 < x_3 < x_3 < \cdots < x_n\}$$

is finite, then apply the cyclic action $1 \cdot x_j = x_{j+1}$ for $j < n$, but $1 \cdot x_n = x_0$. To prove that the action of $\mathbb{Z}$ defined this way is Borel, apply the theorems in Section 2.12.

(v) $\Longrightarrow$ (ii). Assume *w.l.o.g.* that $\mathbb{X} = 2^{\mathbb{N}}$. An increasing sequence of equivalence relations F_n whose union is E is defined separately on each E-class C. They "integrate" into Borel equivalence relations F_n defined on the whole of $2^{\mathbb{N}}$ because the action allows us to replace quantifiers over an E-class C by quantifiers over $\mathbb{Z}$.

Let C be the E-class of an element $x \in X$. Note that if $x_C \in C$ can be chosen in some Borel-definable way, then we can define $x \mathrel{\mathsf{F}_n} y$ iff there exist integers $j, k \in \mathbb{Z}$ with $|j| \le n$, $|k| \le n$, and $x = j \cdot x_C$, $y = k \cdot x_C$. This applies, for instance, when C is finite, thus, we can assume that C is infinite. Let $<_{\mathtt{lex}}$ be the lexicographical ordering of $2^{\mathbb{N}}$, and $<_{\mathtt{act}}$ be the partial order induced by the action, *i.e.*, $x <_{\mathtt{act}} y$ iff $y = j \cdot x$, $j > 0$. By the same reason, we can assume that neither $a = \mathtt{inf}_{<_{\mathtt{lex}}} C$ nor $b = \mathtt{sup}_{<_{\mathtt{lex}}} C$ belongs to C. Let C_n be the set of all $x \in C$ with $x \restriction n \ne a \restriction n$ and $x \restriction n \ne b \restriction n$. Define $x \mathrel{\mathsf{F}_n} y$ iff x, y belong to one and the same $<_{\mathtt{lex}}$-interval in C lying entirely within C_n, or just $x = y$. In our assumptions, every equivalence relation F_n has finite classes, and for any two $x, y \in C$ there is n with $x \mathrel{\mathsf{F}_n} y$.

(v) $\Longrightarrow$ (i). This implication is somewhat more complicated. A preliminary step is to show that $\mathsf{E} \le_{\mathsf{B}} \mathbf{E}(\mathbb{Z}, 2^{\mathbb{N}})$, where $\mathbf{E}(\mathbb{Z}, 2^{\mathbb{N}})$ is the orbit equivalence relation induced by the shift action of $\mathbb{Z}$ on $(2^{\mathbb{N}})^{\mathbb{Z}}$; that is, $(k \cdot x)(j) = x(j-k)$ for $k, j \in \mathbb{Z}$ and $x \in (2^{\mathbb{N}})^{\mathbb{Z}}$. Assuming *w.l.o.g.* that E is an equivalence relation on $2^{\mathbb{N}}$, we obtain a Borel reduction of E to $\mathbf{E}(\mathbb{Z}, 2^{\mathbb{N}})$ by $\vartheta(a) = \{j \cdot a\}_{j \in \mathbb{Z}}$, where $\cdot$ is a Borel action of $\mathbb{Z}$ on $2^{\mathbb{N}}$ which induces E. That is, if $a \in 2^{\mathbb{N}}$, then $\vartheta(a) = x \in (2^{\mathbb{N}})^{\mathbb{Z}}$ and $x(j) = j \cdot a$ for all $j \in \mathbb{Z}$. Thus we can assume that simply E is $\mathbf{E}(\mathbb{Z}, 2^{\mathbb{N}})$, and the goal is to prove that $\mathsf{E} = \mathbf{E}(\mathbb{Z}, 2^{\mathbb{N}}) \le_{\mathsf{B}} \mathsf{E}_0$.

Beginning with some necessary definitions and terminology, we put $W_n = 2^{n \times n}$, thus W_n consists of all functions with $\mathtt{dom}\, f = [0, n) \times [0, n)$ and $\mathtt{ran}\, f \subseteq 2 = \{0, 1\}$. Fix an order $<_n$ on each set W_n so that $u <_{n+1} v$ implies $u \restriction n <_n v \restriction n$ for all $u, v \in W_{n+1}$, where $u \restriction n \in W_n$ is defined for $u \in W_{n+1}$ so that $(u \restriction n)(k, i) = u(k, i)$ for all $i, k < n$. Put $W = \bigcup_n W_n$.

If $x \in (2^{\mathbb{N}})^{\mathbb{Z}}$ and $w \in W_n$, then let $A^x(w)$ be the set of all integers $a \in \mathbb{Z}$ satisfying $x(a+k)(i) = w(k, i)$ for all $k, i < n$.[4] If $x \in (2^{\mathbb{N}})^{\mathbb{Z}}$ and $n \in \mathbb{N}$, then let w_n^x be the $<_n$-least element $w \in W_n$ such that $A^x(w) \ne \varnothing$. Then clearly $w_n^x \subset w_{n+1}^x$, and hence there is an element $\psi^x \in (2^{\mathbb{N}})^{\mathbb{N}}$ such that $\psi^x(k)(i) = w_n^x(k, i)$ for all $k, i < n$. Finally we let A^x denote the set of all integers $a \in \mathbb{Z}$ satisfying $x(a+k) = \psi^x(k)$ for all $k \in \mathbb{N}$.

Our plan is to define a partition $(2^{\mathbb{N}})^{\mathbb{Z}} = X_1 \cup X_2 \cup X_3 \cup Y$ into Borel pairwise disjoint sets X_i and Y such that $\mathsf{E} \restriction X_i$ is smooth for $i = 1, 2, 3$, while $\mathsf{E} \restriction Y \le_{\mathsf{B}} \mathsf{E}_0$. Then we conclude that $\mathsf{E} \le_{\mathsf{B}} \mathsf{E}_0$ by Corollary 7.3.2.

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[4] The condition $a \in A^x(w)$ is the same as "w occurs in x at a" in **[DJK94]**.

Thus let

$$\begin{aligned}
X_1' &= \{x \in (2^{\mathbb{N}})^{\mathbb{Z}} : \exists w \in W\ (A^x(w) \text{ is } \neq \varnothing \text{ and bounded in } \mathbb{Z} \text{ from below})\};\\
X_1'' &= \{x \in (2^{\mathbb{N}})^{\mathbb{Z}} : \exists w \in W\ (A^x(w) \text{ is } \neq \varnothing \text{ and bounded in } \mathbb{Z} \text{ from above})\};\\
X &= (2^{\mathbb{N}})^{\mathbb{Z}} \smallsetminus (X_1' \cup X_2');\\
X_2 &= \{x \in X : A^x \text{ is } \neq \varnothing \text{ and bounded in } \mathbb{Z} \text{ from below}\};\\
X_3 &= \{x \in X : A^x \text{ is } \neq \varnothing \text{ and unbounded in } \mathbb{Z} \text{ from below}\};\\
Y &= \{x \in X : A^x = \varnothing\} = X \smallsetminus (X_2 \cup X_3).
\end{aligned}$$

We leave it to the reader to check that all these sets, as well as different intermediate objects, are Borel, using the theorems in Section 2.12.

Step 1. We claim that the restricted equivalence relation $\mathsf{E} \restriction X_1'$ is smooth. Indeed if $x \in X_1'$, then let w^x be the least, in the sense of a fixed ω-ordering of W, element of W such that the set $A^x(w)$ is non-empty and bounded from below. Note that the definition of w^x is invariant in the sense that $x \mathsf{E} y \Longrightarrow w^x = w^y$.[5] Obviously the equivalence class $[x]_{\mathsf{E}}$ contains a unique element y such that 0 is the least number in the set $A^y(w^x) = A^y(w^y)$. We conclude that $\mathsf{E} \restriction X_1'$ admits a Borel transversal

$$\{y \in X_1' : 0 \text{ is the least number in } A^y(w^y)\},$$

and hence is smooth by the same reasons $\mathsf{E} \restriction X_1''$ is smooth. Therefore, $\mathsf{E} \restriction X_1$ is smooth by Corollary 7.3.2, where $X_1 = X_1' \cup X_1''$.

Step 2. We claim that $\mathsf{E} \restriction X_2$ is smooth. Indeed if $x \in X_2$, then A^x contains a least element, say d ($d \in \mathbb{Z}$), and hence $[x]_{\mathsf{E}}$ contains a unique point y such that 0 is the least element in A^y (y is the $(-d)$-shift of x). Once again this leads to a Borel transversal. (To prove the uniqueness of y, suppose toward the contrary that $y, z \in [x]_{\mathsf{E}}$ and 0 is the least element in both A^y and A^z. Then easily $y(n) = z(n)$ for all $n \geq 0$. This still does not imply $y = z$ because there could be a number $d \geq 1$ such that say z is the $-d$-shift of y and at the same time d-periodic above, but not below $-d$. Yet in this case obviously $-d \in A^z$, a contradiction.)

Step 3. We claim that $\mathsf{E} \restriction X_3$ is smooth too. Indeed, if $x \in X_3$, then A^x is unbounded from below, and by obvious reasons x is periodic: there is a natural number $d \geq 1$ such that $x(a+d)(i) = x(a)(i)$ for all $a \in \mathbb{Z}$ and $i \in \mathbb{N}$. It follows that $[x]_{\mathsf{E}}$ is a finite set. Then apply Proposition 7.2.1(ii),(i).

Step 4. We finally claim that $\mathsf{E} \restriction Y \leq_{\mathrm{B}} \mathsf{E}_0$.

Suppose that $x \in Y$. Then $x \in X$; therefore, for every n the set $A^x(w_n^x)$ is unbounded in $\mathbb{Z}$ in both directions. In addition $A^x(w_{n+1}^x) \subseteq A^x(w_n^x)$, and $\bigcap_n A^x(w_n^x) = \varnothing$, since otherwise a number $a \in \mathbb{Z}$ that belongs to all $A^x(w_n^x)$ would also belong to A^x. (But $A^x = \varnothing$ because $x \in Y$.) Define integers $a_n^x \in \mathbb{Z}$ so that $a_0^x = 0$ and

$$(1)\quad \begin{cases} a_{2n+1}^x &= \text{the least element of } A^x(w_{2n+1}^x) \text{ bigger than } 0, \quad \text{and} \\ a_{2n+2}^x &= \text{the largest element of } A^x(w_{2n+2}^x) \text{ smaller than } 0. \end{cases}$$

Then $a_{2n+2}^x \leq a_{2n}^x$ because $A^x(w_{2n+2}^x) \subseteq A^x(w_{2n}^x)$, and similarly $a_{2n+1}^x \leq a_{2n+3}^x$.

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[5] The definition of $A^x(w)$ is not invariant in the same sense; however, if $x \mathsf{E} y$, then there exists $b \in \mathbb{Z}$ such that the sets $A^x(w)$ and $b + A^y(w)$ coincide.

It follows that

$$(2) \qquad \cdots \le a^x_4 \le a^x_2 \le a^x_0 = 0 < a^x_1 \le a^x_3 \le a^x_5 \le \cdots \quad .$$

Moreover, there exist infinitely many strict relations $<$ in (2) both to the left and to the right of 0, since otherwise $\bigcap_n A^x(w^x_n) \ne \varnothing$.

EXERCISE 8.2.1. Prove that if 0 is changed to a^x_{2n} in the first line of (1) and to a^x_{2n+1} in the second line, then the results will be the same. □

If $x \in Y$ and $n \in \mathbb{N}$, then put $d^x_n = |a^x_{n+1} - a^x_n|$ and $r^x_n = \langle r^x_{n0}, r^x_{n1}, \dots, r^x_{nm}\rangle$, where $m = d^x_n$ and $r^x_{ni} = x_{\min\{a^x_{n+1}, a^x_n\}+i} \restriction m$. Put $\vartheta(x)(n) = r^x_n$, this $\vartheta(x) \in W^{\mathbb{N}}$.

The following lemma ends the proof of the implication (v) $\Longrightarrow$ (i).

LEMMA 8.2.2. *Suppose that $x, y \in Y$. Then $x \mathsf{E} y$ iff $\vartheta(x)(n) = \vartheta(y)(n)$ for all but finite $n \in \mathbb{N}$. In other words, ϑ reduces $\mathsf{E} \restriction Y$ to the equivalence relation $\mathsf{E}_0(W)$, and hence $\mathsf{E} \restriction Y \le_{\mathsf{B}} \mathsf{E}_0$ by Lemma* 5.1.3.

PROOF. Let $x \mathsf{E} y$. As $\mathsf{E} = \mathbb{E}(\mathbb{Z}, 2^{\mathbb{N}})$, there is an integer $p \in \mathbb{Z}$ such that $x(a+p) = y(a)$ for all $a \in \mathbb{Z}$. Then $w^x_n = w^y_n$ and

$$(3) \qquad a + p \in A^x(w^x_n) \iff a \in A^y(w^y_n): \qquad \text{for all } a \in \mathbb{Z} \text{ and } n\,.$$

Assume w.l.o.g. that $p > 0$. Then $a^x_{2n_0+1} > p$ for some n_0. It follows from (3) that $a^x_{2n+1} = a^y_{2n+1} + p$ for all $n \ge n_0$. We conclude that $d^x_n = d^y_n$ and $r^x_n = r^y_n$ for all $n \ge n_0$, as required.

To prove the converse, assume that $\vartheta(x)(n) = \vartheta(y)(n)$ for all n. Then $d^x_n = d^y_n$ and $r^x_n = r^y_n$ for each $n \ge n_0$. It follows from the first equality that $a^x_{2n+1} = a^y_{2n+1} + p$ for all $n \ge n_0$, where $p \in \mathbb{Z}$ does not depend on n. (The result of Exercise 8.2.1 is applied.) The second equality implies $x(a+p) = y(a)$ for all $a \in \mathbb{Z}$, and hence $x \mathsf{E} y$. □ (Lemma 8.2.2)

(vi) $\Longrightarrow$ (v). Suppose that $\mathsf{E} = \mathsf{F} \vee \mathsf{R}$, where F, R are type-2 equivalence relations on $2^{\mathbb{N}}$. Let an F-pair be any pair $\{a, b\}$ in $2^{\mathbb{N}}$ such that $a \mathsf{F} b$. Let an F-singleton be any $x \in 2^{\mathbb{N}}$ F-equivalent only to itself. Then every $x \in 2^{\mathbb{N}}$ is either an F-singleton or a member of a unique F-pair.

Fix an arbitrary $x \in 2^{\mathbb{N}}$. We now define an oriented chain $\to$ on the equivalence class $[x]_{\mathsf{E}}$. For every F-pair $\langle a, b\rangle$ in $\mathbb{X}$, define $a \to b$ whenever $a <_{\mathtt{lex}} b$, where $<_{\mathtt{lex}}$ is the lexicographical order on $2^{\mathbb{N}}$. If $\{a <_{\mathtt{lex}} b\}$ and $\{a' <_{\mathtt{lex}} b'\}$ are different F-pairs, then define $b \to a'$ whenever either $b \mathsf{R} a'$ or $b \mathsf{R} b'$. (These two conditions are obviously incompatible.) If c is an F-singleton, then define $b \to c$ whenever $b \mathsf{R} c$, and $c \to a$ whenever $c \mathsf{R} a$. If finally $c \ne d$ are F-singletons, then define $c \to d$ whenever $c \mathsf{R} d$ and $c <_{\mathtt{lex}} d$.

If $[x]_{\mathsf{E}}$ has no endpoints in the sense of $\to$, then either

$$[x]_{\mathsf{E}} = \{\cdots \to a_{-2} \to a_{-1} \to a_0 \to a_1 \to a_2 \to \cdots\}$$

is a bi-infinite chain or $[x]_{\mathsf{E}} = \{a_1 \to a_2 \to a_3 \to \cdots \to a_n \to a_1\}$ is a finite cyclic chain. In the first subcase, we straightforwardly define an action of $\mathbb{Z}$ on $[x]_{\mathsf{E}}$ by $1 \cdot a_n = a_{n+1}$, $\forall n \in \mathbb{Z}$. In the second subcase, put $1 \cdot a_k = a_{k+1}$ for $k < n$, and $1 \cdot a_n = a_1$. If $[x]_{\mathsf{E}} = \{a_1 \to a_2 \to a_3 \to \cdots \to a_n\}$ is a chain with two endpoints, then the action is defined the same way. If finally $[x]_{\mathsf{E}}$ is a chain with just one endpoint, say $[x]_{\mathsf{E}} = \{a_0 \to a_1 \to a_2 \to \cdots\}$, then put $1 \cdot a_{2n} = a_{2n+2}$, $1 \cdot a_{2n+3} = a_{2n+1}$, and $1 \cdot a_1 = a_0$.

(v) $\Longrightarrow$ (vi). A short proof based on several difficult theorems on hyperfinite equivalence relations is given in [**JKL02**]. Here we present an elementary proof.

Let E be induced by a Borel action of $\mathbb{Z}$. We are going to define F and R on every E-class $C = [x]_{\mathsf{E}}$. If an element $x_C \in C$ can be chosen in some uniform Borel-definable way, then a rather easy construction is possible, which we leave to the reader. This applies, for instance, when C is finite, hence, let us assume that C is infinite. Then the linear order $<_{\mathtt{act}}$ on C induced by the action of $\mathbb{Z}$ is obviously similar to $\mathbb{Z}$. Let $<_{\mathtt{lex}}$ be the lexicographical ordering of $2^{\mathbb{N}} = \operatorname{dom} \mathsf{E}$.

Our goal is to define F on C so that every F-class contains exactly two (distinct) elements. The ensuing definition of R is then rather simple. (First, order pairs $\{x, y\}$ of elements of C in accordance with the $<_{\mathtt{act}}$-lexicographical ordering of pairs $\langle \mathtt{max}_{<_{\mathtt{act}}}\{x,y\}, \mathtt{min}_{<_{\mathtt{act}}}\{x,y\}\rangle$, this is still similar to $\mathbb{Z}$. Now, if $\{x, y\}$ and $\{x', y'\}$ are two F-classes, the latter being the next to the former in the sense just defined, and $x <_{\mathtt{act}} y$, $x' <_{\mathtt{act}} y'$, then define $y \,\mathsf{R}\, x'$.)

Suppose that $W \subseteq C$. An element $z \in W$ is *lmin* (locally minimal) in W if it is $<_{\mathtt{lex}}$-smaller than both of its $<_{\mathtt{act}}$-neighbours in W. Put

$$W_{\mathrm{lmin}} = \{z \in W : z \text{ is lmin in } W\}.$$

If C_{lmin} is *not* unbounded in C in both directions, then an appropriate choice of $x_C \in C$ is possible. (Take the $<_{\mathtt{act}}$-least or $<_{\mathtt{act}}$-largest point in C_{lmin}, or if $C_{\mathrm{lmin}} = \varnothing$, so that, for instance, $<_{\mathtt{act}}$ and $<_{\mathtt{lex}}$ coincide on C, we can choose something like a $<_{\mathtt{lex}}$-middest element of C.) Thus, we can assume that C_{lmin} is unbounded in C in both directions.

Let a *lmin-interval* be any $<_{\mathtt{act}}$-semi-interval $[x, x')$ between two consecutive elements $x <_{\mathtt{act}} x'$ of C_{lmin}. Let $[x, x') = \{x_0, x_1, \dots, x_{m-1}\}$ be the enumeration in the $<_{\mathtt{act}}$-increasing order ($x_0 = x$). Define $x_{2k} \mathsf{F} x_{2k+1}$ whenever $2k+1 < m$. If m is odd, then x_{m-1} remains unmatched. Let C^1 be the set of all unmatched elements. Now, the nontrivial case is when C^1 is unbounded in C in both directions. We define C^1_{lmin} as above, and repeat the same construction, extending F to a part of C^1 with, perhaps, a remainder $C^2 \subseteq C^1$ where F remains indefined. *Et cetera.*

Thus, we define a decreasing sequence $C = C^0 \supseteq C^1 \supseteq C^2 \supseteq \cdots$ of subsets of C, and the equivalence relation F on each difference $C^n \smallsetminus C^{n+1}$ whose classes contain exactly two points each, and the nontrivial case is when every C^n is $<_{\mathtt{act}}$-unbounded in C in both directions. (Otherwise there is an appropriate choice of $x_C \in C$.) If $C^\infty = \bigcap_n C^n = \varnothing$, then F is defined on C and we are done. If $C^\infty = \{x\}$ is a singleton, then $x_C = x$ chooses an element in C. Finally, C^∞ cannot contain two different elements as otherwise one of C^n would contain two $<_{\mathtt{act}}$-neighbours $x <_{\mathtt{act}} y$ which survive in C^{n+1}, which is clearly impossible.

□ (Theorem 8.1.1)

8.3. Hyperfiniteness of tail equivalence relations

This is an interesting series of hyperfinite equivalence relations not directly covered by Theorem 8.1.1. Given a Borel set X, we define the *tail equivalence relation* $\mathsf{E}_{\mathrm{t}}(X)$ on the set $X^{\mathbb{N}}$ as follows:

$$(*) \qquad \text{for } x, y \in X^{\mathbb{N}} : \qquad x \,\mathsf{E}_{\mathrm{t}}(X)\, y \ \text{ iff } \ \exists\, m\, \exists\, n\, \forall\, k\, (x(m+k) = y(n+k))\,.$$

The most elementary representative of this family is the tail equivalence relation $\mathsf{E}_{\mathrm{t}} = \mathsf{E}_{\mathrm{t}}(2)$ defined in Example 4.6.9.

Yet there is a more general approach. Suppose that $\mathbb{X}$ is a Polish space and $U : \mathbb{X} \to \mathbb{X}$ a Borel map. If $x, y \in \mathbb{X}$, then define

$$x \mathsf{E}_{\mathsf{t}}(U)\, y \quad \text{iff} \quad \exists m \,\exists n \,\forall k\, (U^{m+k}(x) = U^{n+k}(y)),$$

so that $\mathsf{E}_{\mathsf{t}}(U)$ is a Borel equivalence relation on $\mathbb{X}$. Then E_{t} coincides with $\mathsf{E}_{\mathsf{t}}(U)$, where $U : (2^{\mathbb{N}})^{\mathbb{N}} \to (2^{\mathbb{N}})^{\mathbb{N}}$ is the shift map; that is, $U(x) = y$ iff we have $y(n) = x(n+1)$ for all n.

THEOREM 8.3.1 (8.1 and 8.2 in **[DJK94]**). (i) *If* $\mathbb{X}$ *is a Polish space and* $U : \mathbb{X} \to \mathbb{X}$ *a Borel map, then* $\mathsf{E}_{\mathsf{t}}(U)$ *is a hypersmooth equivalence relation.*

(ii) *If, in addition,* U *is a countable-to-*1 *map, then* $\mathsf{E}_{\mathsf{t}}(U)$ *is hyperfinite. In particular, if* X *is countable, then* $\mathsf{E}_{\mathsf{t}}(X)$ *is hyperfinite.*

PROOF. (i) First of all we note that $\mathsf{E}_{\mathsf{t}}(U) \le_{\mathrm{B}} \mathsf{E}_{\mathsf{t}}(2^{\mathbb{N}})$: indeed, assuming that $\mathbb{X} = 2^{\mathbb{N}}$, the map $\vartheta(x) = \langle x, U(x), U^2(x), U^3(x), \dots \rangle$ is a reduction required. Thus, we have to show that the tail equivalence relation $\mathsf{E}_{\mathsf{t}}(2^{\mathbb{N}})$ on $(2^{\mathbb{N}})^{\mathbb{N}}$ is hypersmooth. The proof of this fact involves the same ideas and notation as the proof of the implication (v) $\Longrightarrow$ (i) of Theorem 8.1.1, from where we borrow notation like W_n, W, $<_n$, $A^x(w)$. In particular, if $x \in (2^{\mathbb{N}})^{\mathbb{N}}$ and $w \in W_n$, then the set $A^x(w)$ consists of all numbers $a \in \mathbb{N}$ satisfying $x(a+k)(i) = w(k,i)$ for all $k, i < n$. Put

$$w_n^x = \text{the } <_n\text{-least } w \in W_n \text{ such that } A^x(w) \text{ is infinite}$$

for $x \in (2^{\mathbb{N}})^{\mathbb{N}}$ and $n \in \mathbb{N}$, and then $b_n^x = \mathtt{min}\, A^x(w_n^x)$. Note a difference with the definition of w_n^x in the proof of (v) $\Longrightarrow$ (i) of Theorem 8.1.1. This slight change makes the definition of w_n^x to be $\mathsf{E}_{\mathsf{t}}(2^{\mathbb{N}})$-invariant, not only shift-invariant, so that

$$(*) \qquad x \mathsf{E}_{\mathsf{t}}(2^{\mathbb{N}})\, y \Longrightarrow w_n^x = w_n^y \quad \text{for all } n.$$

As for b_n^x, one easily proves that $b_n^x \le b_{n+1}^x$. This leads to the partition of the domain $(2^{\mathbb{N}})^{\mathbb{N}}$ of the equivalence relation $\mathsf{E}_{\mathsf{t}}(2^{\mathbb{N}})$ into the sets

$$\begin{aligned} X &= \{x \in (2^{\mathbb{N}})^{\mathbb{N}} : \text{ the sequence } \{b_n^x\}_{n \in \mathbb{N}} \text{ is eventually constant}\}, \\ Y &= \{x \in (2^{\mathbb{N}})^{\mathbb{N}} : \mathtt{lim}_{n \to \infty} b_n^x = +\infty\}. \end{aligned}$$

We claim that the restricted equivalence relation $\mathsf{E}_{\mathsf{t}}(2^{\mathbb{N}}) \restriction X$ is smooth, and this is witnessed by the Borel map $f(x) = \{w_n^x\}_{n \in \mathbb{N}}$. That f is $\mathsf{E}_{\mathsf{t}}(2^{\mathbb{N}})$-invariant follows from $(*)$. To prove the converse, suppose that $x, y \in X$ satisfy $f(x) = f(y)$; that is, $w_n^x = w_n^y$ for all n, and prove that $x \mathsf{E}_{\mathsf{t}}(2^{\mathbb{N}})\, y$. We have $b_n^x = b$ and $b_n^y = b'$ for all $n \ge n_0$. (The values $b, b', n_0 \in \mathbb{N}$ depend on x, y of course.) In other words, $b \in A^x(w_n^x)$ and $b' \in A^y(w_n^y)$ for $n \ge n_0$. As $w_n^x = w_n^y$, $\forall n$, it easily follows that $x(b+k) = y(b'+k)$ for all $k \in \mathbb{N}$; therefore, $x \mathsf{E}_{\mathsf{t}}(2^{\mathbb{N}})\, y$, as required.

We claim that $\mathsf{E}_{\mathsf{t}}(2^{\mathbb{N}}) \restriction Y \le_{\mathrm{B}} \mathsf{E}_1$. Suppose that $x \in Y$, so that $\mathtt{lim}_{n \to \infty} b_n^x = +\infty$. Let $\vartheta_n(x) = \langle x_{b_n^x}, x_{b_n^x+1}, \dots \rangle$. This is an infinite sequence of elements of $2^{\mathbb{N}}$, essentially an element of $(2^{\mathbb{N}})^{\mathbb{N}}$. We prove now that $x \mathsf{E}_{\mathsf{t}}(2^{\mathbb{N}})\, y$ iff $\vartheta_n(x) = \vartheta_n(y)$ for almost all n. As ϑ can be easily verified to be Borel, this result implies (modulo a Borel bijection between $(2^{\mathbb{N}})^{\mathbb{N}}$ and $2^{\mathbb{N}}$) that $\mathsf{E}_{\mathsf{t}}(2^{\mathbb{N}}) \restriction Y \le_{\mathrm{B}} \mathsf{E}_1$. That this implies hypersmoothness, see Lemma 11.2.1 below.

If $x, y \in Y$ and $\vartheta_n(x) = \vartheta_n(y)$ for almost all (or, that is equivalent, for at least one) n, then $x \mathsf{E}_{\mathsf{t}}(2^{\mathbb{N}})\, y$ is obvious. Suppose that $x \mathsf{E}_{\mathsf{t}}(2^{\mathbb{N}})\, y$. Then by the way $w_n^x = w_n^y$ for all n by $(*)$. Let $p, q \in \mathbb{N}$ be such that $x(p+k) = y(q+k)$, $\forall k$. As $x, y \in Y$, there exists r such that $b_r^x > p$ and $b_r^y > q$. Then easily $b_n^x - p = b_n^y - q$ for all $n \ge r$. Therefore, $\vartheta_n(x) = \vartheta_n(y)$ for $n \ge r$, as required.

(ii) Note that if U is countable-to-1 (that is, the U-preimage $\{x : U(x) = y\}$ of every point $y \in \mathbb{X}$ is at most countable), then $\mathsf{E_t}(U)$ is a countable Borel equivalence relation; therefore, (ii) follows from (i) by Theorem 8.1.1. □

8.4. Classification modulo Borel isomorphism

To give a motivation for this study, let us draw the following corollary of Theorem 8.1.1 and one more theorem.

COROLLARY 8.4.1. *If* E *is a non-smooth hyperfinite (Borel) equivalence relation, then* $\mathsf{E} \sim_{\mathrm{B}} \mathsf{E_0}$, *that is, both* $\mathsf{E} \le_{\mathrm{B}} \mathsf{E_0}$ *and* $\mathsf{E_0} \le_{\mathrm{B}} \mathsf{E}$.

PROOF. $\mathsf{E} \le_{\mathrm{B}} \mathsf{E_0}$ follows from Theorem 8.1.1. On the other hand, it follows from the 1st dichotomy theorem (Theorem 5.7.1, to be proved in Chapter 10) that either E is smooth or $\mathsf{E_0} \le_{\mathrm{B}} \mathsf{E}$. □

Thus all non-smooth hyperfinite (Borel) equivalence relations are $\sim_{\mathrm{B}}$-equivalent to each other. But this does not say much in terms of comparison of the structure of different non-smooth hyperfinite equivalence relations. For instance, it does not follow from Corollary 8.4.1 that these equivalence relations are pairwise Borel isomorphic (via a Borel bijection between the domains; see Section 5.1). And in fact such a conjecture fails: the classification of non-smooth hyperfinite relations turns out to be nontrivial! The following theorem treats the case of aperiodic hyperfinite equivalence relations, where E is *aperiodic*[6] iff every E-equivalence class is infinite.

THEOREM 8.4.2 (Classification theorem, 9.1 in **[DJK94]**). *If* E *is an aperiodic non-smooth hyperfinite equivalence relation, then* E *is Borel isomorphic to exactly one of the following equivalence relations:*

$\mathsf{E_t} = \mathsf{E_t}(2)$: *the tail equivalence relation on* $2^{\mathbb{N}}$;

$\mathsf{E_0} \times \Delta_n$: *where* Δ_n *is the equality on an* n*-element set* $s_n = \{1, 2, \dots, n\}$;

$\mathsf{E_0} \times \Delta_{\mathbb{N}}$: *where* $\Delta_{\mathbb{N}}$ *is the equality on* $\mathbb{N}$;

$\mathsf{Fr}(\mathbb{Z}, 2)$: *the free part of the shift equivalence relation* $\mathsf{E}(\mathbb{Z}, 2)$. □

Here $\mathsf{E_0} \times \Delta_n$ is the equivalence relation on $2^{\mathbb{N}} \times \{1, 2, \dots, n\}$ such that

$$(*) \qquad \langle a, i\rangle \ (\mathsf{E_0} \times \Delta_n) \ \langle b, j\rangle \quad \text{iff} \quad a \,\mathsf{E_0}\, b \wedge i = j\,.$$

$\mathsf{E_0} \times \Delta_{\mathbb{N}}$ is defined on $2^{\mathbb{N}} \times \mathbb{N}$ the same way. Finally, $\mathsf{E}(\mathbb{Z}, 2)$ is induced on $2^{\mathbb{Z}}$ by the shift action of $\mathbb{Z}$ (see Example 4.4.4) and $\mathsf{Fr}(\mathbb{Z}, 2) = \mathsf{E}(\mathbb{Z}, 2) \restriction (2)^{\mathbb{Z}}$, where $(2)^{\mathbb{Z}}$ is the free domain of the action, in this case equal to the set

$$A^* = \{x \in 2^{\mathbb{Z}} : \text{the equivalence class } [x]_{\mathsf{E}(\mathbb{Z},2)} \text{ is infinite}\}$$

of all aperiodic elements $z \in 2^{\mathbb{Z}}$ (A^* is Borel; see below).

EXERCISE 8.4.3. Show that all equivalence relations mentioned in Theorem 8.4.2 are hyperfinite and aperiodic. □

EXERCISE 8.4.4. Using Exercise 8.1.2, Theorem 8.3.1(ii), and Exercises 4.6.11 and 4.6.12, prove that the equivalence relations $\mathsf{E_t}$ and Vit are Borel isomorphic. □

[6] To understand the origins of this notion, consider any equivalence relation E induced by a Polish action of the group of integers $\mathbb{Z}$ on a Borel set X. An element $x \in X$ is *periodic* iff $x = j \cdot x$ holds for some $j \in \mathbb{Z}$, $j \neq 0$, and *aperiodic* otherwise. Obviously, x is aperiodic iff the equivalence class $[x]_{\mathsf{E}} = \{j \cdot x : j \in \mathbb{Z}\}$ is infinite.

Non-aperiodic case.

It takes a bit of extra work to derive a suitable classification of all (not necessarily aperiodic) hyperfinite Borel equivalence relations up to Borel isomorphism.

COROLLARY 8.4.5. *Suppose that* E,F *are non-smooth hyperfinite equivalence relations on Borel sets* X, *resp.*, Y. *Then all sets*

$$X_n = \{x \in X : \mathtt{card}\,[x]_{\mathsf{E}} = n\} \quad \textit{and} \quad Y_n = \{y \in Y : \mathtt{card}\,[y]_{\mathsf{E}} = n\}\,, \quad n \in \mathbb{N}\,,$$

and $X^* = X \smallsetminus \bigcup_n X_n$, $Y^* = Y \smallsetminus \bigcup_n Y_n$ *are Borel.*

In addition, E *is Borel isomorphic to* F *if and only if*

1) *for every* $n \in \mathbb{N}$, $\mathtt{card}\,X_n = \mathtt{card}\,Y_n$; *and*
2) *the aperiodic parts* $\mathsf{E} \restriction X^*$ *and* $\mathsf{F} \restriction Y^*$ *are Borel isomorphic to one and the same equivalence relation in the list of Theorem* 8.4.2.

Note that the cardinality $\mathtt{card}\,X$ of a Borel set X in a Polish space is either a natural number n, or $\aleph_0$, or the cardinality of continuum $\mathfrak{c}$, by Theorem 2.12.6.

PROOF. Let us show that, say, X_2 is Borel. Indeed by Theorem 7.4.1 E is the orbit equivalence relation of a Polish action of a countable group $\mathbb{G}$ on $\mathbb{X}$. Then

$$X_2 = \{x \in \mathbb{X} : \exists\, a,b \in \mathbb{G}\,\big(a\cdot x \neq b \cdot x \wedge \forall\, c \in \mathbb{G}\,(c \cdot x = a \cdot x \vee c\cdot x = b \cdot x)\big)\}\,,$$

clearly a Borel set. By the same argument, all sets X_n and Y_n are Borel, and hence so are the aperiodic subdomains X^* and Y^*. (In particular the set $A^* \subseteq 2^{\mathbb{Z}}$ above is Borel and $\mathsf{Fr}(\mathbb{Z},2)$ is a Borel equivalence relation.)

The "only if" part in the last claim of the corollary is obvious. As for the "if" part, clearly it remains to prove that $\mathsf{E}\restriction X_n$ is Borel isomorphic to $\mathsf{F}\restriction Y_n$ provided $\mathtt{card}\,X_n = \mathtt{card}\,Y_n$. Let $\prec$ be any Borel linear order on $\mathbb{X}$. Since every E-class $[x]_{\mathsf{E}}$ in X_n contains exactly n elements, we define

$$X_n^i = \{x : x \text{ is the } i\text{-th element of } [x]_{\mathsf{E}} \cap X_n \text{ in the sense of } \prec\},$$

and Y_n^i the same way, for $i = 1,\dots,n$. The sets X_n^i, Y_n^i are Borel (see the proof of Proposition 7.2.1(ii)), and obviously still $\mathtt{card}\,X_n^i = \mathtt{card}\,Y_n^i$ for all i,n. By Corollary 2.12.8 there exist Borel bijections $f_n^i : X_n^i \xrightarrow{\text{onto}} Y_n^i$. On the other hand, the sets X_n^i are pairwise disjoint transversals in X_n, as are Y_n^i in Y_n. It follows that $f = \bigcup_{1\le i\le n} f_n^i$ is a Borel isomorphism of $\mathsf{E}\restriction X_n$ onto $\mathsf{F}\restriction Y_n$, as required. □

EXERCISE 8.4.6. Show that under the assumptions of Corollary 8.4.5 the equivalence relations $\mathsf{E}\restriction X_n$ and $\mathsf{F}\restriction Y_n$ are smooth while the equivalence relations $\mathsf{E}\restriction X^*$ and $\mathsf{F}\restriction Y^*$ are non-smooth hyperfinite. Use Corollary 7.3.2. □

8.5. Remarks on the classification theorem

We we are not going to present a proof of Theorem 8.4.2 here. (See a fairly complicated proof in [**DJK94**, theorem 9.1].) But some concepts and results involved in the proof are worth mentioning.

Remarkably, the number $\mathtt{card}\,(\mathbf{EINV}_{\mathsf{E}})$ (a natural number or an infinite cardinal) of all E-invariant E-ergodic probability measures (see Definition Section 4.5) is the true parameter behind the classification in Theorem 8.4.2. Given a Polish space $\mathbb{X}$, the set $\boldsymbol{P}(\mathbb{X})$ of all probability measures on $\mathbb{X}$ (see Definition 4.5.1) is a Polish space itself in the topology generated by all sets of the form

$$U_{\mu\,\varepsilon\,f_1\cdots f_n} = \{\nu \in \boldsymbol{P}(\mathbb{X}) : \left|\int f_i d\nu - \int f_i d\mu\right| < \varepsilon \text{ for all } i = 1,\dots,n\},$$

where $\mu \in \boldsymbol{P}(\mathbb{X})$, $\varepsilon > 0$, and $f_1 \cdots f_n$ are bounded continuous real functions on $\mathbb{X}$; see 17.E in [**Kec95**] or §4 in [**DJK94**].[7] Moreover, if in addition E is a Borel equivalence relation on $\mathbb{X}$, then the set $\mathbf{EINV}_{\mathsf{E}}$ of all E-invariant E-ergodic probability measures (Definition 4.6.2) turns out to be a $\mathbf{G}_\delta$ set in $\boldsymbol{P}(\mathbb{X})$; see 17.33 in [**Kec95**]. It follows that in this case the cardinality $\mathtt{card}\,(\mathbf{EINV}_{\mathsf{E}})$ is either a natural number, or $\aleph_0$, or the continuum $\mathfrak{c}$ (we refer to Theorem 2.12.6).

It turns out that the equivalence relations mentioned in Theorem 8.4.2 have different values of $\mathtt{card}\,(\mathbf{EINV}_{\mathsf{E}})$:

1) $\mathtt{card}\,(\mathbf{EINV}_{\mathsf{E}_t}) = 0$ for the tail equivalence relation by Exercise 4.6.11.

2) $\mathtt{card}\,(\mathbf{EINV}_{\mathsf{E}_0\times\Delta_n}) = n$ for any $n \ge 1$. Indeed we have $\mathtt{card}\,(\mathbf{EINV}_{\mathsf{E}_0}) = 1$. In fact there is a unique E_0-invariant probability measure λ on $2^{\mathbb{N}}$, and it is E_0-ergodic; see Exercise 4.6.5. Now fix $n \ge 1$ and let $s_n = \{1,\dots,n\}$, as above, so that $\mathsf{E}_0 \times \Delta_n$ is an equivalence relation on $2^{\mathbb{N}} \times s_n$ defined by $(*)$. Put $X_k = \{a \in 2^{\mathbb{N}} : \langle a, k\rangle \in X\}$ for $X \subseteq 2^{\mathbb{N}} \times s_n$ and $1 \le k \le n$. The measures $\mu_k(X) = \lambda(X_k)$ obviously belong to $\mathbf{EINV}_{\mathsf{E}_0\times\Delta_n}$. On the other hand the sets $U_k = 2^{\mathbb{N}} \times \{k\}$ are $(\mathsf{E}_0 \times \Delta_n)$-invariant; therefore, if $\mu \in \mathbf{EINV}_{\mathsf{E}_0\times\Delta_n}$, then for exactly one k we have $\mu(U_k) = 1$, so that $\mu = \mu_k$.

3) $\mathtt{card}\,(\mathbf{EINV}_{\mathsf{E}_0\times\Delta_{\mathbb{N}}}) = \aleph_0$ by exactly the same reasons.

4) $\mathtt{card}\,(\mathbf{EINV}_{\mathsf{E}(\mathbb{Z},2)}) = \mathfrak{c}$ by Example 4.6.6, and we leave it as an *exercise* for the reader to show that $\mathtt{card}\,(\mathbf{EINV}_{\mathsf{Fr}(\mathbb{Z},2)}) = \mathfrak{c}$ as well.

Since obviously $\mathtt{card}\,(\mathbf{EINV}_{\mathsf{E}}) = \mathtt{card}\,(\mathbf{EINV}_{\mathsf{F}})$ whenever Borel equivalence relations E, F are Borel isomorphic, it follows that all particular equivalence relations mentioned in Theorem 8.4.2 are pairwise Borel non-isomorphic. This is the first, comparably easy part of Theorem 8.4.2. The second, complicated part is to prove that if E is a hyperfinite aperiodic equivalence relation, then it is Borel isomorphic to an equivalence relation F in the list of Theorem 8.4.2, namely, to that one which has the same number $\mathtt{card}\,(\mathbf{EINV}_{\mathsf{F}})$ of invariant ergodic measures as E. The proof in [**DJK94**] is based on several difficult theorems of ergodic theory, and among them the following result originally established in [**Dye63**]. For a modern proof, see Theorem 7.13 in [**KM04**].

THEOREM 8.5.1 (DYE's theorem). *Suppose that E, F are hyperfinite equivalence relations on Borel sets X, Y, and μ, ν are invariant ergodic measures on X, Y. Then there exist an E-invariant Borel set $X' \subseteq X$ and an F-invariant Borel set $Y' \subseteq Y$ with $\mu(X') = \nu(Y') = 1$, and a Borel isomorphism $\vartheta : X' \xrightarrow{\text{onto}} Y'$ that sends $\mathsf{E} \restriction X'$ onto $\mathsf{F} \restriction Y'$ and μ to ν.* □

Thus, hyperfinite equivalence relations, generally speaking Borel non-isomorphic by Theorem 8.4.2, become isomorphic on suitable invariant Borel sets of full measure in the sense of given invariant ergodic measures. By the way, Theorem 8.5.1

[7] That $\mathbb{X}$ is assumed to be a Polish space here does not restrict generality *w.r.t.* the case of measures on an arbitrary Borel set X in a Polish space. Indeed it follows from Corollary 1.2.3 that we can strengthen the relative topology on such a set X to a Polish topology on X which produces exactly the same Borel sets.

is not applicable for the tail equivalence relation E_t since there is no invariant ergodic measures for E_t by Exercise 4.6.11, and generally speaking it is not applicable for non-hyperfinite countable equivalence relations by Corollary 9.3.3(iv) below.

8.6. Which groups induce hyperfinite equivalence relations?

This question belongs to a series of very interesting and mostly very difficult and largely open problems in this area. Let us mention several known results of this kind without going into detail. First of all, for an equivalence relation E induced by a Borel action of a countable group $\mathbb{G}$ to be hyperfinite, each of the following conditions is sufficient:

(1) $\mathbb{G} = \mathbb{Z}$, the additive group of integers, by Theorem 8.1.1.

(2) $\mathbb{G} = \mathbb{Z}^n$ $(n \geq 2)$, by an unpublished theorem of WEISS.

(3) $\mathbb{G}$ is abelian, by a recent result of SU GAO and JACKSON [**GJ07**].

(4) $\mathbb{G}$ is a finitely generated group of polynomial growth,[8] see Theorem 11.1 in [**KM04**].

The last-mentioned result is a corollary of a somewhat more general theorem in [**JKL02**], saying that locally compact Polish groups of polynomial growth over a compact kernel induce essentially hyperfinite equivalence relations. A related result of [**Kec92**]: if $\mathbb{G}$ is a locally compact Polish group of any kind, then $\mathbb{G}$ induces only essentially countable equivalence relations. (Note that countable groups are locally compact in discrete topology.)

QUESTION 8.6.1. Let $\mathbb{G}$ consist of all eventually constant sequences in $\mathbb{Z}^{\mathbb{N}}$. This is a countable abelian group (with the operation of component-wise addition), a subgroup of $\mathbb{R}^{\mathbb{N}}$. The equivalence relation E induced by the component-wise addition action of $\mathbb{G}$ on $\mathbb{Z}^{\mathbb{N}}$ is obviously countable, moreover, hyperfinite by (3) above. Is the hyperfiniteness of E provable by more elementary methods (like those used in the proof of Theorem 8.1.1)? □

In the opposite direction, it turns out that if all orbit equivalence relations of Borel actions of a Polish group $\mathbb{G}$ are smooth, then $\mathbb{G}$ is a compact group; see [**Sol00**]. And if all orbit equivalence relations of Borel actions of a countable group are hyperfinite, then the group belongs to the class of *amenable* (countable) groups considered in the next chapter.

We finish with a problem considered as one of the central problems in the field of hyperfinite equivalence relations. It has been cited in the literature since the early 1990s at least.

QUESTION 8.6.2. Suppose that $\mathsf{F}_0 \subsetneq \mathsf{F}_1 \subsetneq \cdots \subsetneq \mathsf{F}_n \subsetneq \cdots$ is an $\subsetneq$-increasing sequence of hyperfinite equivalence relations F_n. Is their union $\mathsf{F} = \bigcup_n \mathsf{F}_n$ hyperfinite? □

[8] A finitely generated group $\mathbb{G}$ is a group *of polynomial growth* iff it admits a finite symmetric set of generators $\{g_1, \ldots, g_k\}$ such that the number of elements of length n is bounded above by a polynomial function $p(n)$. Elements of length n are those of the form $g_{i_1} g_{i_2} \cdots g_{i_n}$, where $1 \leq i_r \leq n$ for all $r = 1, \ldots, k$.

CHAPTER 9

More on countable equivalence relations

Recall that $\mathsf{E}_\infty = \mathsf{E}(F_2, 2)$, the orbit equivalence relation of the shift action of F_2 on 2^{F_2}, is a $\le_{\mathrm{B}}$-largest countable Borel equivalence relation in the sense that any other countable Borel equivalence relation E satisfies $\mathsf{E} \le_{\mathrm{B}} \mathsf{E}_\infty$. On the other hand, E_0 is a $\le_{\mathrm{B}}$-least non-smooth Borel equivalence relation by the 1st dichotomy theorem (Theorem 5.7.1 or 10.1.1). It follows that any countable non-smooth Borel equivalence relation E belongs to the $\le_{\mathrm{B}}$-interval between E_0 and E_∞. That $\mathsf{E}_0 <_{\mathrm{B}} \mathsf{E}_\infty$ strictly follows from Theorem 7.5.1. This leads to the question, what is the $\le_{\mathrm{B}}$-structure of this interval?

It was once considered plausible that the interval is in fact empty. (See *e.g.* a question on the middle of p. 896 in [**Kec93**].) However a couple of *intermediate* (that is, strictly $\le_{\mathrm{B}}$-between E_0 and E_∞) countable equivalence relations were discovered in the mid-1990s. Further studies (for instance [**AK00, HK05**]) demonstrated that in fact the $\le_{\mathrm{B}}$-structure between E_0 and E_∞ is extremely rich. In particular, it contains continuum-many different and $\le_{\mathrm{B}}$-incomparable countable Borel equivalence relations. In fact one of the main aspects of the study of countable Borel equivalence relations from the descriptive set theoretic standpoint is the discovery of different types of intermediate equivalence relations and $\le_{\mathrm{B}}$-connections between them.

We are not going to present these remarkable results in detail or with at least key proofs since the technique they depend upon includes much deeper methods and results in egrodic theory and algebra than is suitable in this book. Nevertheless we decided to add this chapter as a sort of introduction to these studies that does not go into technical details.

There are two most powerful methods of definition of countable Borel equivalence relations, that is, 1) consider actions of various countable groups, and 2) consider countable Borel equivalence relations which admit an invariant Borel assignment of a structure of a certain type to every equivalence class. Both approaches can be traced down to the study of hyperfinite equivalence relations: indeed, by Theorem 8.1.1, this class can be characterized both as orbit equivalence relations of Borel actions of $\mathbb{Z}$ and as those which admit an invariant Borel assignment of a linear order of C, similar to a subset of $\mathbb{Z}$, to every equivalence class C.

We begin our short survey with the second approach.

Obviously there exist a lot of different structures that can be reasonably considered instead of linear orders embeddable in $\mathbb{Z}$. Two of them play a distinguished role in the development of the theory of countable Borel equivalence relations. They are finitely additive measures leading to *amenable* groups and equivalence relations, and trees leading to *treeable* equivalence relations. The first type (amenable equivalence relations) has not yet produced any intermediate countable Borel equivalence relations, nor is there any conclusive evidence that all equivalence relations of this class are hyperfinite, yet it certainly deserves at least a brief intoduction here.

9.1. Amenable groups

This is a widely studied type of groups (especially among countable ones).

DEFINITION 9.1.1. A group $\mathbb{G}$ is *amenable* iff there is a finitely additive probability measure (*f.a.p.m.*) $\varphi : \mathscr{P}(\mathbb{G}) \longrightarrow [0,1]$, left-invariant in the sense that $\varphi(X) = \varphi(aX)$ for any $X \subseteq \mathbb{G}$ and $a \in G$, where $aX = \{ax : x \in X\}$. □

If $\varphi_{\mathtt{left}}$ is a left-invariant measure, then $\varphi_{\mathtt{right}}(A) = \varphi_{\mathtt{left}}(A^{-1})$ is a right-invariant one, and $\varphi(A) = \int \varphi_{\mathtt{left}}(A\gamma^{-1})\, d\varphi_{\mathtt{right}}(\gamma)$ is a two-sided invariant one.

LEMMA 9.1.2. *The additive group of integers $\mathbb{Z}$ is amenable.*

PROOF. Fix a non-trivial (*i.e.*, not containing singletons) ultrafilter $U \subseteq \mathscr{P}(\mathbb{N})$. Take any set $A \subseteq \mathbb{Z}$. To define $\varphi(A)$, put $\varphi_n(A) = \frac{\mathtt{card}\, A\cap[-n,n]}{2n+1}$ for all n. If p is a real in $[0,1]$, then one of the two complementary sets

$$N_p^+(A) = \{n : p > \varphi_n(A)\}\,, \quad N_p^-(A) = \{n : p \le \varphi_n(A)\}\,,$$

belongs to U. The sets

$$P^+(A) = \{p : N_p^+(A) \in U\}\,, \quad P^-(A) = \{p : N_p^-(A) \in U\}$$

form a Dedekind cut in $[0,1]$. Let $\varphi(A)$ be either the least real in $P^+(A)$ or the largest one in $P^-(A)$.[1] The left-invariance of φ follows from the fact that for any $A \subseteq \mathbb{Z}$, $z \in \mathbb{Z}$, $\varepsilon > 0$ we have $|\varphi_n(A) - \varphi_n(z+A)| < \varepsilon$ for all sufficiently large natural n. □

In fact we have the following much more general result:

THEOREM 9.1.3 (see *e.g.* 5.7 in [**KM04**]). *All countable abelian, and even solvable, groups are amenable.* □

The proof of 9.1.2 shows that the verification of amenability is not entirely elementary even in the most elementary cases, and the *f.a.p.m.*'s involved usually require the Axiom of Choice.

On the other hand, non-amenability is sometimes rather obvious.

LEMMA 9.1.4. *F_2, the free group with two generators, is not amenable.*

PROOF. Let a, b be the generators. Suppose that $\varphi : \mathscr{P}(F_2) \longrightarrow [0,1]$ is a left-invariant *f.a.p.m.* Let, for $x \in \{a, b, a^{-1}, b^{-1}\}$, W_x denote the set of all words in F_2 beginning with x. As obviously $F_2 = W_a \cup aW_{a^{-1}}$, we have $\varphi(W_a) + \varphi(W_{a^{-1}}) = 1$. Similarly, $\varphi(W_b) + \varphi(W_{b^{-1}}) = 1$. On the other hand, $F_2 = \{1\} \cup W_a \cup W_{a^{-1}} \cup W_b \cup W_{b^{-1}}$, thus $\varphi(W_a) + \varphi(W_{a^{-1}}) + \varphi(W_b) + \varphi(W_{b^{-1}}) \le 1$; a contradiction. □

This lemma also admits a much more general form. Suppose that G is a group. If $X, Y \subseteq \mathbb{G}$, then $X \sim Y$ means that there are partitions $X_1, \dots, X_n$ of X and $Y_1, \dots, Y_n$ of Y and elements $g_1, \dots, g_n \in \mathbb{G}$ such that $g_i X_i = Y_i$ for all i. And $\mathbb{G}$ is called *paradoxical* if there exist disjoint sets $A, B \subseteq \mathbb{G}$ such that $A \sim B \sim \mathbb{G}$. Obviously, $\mathbb{G}$ is not amenable in this case. Remarkably the converse is true as well:

[1] In other words, $\varphi(A)$ is the limit over U of the sequence of numbers $\varphi_n = \varphi_n(A)$.

THEOREM 9.1.5 (TARSKI, see, e.g., [**Wag93**]). *A group* $\mathbb{G}$ *is amenable iff it is not paradoxical.* □

It follows that being amenable is *a projective notion* for countable groups; that is, it is expressible in terms of the existence or non-existence of certain subsets of $\mathbb{G}$, or, in other words, by an analytic formula in the sense of Section 1.4, in the natural assumption that the underlying set of $\mathbb{G}$ is $\mathbb{N}$. This property is not clear from the definition of an amenable group.

There is a slightly different (but equivalent) approach to amenability. Instead of f.a.p.m.'s, it is based on means.

DEFINITION 9.1.6. A *mean* on a countable set C is a positive linear functional $m : \ell^\infty(C) \to \mathbb{R}$ defined on the set $\ell^\infty(C)$ of all bounded maps $f : C \to \mathbb{R}$ and such that $m(1) = 1$, where $1 \in \ell^\infty(C)$ is the constant 1. (The *positivity* means that if $f(c) \geq 0$ for all $c \in C$, then $m(f) \geq 0$.)

Suppose that $C = \mathbb{G}$ is a countable group. A mean $m : \ell^\infty(\mathbb{G}) \to \mathbb{R}$ is *left-invariant* if $m(f) = m(a \cdot f)$ for all $a \in \mathbb{G}$ and f, where $a \cdot f \in \ell^\infty(\mathbb{G})$ is defined so that $(a \cdot f)(b) = f(a^{-1}b)$ for all $b \in \mathbb{G}$. □

The means and f.a.p.m.'s are naturally connected. Indeed, if m is a left-invariant mean on a countable group $\mathbb{G}$, then $\varphi(X) = m(1_X)$ is a left-invariant f.a.p.m., where 1_X is the characteristic function of X. And conversely, if φ is a left-invariant f.a.p.m. on $\mathbb{G}$, then $m(f) = \int f\, d\varphi$ is a left-invariant mean.

COROLLARY 9.1.7. *A countable group* $\mathbb{G}$ *is amenable iff it admits a left-invariant mean iff it admits a right-invariant mean.* □

Similar to f.a.p.m.'s, a left-invariant mean can be converted to a right-invariant mean (and conversely), and both can be combined to get a two-sided invariant mean.

9.2. Amenable equivalence relations

One may want to define an equivalence relation to be amenable iff it is induced by a Borel action of a countable amenable group. The actual definition, invented in ergodic theory, is more complicated in particular because it involves probability measures on the underlying spaces.

DEFINITION 9.2.1. Let $\mathbb{X}, \mathbb{Y}$ be Polish spaces, and μ a Borel measure on $\mathbb{X}$. A map $f : \mathbb{X} \to \mathbb{Y}$ is *μ-measurable* iff all f-preimages of open (equivalently, of Borel) sets in $\mathbb{Y}$ are μ-measurable. (Compare with Remark 1.9.4.) A map f is *universally measurable* if it is μ-measurable for every Borel measure μ on $\mathbb{X}$. □

DEFINITION 9.2.2 (see e.g. [**CFW81, Kec93, JKL02, HK05**]). Suppose that μ is a probability measure on a Polish space $\mathbb{X}$. A countable Borel equivalence relation E on $\mathbb{X}$ is *μ-amenable*, resp., *amenable*,[2] if there is an assignment of a mean m_C to every E-equivalence class $C \subseteq \mathbb{X}$ such that for any Borel $f : \mathsf{E} \to [-1, 1]$ the map $F(x) = m_{[x]_\mathsf{E}}(f_x)$ is μ-measurable, resp., universally measurable. □

[2] There are other and not necessarily equivalent definitions of amenable equivalence relations; see [**KM04**] or *measure amenability* in [**JKL02**]. An essentially different concept of *Frechet amenability* was developed in [**JKL02**].

Here $f_x(y) = f(x,y)$ for all $x \in \mathbb{X}$ and $y \in [x]_{\mathsf{E}}$, so that $f_x : [x]_{\mathsf{E}} \to [-1,1]$ and hence $f_x \in \ell^\infty([x]_{\mathsf{E}})$; therefore, $F(x) = m_{[x]_{\mathsf{E}}}(f_x)$ is defined for all $x \in \mathbb{X}$. Finally, by the assumption of positivity F maps $\mathbb{X}$ to $[-1,1]$.

However, the next theorem shows that there is an intrinsic connection between amenable groups and amenable equivalence relations.

THEOREM 9.2.3. *Suppose that* E *is induced by a Borel action of a countable group* $\mathbb{G}$ *on a Polish space* $\mathbb{X}$, *and* μ *is a probability measure on* $\mathbb{X}$. *Then*

(i) *if* $\mathbb{G}$ *is amenable, then* E *is* μ*-amenable*;

(ii) *in particular if* E *is hyperfinite, then it is amenable*;

(iii) *if* μ *is* E*-invariant,* $X \subseteq \mathbb{X}$ *is a Borel set,* $\mu(X) = 1$, *the action is free on* X, *that is,* $g \neq 1 \Longrightarrow g \cdot x \neq x$ *for all* $x \in X$, *and* E *is* μ*-amenable, then the group* $\mathbb{G}$ *itself is amenable.*

PROOF (sketch given in [**JKL02, Kec91**]). (i) First of all, note that the claim can be expressed by an analytic formula. Indeed regarding the notion of group amenability see the remark after 9.1.5. Regarding the μ-amenability of E, analytic expressibility is based on the existence of different but equivalent definitions of the notion of μ-amenable equivalence relation, as *e.g.* 2.3 in [**JKL02**] or the definition just before 9.1 in [**KM04**], the analytic (in the sense considered) character of which becomes clear after a lengthy inspection. See for instance pp. 901 – 902 in [**Kec93**].

Second, to prove an analytically expressed claim P it suffices to prove P under **CH**, the continuum hypothesis. This reduction can be established by the following metamathematical argument. To prove P means to demonstrate that P holds in every set theoretic universe V. (We refer to the completeness theorem in model theory.) However COHEN's forcing method allows to extend V to a bigger universe V' still satisfying the axioms of set theory, containing the same reals (here: subsets of $\mathbb{N}$ and elements of $\mathbb{N}^{\mathbb{N}}$) and satisfying **CH**. Then P is true in V' because of **CH**. Therefore P is true in V since the reals are the same.

Conclusion: it suffices to prove (i) under **CH**. And this reduced claim is based on the following mean-existence result due to Christensen and Mokobodzki.

PROPOSITION 9.2.4 (see 2.1 and 2.2 in [**KM04**] and references there). *Assuming* **CH**, *there is a universally measurable mean on* $\mathbb{N}$.[3] *Moreover, if* $\mathbb{G}$ *is a countable group, then there is a universally measurable left-invariant mean on* $\mathbb{G}$. □

To apply this result in the proof of (i), we transform a universally measurable left-invariant mean on $\mathbb{G}$ to a universally measurable right-invariant mean on $\mathbb{G}$, let it be m, and then define m_C for each E-equivalence class $C = [x]_{\mathsf{E}} \subseteq \mathbb{X}$ as follows. If $f \in \ell^\infty(C)$, then $m_C(f) = m(f')$, where $f' \in \ell^\infty(\mathbb{G})$ is defined by $f'(a) = f(a \cdot x)$, $\forall a \in \mathbb{G}$. By the right invariance of m, this does not depend on the choice of $x \in C$.

[3] The measurability of means deserves some comments. By definition a mean m on $\mathbb{N}$ is a linear map $\ell^\infty(\mathbb{N}) \to \mathbb{R}$. The restriction $m \restriction [-1,1]^{\mathbb{N}}$ of m to the set $[-1,1]^{\mathbb{N}} \subseteq \ell^\infty(\mathbb{N})$ of all sequences of reals in $[-1,1]$ easily defines m as a whole. On the other hand $[-1,1]^{\mathbb{N}}$ is an infinite product of Polish spaces and therefore, a Polish space itself. Define m to be μ-measurable, resp., universally measurable, if the restriction $m \restriction [-1,1]^{\mathbb{N}}$ is universally measurable in the sense of Definition 9.2.1.

(ii) Hyperfinite equivalence relations are induced by Borel actions of $\mathbb{Z}$ by Theorem 8.1.1, and $\mathbb{Z}$ is amenable by Lemma 9.1.2.

(iii) If $C \mapsto m_C$ witnesses the amenability of E, then $\varphi(A) = \int \varphi_{[x]_{\mathsf{E}}}(A \cdot x)\, d\mu(x)$ is the right-invariant *f.a.p.m.* on $\mathbb{G}$, where φ_C is the *f.a.p.m.* associated to the mean m_C. The integral is restricted to an invariant set of μ-measure 1 on which the action of $\mathbb{G}$ is free. □

9.3. Hyperfiniteness and amenability

Theorem 9.2.3 leads us to the following classes of Borel equivalence relations:

HF: hyperfinite equivalence relations;

AGA: those induced by Borel actions of countable amenable groups;

$\forall\mu$A: μ-amenable for each relevant Borel probability measure μ;

A: amenable.

We have **HF** $\subseteq$ **AGA** $\subseteq$ **$\forall\mu$A** by Theorem 9.2.3 and **A** $\subseteq$ **$\forall\mu$A** by obvious reasons.

Not much is known beyond this. For instance whether **AGA** = **HF**, **A** $\subseteq$ **HF**, **A** = **$\forall\mu$A** hold are long-standing problems (see, *e.g.*, [**Kec93**] or a commented list of problems in the end of [**JKL02**]). According to a recent theorem in [**GJ07**], all equivalence relations induced by Borel actions of countable abelian groups are hyperfinite, but amenable groups are not necessarily abelian. The continuum hypothesis **CH** simplifies the picture to some extent: then **$\forall\mu$A** = **A** by the next theorem (3.2 in [**Kec93**]).

THEOREM 9.3.1 (under **CH**). *If E is a Borel countable equivalence relation on a Polish space $\mathbb{X}$, then the following are equivalent:*

(1) E *is amenable*;

(2) E *belongs to* **$\forall\mu$A**;

(3) *for every Borel probability measure μ on $\mathbb{X}$, there is a Borel E-invariant set $Y \subseteq \mathbb{X}$ with $\mu(X) = 1$ such that $\mathsf{E} \restriction Y$ is hyperfinite*;

(4) E *is "universally measurably hyperfinite", that is, induced by a universally measurable action of $\mathbb{Z}$.* □

The equivalence (2) $\Longleftrightarrow$ (3) in this theorem is a corollary of the following earlier result in [**CFW81**]:

THEOREM 9.3.2. *If E is induced by a Borel action of a countable group $\mathbb{G}$ on a Polish space $\mathbb{X}$, μ is a probability measure on $\mathbb{X}$, and E is μ-amenable, then there is a Borel E-invariant set $X \subseteq \mathbb{X}$ such that $\mu(X) = 1$ and $\mathsf{E} \restriction X$ is hyperfinite.* □

We finish with another corollary of Theorem 9.2.3.

COROLLARY 9.3.3. *Suppose that $\mathbb{G}$ is a countable group acting on $2^{\mathbb{G}}$ by shift, and $\lambda = \lambda_{1/2}$ is the product $(\frac{1}{2}, \frac{1}{2})$-measure on $2^{\mathbb{G}}$ (see Examples 4.4.4 and 4.5.4). Then*:

(i) *if the shift equivalence relation $\mathsf{E}(\mathbb{G}, 2)$ is λ-amenable, then $\mathbb{G}$ is amenable*;

(ii) *in particular if $\mathsf{E}(\mathbb{G}, 2)$ is hyperfinite, then $\mathbb{G}$ is amenable.*

In the case when $\mathbb{G} = F_2$, *the free group with two generators*:

(iii) *The shift equivalence relation* $\mathsf{E}_\infty = \mathbb{E}(F_2, 2)$ *is not hyperfinite*;

(iv) [**JKL02**, 1.7, 1.8] *if* $X \subseteq 2^{F_2}$ *is an* E_∞*-invariant Borel set,* $\lambda(X) = 1$, *and the shift action of* F_2 *on* X *is free, then* $\mathsf{E}_\infty \restriction X$ *is not hyperfinite*;

(v) *in particular, for* $X = (2)^{F_2}$, *the free part* $\mathsf{E}_{\infty T} = \mathbb{Fr}(F_2, 2) = \mathsf{E}_\infty \restriction (2)^{F_2}$ *of* $\mathsf{E}_\infty = \mathbb{E}(F_2, 2)$ *is not hyperfinite since* $\lambda(X) = 1$ *by Exercise* 4.5.4.

PROOF (sketch). (i) The set $X = (2)^{\mathbb{G}}$ defined as in Exercise 4.5.4 is an E-invariant Borel set satisfying $\lambda(X) = 1$, and the shift action is free on X. Therefore, the group $\mathbb{G}$ is amenable by Theorem 9.2.3(iii).

(ii) Hyperfinite equivalence relations are λ-amenable by Theorem 9.2.3(ii).

(iii) The group F_2 is not amenable by Lemma 9.1.4.

(iv) Note that hyperfiniteness implies λ-amenability in this case by Theorem 9.2.3(ii), and hence the result follows from Theorem 9.2.3(iii). □

Recall that any countable Borel equivalence relation is hyperfinite on a suitable comeager invariant set by Theorem 7.5.4, but this is not necessarily true in the measure theoretic context because the countable Borel equivalence relation E_∞ is not hyperfinite on invariant sets of full measure by Corollary 9.3.3(iv). The property of being hyperfinite on a suitable invariant set of full measure characterizes equivalence relations in the class $\forall\mu\mathbf{A}$ by Theorem 9.3.2. However, as mentioned above, it is not yet known whether at least one of the classes $\forall\mu\mathbf{A}$, $\mathbf{A}$, $\mathbf{AGA}$ really extends the class $\mathbf{HF}$ of hyperfinite equivalence relations!

9.4. Treeable equivalence relations

The idea behind this type of Borel countable equivalence relation is to replace linear orders embeddable in $\mathbb{Z}$ in the characterization in Theorem 8.1.1(vii) of hyperfinite equivalence relations by connected trees. That is, a Borel countable equivalence relation is treeable if a connected tree T_C on C can be associated in a Borel way to every E-class C. Now let us give a more detailed definition.

DEFINITION 9.4.1. A *forest* on a set X is any graph $\Gamma \subseteq X \times X$ satisfiyng $x \mathrel{\Gamma} y \iff y \mathrel{\Gamma} x$ and $x \not\mathrel{\Gamma} x$ for all $x \in X$, and *acyclic* in the sense that there are no chains of the form $x_1 \mathrel{\Gamma} x_2 \mathrel{\Gamma} \cdots \mathrel{\Gamma} x_n \mathrel{\Gamma} x_1$, where $n \geq 2$ and $x_i \in X$.

A forest Γ is *locally countable*, resp., *locally finite*, if for any $x \in X$ the set $\{y : x \mathrel{\Gamma} y\}$ of all Γ-neighbours of x is at most countable, resp., finite.

Elements $x, y \in X$ are Γ*-connected* if $x = y$ or there is a chain $x = x_1 \mathrel{\Gamma} x_2 \mathrel{\Gamma} \cdots \mathrel{\Gamma} x_n = y$, where $n \geq 2$ and $x_i \in X$. Being Γ-connected is an equivalence relation, and its equivalence classes are called *connected components* of Γ. For instance, if $x \in X$ satisfies $x \not\mathrel{\Gamma} y$ for all $y \in X$, then the singleton $\{x\}$ is a connected component. If Γ is locally countable, then all connected components are at most countable sets.

A forest Γ is a *tree* iff it is connected. For instance connected components of a forest are trees. □

DEFINITION 9.4.2. A Borel countable equivalence relation E on a (Borel) set X is *treeable* iff there is a Borel forest Γ on X whose connected components are precisely E-classes. By the countability of E such an Γ must be locally countable, and in fact we will consider only locally countable forests and trees below. □

Thus the treeability means that a tree structure can be assigned in a Borel way to every E-equivalence class $[x]_{\mathsf{E}}$.

EXERCISE 9.4.3. Using (vii) of Theorem 8.1.1, prove that all hyperfinite equivalence relations are treeable. □

We obtain intermediate (between E_0 and E_∞) Borel countable equivalence relations among treeable ones by the following theorem (see *e.g.* [**JKL02**], § 3).

THEOREM 9.4.4. *The free part* $\mathsf{E}_{\infty T} = \mathbf{Fr}(F_2, 2) = \mathsf{E}_\infty \restriction (2)^{F_2}$ *of the equivalence relation* $\mathsf{E}_\infty = \mathbf{E}(F_2, 2)$ *is a Borel treeable equivalence relation satisfying* $\mathsf{E}_0 <_{\rm B} \mathsf{E}_{\infty T} <_{\rm B} \mathsf{E}_\infty$. *In addition* $\mathsf{E}_{\infty T}$ *is a* $\le_{\rm B}$*-largest Borel treeable equivalence relation.*

PROOF (sketch of general scheme). The treeability is quite obvious: for any $x, y \in X = (2)^{F_2}$ put $x \,\Gamma\, y$ iff one of the following equalities holds:

$$x = a \cdot y\,, \quad x = b \cdot y\,, \quad x = a^{-1} \cdot y\,, \quad x = b^{-1} \cdot y\,,$$

where a, b are the generators of F_2. This is a tree (since it is restricted to the free domain X of E_∞), in fact a locally finite one, and its connected components are precisely $\mathsf{E}_{\infty T}$-classes.

That $\mathsf{E}_{\infty T}$ is non-hyperfinite (equivalently $\mathsf{E}_0 <_{\rm B} \mathsf{E}_{\infty T}$) follows from Corollary 9.3.3(iv).

Regarding the $\le_{\rm B}$-maximality of $\mathsf{E}_{\infty T}$ in the treeable category, see Theorem 3.17 in [**JKL02**].

The proof that $\mathsf{E}_{\infty T} <_{\rm B} \mathsf{E}_\infty$ strictly in [**JKL02**] consists of two separate claims. First [**JKL02**, 3.3(ii)] if $\mathsf{E} \le_{\rm B} \mathsf{F}$ are Borel countable equivalence relations and F is treeable, then E is treeable as well. Thus if, to the contrary, $\mathsf{E}_{\infty T} <_{\rm B} \mathsf{E}_\infty$ fails, then $\mathsf{E}_\infty \le_{\rm B} \mathsf{E}_{\infty T}$, and hence all countable Borel equivalence relations are treeable since E_∞ is $\le_{\rm B}$-largest in this class. However (this is the second key result, [**JKL02**, 3.28]), there exist non-treeable countable Borel equivalence relations, for instance $\mathsf{E}_0 \times \mathsf{E}_{\infty T}$ and $\mathsf{E}^2_{\infty T} = \mathsf{E}_{\infty T} \times \mathsf{E}_{\infty T}$. In particular $\mathsf{E}_{\infty T} <_{\rm B} \mathsf{E}^2_{\infty T}$ strictly. □

Theorem 9.4.4 distinguishes two $\le_{\rm B}$-intervals, $[\mathsf{E}_0, \mathsf{E}_{\infty T}]$ and $[\mathsf{E}_{\infty T}, \mathsf{E}_\infty]$, within the domain $\mathsf{E}_0 \le_{\rm B} \mathsf{E} \le_{\rm B} \mathsf{E}_\infty$ of all non-smooth countable Borel equivalence relations, together with the subdomain of countable Borel equivalence relations $\le_{\rm B}$-incomparable with $\mathsf{E}_{\infty T}$.[4] While considerable progress has been achieved regarding the upper interval $[\mathsf{E}_{\infty T}, \mathsf{E}_\infty]$ (see below), the lower one (the one that consists of treeable Borel equivalence relations) continues to be very tough to study. So far the only known countable Borel equivalence relation E satisfying $\mathsf{E}_0 <_{\rm B} \mathsf{E} <_{\rm B} \mathsf{E}_{\infty T}$ has been recently discovered in [**Hjo05**].

QUESTION 9.4.5. What is the $\le_{\rm B}$-structure of Borel treeable countable equivalence relations? □

9.5. Above treeable. Free Borel countable equivalence relations

The existence of at least one countable Borel equivalence relation satisfying $\mathsf{E}_{\infty T} <_{\rm B} \mathsf{E} <_{\rm B} \mathsf{E}_\infty$ strictly was established in the mid-1990s. The key contribution was achieved in [**HK96**]: $\mathsf{E}^2_{\infty T} <_{\rm B} \mathsf{E}_\infty$ strictly. With the above-mentioned inequality $\mathsf{E}_{\infty T} <_{\rm B} \mathsf{E}^2_{\infty T}$, this implies $\mathsf{E}_{\infty T} <_{\rm B} \mathsf{E}^2_{\infty T} <_{\rm B} \mathsf{E}_\infty$.

[4] S. THOMAS noted to the author that the latter is non-empty. In particular, all equivalence relations introduced by Theorem 5.1 in [**Tho03b**] are $\le_{\rm B}$-incomparable with $\mathsf{E}_{\infty T}$.

Further advances in this direction were connected with applications of ZIMMER's superrigidity results [**Zim84**] in ergodic theory, which demonstrate, loosely speaking, that ergodic properties of the equivalence relation induced by an action of a countable group, can encode quite a lot of information concerning the group and the action. Moreover, it turns out that a lot of information concerning the group can be encoded in the $\sim_{\mathrm{B}}$-class of the induced equivalence relation in a purely descriptive set theoretic context. In other words, for various series of mathematically meaningful countable groups and their actions, it has been proved that equivalence relations induced by different groups are $\sim_{\mathrm{B}}$-inequivalent.

The first result of this sort is obtained by ADAMS and KECHRIS [**AK00**]: there is a map $A \mapsto \mathsf{E}_A$ assigning a countable Borel equivalence relation E_A to each Borel subset $A \subseteq 2^{\mathbb{N}}$ such that $A \subseteq B \iff \mathsf{E}_A \le_{\mathrm{B}} \mathsf{E}_B$. Each E_A here is a direct sum of more elementary equivalence relations E_x, $x \in A$, where each E_x is equal to $\mathsf{Fr}(\Gamma_x, 2)$, the free part of the shift action of a certain countable group Γ_x that consists of 7×7 invertible matrices. And this is based on the following key fact established in [**AK00**]. Let $\Gamma_p = \mathrm{SO}_7(\mathbb{Z}[1/p])$.[5] Then for any pair of primes $p \ne q$ the free parts $\mathsf{Fr}(\Gamma_p, 2)$ and $\mathsf{Fr}(\Gamma_q, 2)$ of the shift actions of these groups on resp. 2^{Γ_p} and 2^{Γ_q} are $\le_{\mathrm{B}}$-incomparable. In other words $\mathsf{Fr}(\Gamma_p, 2)$ encodes p. According to [**HK05**], the same effect holds for the family of groups $\Gamma_p = (\mathbb{Z}_p * \mathbb{Z}_p) \times \mathbb{Z}$, p prime, where $\mathbb{Z}_p$ is the cyclic group of p elements and $*$ is the free product, so that still $\mathsf{Fr}(\Gamma_p, 2)$ and $\mathsf{Fr}(\Gamma_q, 2)$ are $\le_{\mathrm{B}}$-incomparable whenever $p \ne q$ are primes. But on the other hand $\mathsf{Fr}(\mathbb{Z}_p * \mathbb{Z}_p, 2) \sim_{\mathrm{B}} \mathsf{E}_{\infty T}$ for all p.

Such a phenomenon, called *set theoretic rigidity* in [**HK05**], should be compared with the opposite phenomenon of *elastisity* for some other classes of groups. For instance it is known from [**GJ07**] that abelian countable groups induce hyperfinite equivalence relations, and hence all non-smooth orbit equivalence relations of Borel actions of abelian countable groups are $\sim_{\mathrm{B}}$ to each other by Corollary 8.4.1. It follows that there is no way to tell apart non-smooth actions of, say, $\mathbb{Z}$ and $\mathbb{Q}$ in terms of $\le_{\mathrm{B}}$: the actions "do not remember" the acting groups.

Let us mention one more result in [**AK00**]. For $n \ge 1$ consider $\mathbb{Q}^n$ as a group under component-wise addition. Let $S(\mathbb{Q}^n)$ be the space of all subgroups of $\mathbb{Q}^n$. This is a closed set in $2^{\mathbb{Q}^n}$, hence a Polish space. Recall that $\mathrm{GL}_n(\mathbb{Q})$ is the group of all invertible $n \times n$ rational matrices (with the matrix multiplication as the operation). Every matrix $M \in \mathrm{GL}_n(\mathbb{Q})$ acts on $S(\mathbb{Q}^n)$ so that, for every group $G \subseteq \mathbb{Q}^n$, $M \cdot G = \{M\vec{r} : \vec{r} \in G\}$, where $\vec{r} = \langle r_1, \ldots, r_n \rangle$ is an arbitrary n-tuple of rationals. The induced countable Borel equivalence relation is denoted by $\cong_n$. Clearly, $G \cong_n G'$ if and only if G, G' are isomorphic, so $\cong_n$ is the isomorphism relation of subgroups of $\mathbb{Q}^n$ (algebraically — of torsion-free abelian groups of rank n). The action does not have a free part since $r \mapsto -r$ is an automorphism of every group $G \in S(\mathbb{Q}^n)$, but it has a *rigid* part. Call $G \in S(\mathbb{Q}^n)$ rigid if it has no automorphism except for $r \mapsto -r$ and the identity, and let $\cong_n^*$ be the restriction of $\cong_n$ to the set of all rigid $G \in S(\mathbb{Q}^n)$.

The $\le_{\mathrm{B}}$-properties of $\cong_n$ and $\cong_n^*$ have been the subject of study for quite some time. In particular, by an old result of BAER [**Bae37**], $\cong_1$ and $\cong_1^*$ are $\sim_{\mathrm{B}}$-equivalent to E_0. And it is proved in [**AK00**] that $(\cong_n^*) <_{\mathrm{B}} (\cong_{n+1}^*)$ strictly for all $n \ge 1$. THOMAS [**Tho03a**] proved the same for the non-rigid version: $(\cong_n) <_{\mathrm{B}}$

[5] 7×7 matrices A with determinant ± 1 and satisfying $AA^{\top} = I$, and with entries from the ring $\mathbb{Z}[1/p]$ of all rationals whose denominators are degrees of p.

$(\cong_{n+1})$ strictly for all $n \geq 1$, and in fact the strict subintervals $(\cong_n) <_{\mathrm{B}} \mathsf{E} <_{\mathrm{B}} (\cong_{n+1})$ are non-empty, too! It is known (see [**AK00**, § 6]) that all equivalence relations $\cong_n$, $\cong_n^*$, $n \geq 2$, with a possible exception of $\cong_2^*$, are non-treeable; hence, they belong to the strict $\leq_{\mathrm{B}}$-interval $(\mathsf{E}_{\infty T}, \mathsf{E}_\infty)$.

And one more non-trivial series of countable equivalence relations was discovered in [**AK00**, § 7], perhaps, the most elementary one. Consider the canonical action of $\mathrm{GL}_n(\mathbb{Z})$ (invertible $n \times n$ matrices of integers) on $\mathbb{T}^n$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Let R_n be the induced countable Borel equivalence relation. It turns out that $\mathsf{R}_m <_{\mathrm{B}} \mathsf{R}_n$ whenever $1 \leq m < n$!

A related result belongs to THOMAS [**Tho02**]: there exists a countable Borel equivalence relation E on a Polish space $\mathbb{X}$ such that $n\,\mathsf{E} <_{\mathrm{B}} (n+1)\,\mathsf{E}$ for all n, where $n\,\mathsf{E}$ is the disjoint union of n copies of E, *i.e.*, an equivalence relation on $\{1, 2, \ldots, n\} \times \mathbb{X}$ defined so that $\langle i, x\rangle\ (n\,\mathsf{E})\ \langle j, y\rangle$ iff $i = j$ and $x \,\mathsf{E}\, y$. Typical Borel equivalence relations usually satisfy $\mathsf{E} \sim_{\mathrm{B}} n\,\mathsf{E} \sim_{\mathrm{B}} \mathbb{N}\,\mathsf{E}$ for all $n \geq 1$. According to [**HK05**], such an equivalence relation E can be obtained in the form of a Borel action of $F_2 \times F_2$ somewhat more complicated than its shift action on $2^{F_2 \times F_2}$.

Further results in this direction are obtained in [**HK05**] with the help of a technique that depends to the lesser extent on advanced results in ergodic theory than the proofs in the above-mentioned papers [**AK00**] *etc.* For instance, coming back to the equivalence relation $\mathsf{E}_{\infty T} = \mathbf{Fr}(F_2, 2)$ (the free part of the shift action of F_2 on 2^{F_2} and the $\leq_{\mathrm{B}}$-largest treeable Borel equivalence relation), HJORTH and KECHRIS proved in [**HK05**] that strictly

$$(\mathsf{E}_{\infty T})^n <_{\mathrm{B}} \mathbf{Fr}({F_2}^n, 2) <_{\mathrm{B}} \mathbf{Fr}({F_2}^{n+1}, 2) \quad \text{and} \quad (\mathsf{E}_{\infty T})^n <_{\mathrm{B}} (\mathsf{E}_{\infty T})^{n+1}$$

for all n, but $\mathbf{Fr}({F_2}^n, 2)$ is $\leq_{\mathrm{B}}$-incomparable with $(\mathsf{E}_{\infty T})^m$ whenever $m > n \geq 2$. This is graphically presented on Figure 5 borrowed in [**Kec07**].

Thus, the family of non-treeable Borel countable equivalence relations has a rather complicated $\leq_{\mathrm{B}}$-structure with plenty of $\sim_{\mathrm{B}}$-inequivalent and $\leq_{\mathrm{B}}$-different equivalence relations, including infinite series of mathematically meaningful equivalence relations. Most of the examples cited above in this respect have something in common: they are induced by *free* Borel actions of countable groups on appropriate Borel sets, usually defined as just free domains of generally non-free actions. (A notable exception is E_∞, a $\leq_{\mathrm{B}}$-largest Borel countable equivalence relation.)

DEFINITION 9.5.1. A Borel countable equivalence relation E on a Borel set X is *free* if it is induced by a free Borel action of a countable group on X. A (Borel) equivalence relation E is *essentially free* if there is a (Borel countable) free equivalence relation F such that $\mathsf{E} \leq_{\mathrm{B}} \mathsf{F}$. □

In particular all equivalence relations on Figure 5 (except for E_∞) are free by definition. (And treeable ones are at least essentially free because $\mathsf{E}_{\infty T}$, a $\leq_{\mathrm{B}}$-largest of them, is free by definition.) In a very recent study, THOMAS [**Tho07**] demonstrates that there are continuum-many $\sim_{\mathrm{B}}$-inequivalent non-treeable essentially free countable Borel equivalence relations, and on the other hand E_∞ is not essentially free. Moreover, there exist continuum-many $\sim_{\mathrm{B}}$-inequivalent non-essentially free countable Borel equivalence relations. In addition, the class of essentially free Borel equivalence relations does *not* contain a $\leq_{\mathrm{B}}$-largest element. These results are displayed in Figure 6.

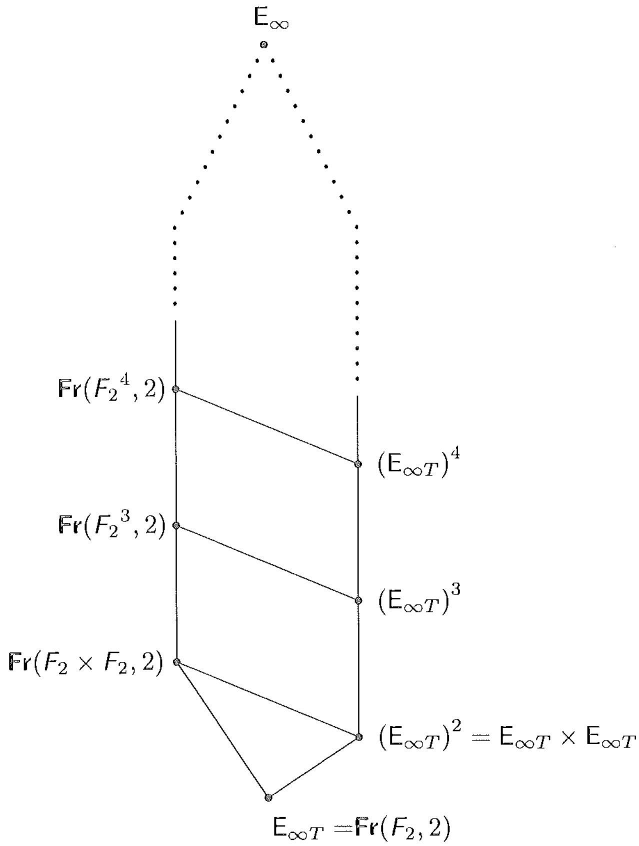

FIGURE 5. Equivalence relations $\mathbf{Fr}(F_2{}^n, 2)$ and $(\mathsf{E}_{\infty T})^n = (\mathbf{Fr}(F_2, 2))^n$

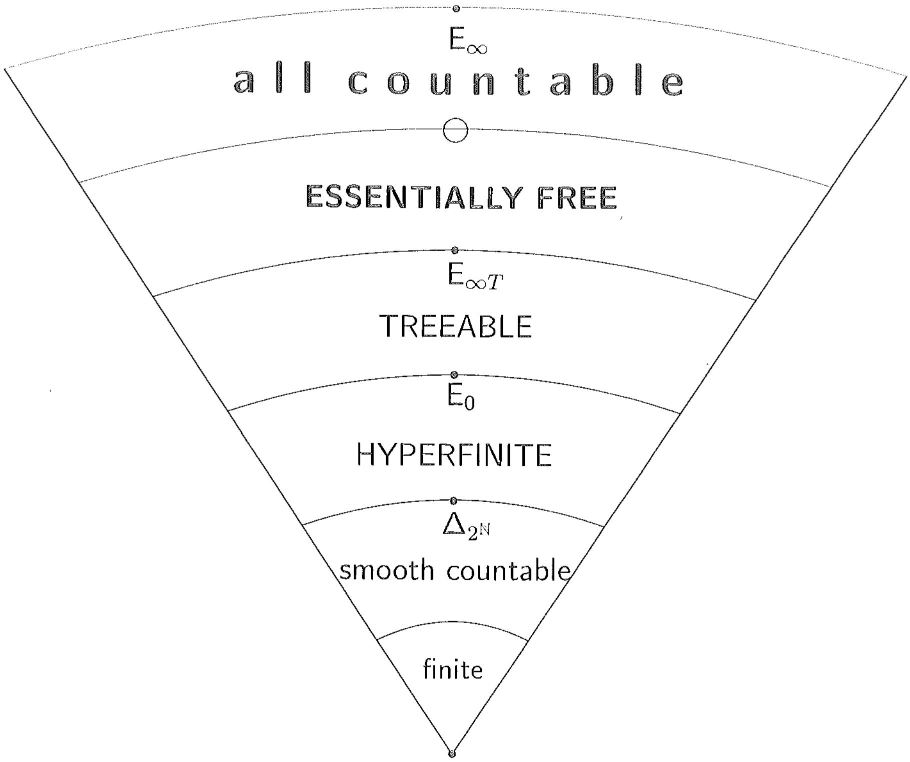

FIGURE 6. fin. $\subsetneq$ smooth ctble $\subsetneq$ hyp.fin. $\subsetneq$ treeable $\subsetneq$ ess. free $\subsetneq$ ctble

CHAPTER 10

The 1st and 2nd dichotomy theorems

This chapter presents proofs of Theorems 5.7.1 and 5.7.2, known as the 1st and 2nd dichotomy theorems. The proofs involve methods of effective descriptive set theory, in particular, on the Gandy–Harrington topology briefly considered in Section 2.10 and the associated forcing discussed in Appendix A.5. An interesting forcing notion that consists of all Borel sets $X \subseteq 2^{\mathbb{N}}$ such that $\mathsf{E}_0 \restriction X$ is non-trivial, is considered in the end of this chapter.

10.1. The 1st dichotomy theorem

We prove the following slightly more detailed form of Theorem 5.7.1.

THEOREM 10.1.1 (SILVER [**Sil80**]). *Every* $\mathbf{\Pi}^1_1$ *(therefore every Borel) equivalence relation* E *on* $\mathbb{N}^{\mathbb{N}}$ either *has at most countably many equivalence classes* or *admits a perfect set of pairwise* E*-inequivalent reals.*

In other words, either $\mathsf{E} \le_{\mathrm{B}} \Delta_{\mathbb{N}}$ or $\Delta_{2^{\mathbb{N}}} \sqsubseteq_{\mathrm{C}} \mathsf{E}$.

Recall that $\sqsubseteq_{\mathrm{C}}$ in the "or" part means the reducibility via a continuous injective map. Obviously $\sqsubseteq_{\mathrm{C}}$ implies $\le_{\mathrm{B}}$.

PROOF. [1] Fix a $\mathbf{\Pi}^1_1$ equivalence relation E on $\mathbb{N}^{\mathbb{N}}$. Then E belongs to $\Pi^1_1(p)$ for some parameter $p \in \mathbb{N}^{\mathbb{N}}$. As usual, we can suppose that E is in fact a lightface Π^1_1 relation. The case of an arbitrary parameter $p \in \mathbb{N}^{\mathbb{N}}$ does not differ in any essential detail since p uniformly passes all arguments.

Case 1. Every $x \in \mathbb{N}^{\mathbb{N}}$ belongs to a pairwise E-equivalent Δ^1_1 set X. (A set X is pairwise E-equivalent iff all elements of X are E-equivalent to each other.) In other words, it is assumed that the union $\mathbb{S}$ of all pairwise E-equivalent Δ^1_1 sets $X \subseteq \mathbb{N}^{\mathbb{N}}$ is equal to $\mathbb{N}^{\mathbb{N}}$.

Then E has at most countably many equivalence classes, the "either" case of Theorem 10.1.1. It remains to consider:

Case 2. Otherwise, that is, the set $\mathbf{H} = \mathbb{N}^{\mathbb{N}} \setminus \mathbb{S}$ of all points $x \in \mathbb{N}^{\mathbb{N}}$, which do *not* belong to a pairwise E-equivalent Δ^1_1 set, is non-empty.

We are going to prove that then the **or** case of Theorem 10.1.1 holds.

CLAIM 10.1.2. $\mathbf{H}$ *is* Σ^1_1. *If* $X \subseteq \mathbf{H}$ *is a non-empty* Σ^1_1 *set, then it is not pairwise* E*-equivalent.*

---

[1] We present a forcing-style proof originally due to HARRINGTON, in the version of MILLER [**Mil95**], with some simplifications. See [**MW85**] for another proof, based rather on topological technique. In fact both proofs involve very similar combinatorial arguments.

PROOF. We make use of an enumeration of Δ^1_1 sets provided by Theorem 2.8.1. Let sets $E = \mathtt{Cod}(\Delta^1_1) \subseteq \mathbb{N}$ and $W, W' \subseteq \mathbb{N} \times \mathbb{N}^{\mathbb{N}}$ be as in Theorem 2.8.1. Suppose that $x \in \mathbb{N}^{\mathbb{N}}$. Then obviously $x \in \mathbf{H}$ iff for every $e \in E$:

if $x \in (W)_e$, then $(W')_e$ is not E-equivalent.

The "if" part of this characterization is Π^1_1 while the "then" part is Σ^1_1 because we can express it as $\exists x, y\, (W'(e,x) \wedge W'(e,y) \wedge x \not\mathsf{E}\, y)$.

If $X \neq \varnothing$ is a pairwise E-equivalent Σ^1_1 set, then $B = \bigcap_{x \in X} [x]_{\mathsf{E}}$ is a Π^1_1 E-equivalence class and $X \subseteq B$. By Separation (Theorem 2.3.2), there is a Δ^1_1 set C with $X \subseteq C \subseteq B$. Then, if $X \subseteq \mathbf{H}$, then $C \subseteq \mathbf{H}$ is a Δ^1_1 pairwise E-equivalent set, a contradiction to the definition of $\mathbf{H}$. □ (Claim)

The continuation of the proof involves the Gandy–Harrington forcing $\mathbb{P}$. Recall that $\mathbb{P}$ consists of all non-empty Σ^1_1 sets $X \subseteq \mathbb{N}^{\mathbb{N}}$.

DEFINITION 10.1.3. Let us fix a countable transitive model $\mathfrak{M}$ of $\mathbf{ZFC}^-$ which is an elementary submodel of the universe *w.r.t.* all analytic formulas with parameters in $\mathfrak{M}$. Such a model exists by Corollary A.1.5. The model $\mathfrak{M}$ will be the ground model for the forcing notion $\mathbb{P}$. □

See Appendix A.5 for detail and explanation.

Recall that if $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over $\mathfrak{M}$, then by Theorem A.5.4 the intersection $\bigcap G$ is a singleton whose only element is denoted by x_G. Let $\mathring{x}$ be the name for x_G in the machinery of the Gandy–Harrington forcing $\mathbb{P}$. Then every condition $A \in \mathbb{P}$ forces that $\mathring{x} \in A$.

The forcing product $\mathbb{P}^2$ can be defined as just the cartesian product $\mathbb{P} \times \mathbb{P}$, but it is somewhat more convenient here to define it as the set of all rectangles $X \times Y$ with $X, Y \in \mathbb{P}$. (And $X \times Y$ is identified with $\langle X, Y \rangle \in \mathbb{P} \times \mathbb{P}$.) It follows from Theorem A.5.4 by the product forcing lemma (Theorem A.3.2) that every set $G \subseteq \mathbb{P}^2$, $\mathbb{P}^2$-generic over $\mathfrak{M}$, produces a pair of reals (a $\mathbb{P}^2$-*generic pair*), say, $x^G_{\mathtt{left}}$ and $x^G_{\mathtt{right}}$, so that $\langle x^G_{\mathtt{left}}, x^G_{\mathtt{right}} \rangle \in W$ for all $W \in G$. Let $\mathring{x}_{\mathtt{left}}$ and $\mathring{x}_{\mathtt{right}}$ be their names. The following is the key fact:

LEMMA 10.1.4. *The set* $\mathbf{H} \times \mathbf{H}$ $\mathbb{P}^2$*-forces* $\mathring{x}_{\mathtt{left}} \not\mathsf{E}\, \mathring{x}_{\mathtt{right}}$.

PROOF. Otherwise there is a condition $X \times Y \in \mathbb{P}^2$ with $X \cup Y \subseteq \mathbf{H}$ that $\mathbb{P}^2$-forces $\mathring{x}_{\mathtt{left}} \mathsf{E}\, \mathring{x}_{\mathtt{right}}$, and hence all $\mathbb{P}^2$-generic pairs $\langle x, y \rangle \in X \times Y$ satisfy $x \mathsf{E}\, y$. Fix X, Y and prove two auxiliary claims.

CLAIM 10.1.5. *If* $x, x' \in X$ *are* $\mathbb{P}$*-generic over* $\mathfrak{M}$, *then there is* $y \in Y$ *such that both* $\langle x, y \rangle$ *and* $\langle x', y \rangle$ *are* $\mathbb{P}^2$*-generic* pairs *over* $\mathfrak{M}$.

PROOF (Claim). The proof is a rather lengthy routine. Let us enumerate $\{D_n : n \in \mathbb{N}\}$ all open dense sets $D \subseteq \mathbb{P}^2$ coded in $\mathfrak{M}$ in the sense of Appendix A.5. By induction we define sets $X_n, X'_n, Y_n \in \mathbb{P}$ such that

(1) $X_{n+1} \subseteq X_n \subseteq X_0 = X$, $X'_{n+1} \subseteq X'_n \subseteq X'_0 = X$, $Y_{n+1} \subseteq Y_n \subseteq Y_0 = Y$;

(2) if $n \geq 1$, then the sets $X_n \times Y_n$ and $X'_n \times Y_n$ belong to D_{n-1};

(3) $x \in \bigcap_n X_n$ and $x' \in \bigcap_n X'_n$.

As soon as this is done, by Theorem A.5.4 (and A.3.2) the intersections $\bigcap_n X_n$, $\bigcap_n X'_n$, $\bigcap_n Y_n$, are singletons, say $\{\xi\}$, $\{\xi'\}$, $\{y\}$, where $\xi, \xi' \in X$ and $y \in Y$, and the pairs $\langle \xi, y\rangle$ and $\langle \xi', y\rangle$ are $\mathbb{P}^2$-generic over $\mathfrak{M}$. Note that $\xi = x$, $\xi' = x'$ by (3). It follows that the pairs $\langle x, y\rangle$ and $\langle x', y\rangle$ are $\mathbb{P}^2$-generic over $\mathfrak{M}$, as required.

Thus it remains to accomplish the construction.

Suppose that X_n, X'_n, Y_n are defined. Recall that $D_n \subseteq \mathbb{P}^2$ is a dense open set in $\mathbb{P}^2$ coded in $\mathfrak{M}$. We claim that the set

$$D_n^* = \{A \in \mathbb{P} : \exists B\,(B \subseteq Y_n \wedge A \times B \in D_n)\}$$

is dense in $\mathbb{P}$. Indeed suppose that $A_0 \in \mathbb{P}$. Then $A_0 \times Y_n \in \mathbb{P}^2$, and by the density there exists a set $A \times B \in D_n$ such that $A \subseteq A_0$ and $B \subseteq Y_n$. Then $A \in D_n^*$, as required. Moreover D_n^* is coded in $\mathfrak{M}$ since so is D_n.

It follows, by the genericity of x, that there is a set $A \in D_n^*$ such that $x \in A$. By definition, there is a set $B \subseteq Y_n$ such that $A \times B \in D_n$. And we can *w.l.o.g.* assume that $A \subseteq X_n$ because otherwise the set $A' = A \cap X_n$ (still a non-empty set since $x \in A'$) can replace A.

By quite the same argument there exists a condition $A' \times B' \in D_n$ such that $B' \subseteq B$, $A' \subseteq X'_n$, and $x' \in A'$. To end the proof, put $X_{n+1} = A$, $X'_{n+1} = A'$, $Y_{n+1} = B'$. □ (Claim)

CLAIM 10.1.6. *If $x, x' \in X$ are $\mathbb{P}$-generic over $\mathfrak{M}$, then $x \mathsf{E} x'$.*

PROOF (Claim). Let y be given by Claim 10.1.5. Then $x \mathsf{E} y$ and $x' \mathsf{E} y$ by the above. □ (Claim)

In the continuation of the proof of Lemma 10.1.4, note that the set $\mathbb{P}_2$ of all non-empty Σ^1_1 subsets of $\mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}$ is just a copy of $\mathbb{P}$ (not of $\mathbb{P}^2$!) as a forcing.

CLAIM 10.1.7. (i) *If a set $G \subseteq \mathbb{P}_2$ is $\mathbb{P}_2$-generic over $\mathfrak{M}$, then there is a unique pair of reals* ($\mathbb{P}_2$-generic pair) $\langle x^G_{\texttt{left}}, x^G_{\texttt{right}}\rangle$ *which belongs to every $W \in G$.*

(ii) *Moreover in this case, both $x^G_{\texttt{left}}$ and $x^G_{\texttt{right}}$ are $\mathbb{P}$-generic over $\mathfrak{M}$.*

PROOF (Claim). (i) is similar to Theorem A.5.4. To prove (ii), it suffices to note that for any set $D \subseteq \mathbb{P}$ dense in $\mathbb{P}$, the set $D' = \{W \in \mathbb{P}_2 : \mathtt{dom}\, W \in D\}$ is dense in $\mathbb{P}$, and if D is coded in $\mathfrak{M}$, then so is D'. (We leave the details to the reader.) But then given a set $G \subseteq \mathbb{P}_2$ that is $\mathbb{P}_2$-generic over $\mathfrak{M}$, the set G' of all projections $\mathtt{dom}\, W$ of sets $W \in G$ is clearly $\mathbb{P}$-generic. □ (Claim)

Note that $P = X^2 \smallsetminus \mathsf{E}$ is a non-empty (by Claim 10.1.2) Σ^1_1 set, hence a condition in $\mathbb{P}_2$. It follows that there is a set $G \subseteq \mathbb{P}_2$, $\mathbb{P}_2$-generic over $\mathfrak{M}$. Then $\langle x^G_{\texttt{left}}, x^G_{\texttt{right}}\rangle \in P$ by 10.1.7(i), thus we have $x^G_{\texttt{left}} \not\mathsf{E}\, x^G_{\texttt{right}}$. However both $x^G_{\texttt{left}}$ and $x^G_{\texttt{right}}$ are $\mathbb{P}$-generic elements of X by 10.1.7(ii) (because $P \subseteq X \times X$). We conclude that $x^G_{\texttt{left}} \mathsf{E}\, x^G_{\texttt{right}}$ by 10.1.6, a contradiction. □ (Lemma 10.1.4)

10.2. Splitting system

Now to complete the proof of Theorem 10.1.1 we fix enumerations $\{\mathscr{D}_n\}_{n\in\mathbb{N}}$ and $\{\mathscr{D}_2^n\}_{n\in\mathbb{N}}$ of all open dense subsets, of forcings resp. $\mathbb{P}$ and $\mathbb{P}^2$, which are coded in $\mathfrak{M}$ (a model chosen by Definition 10.1.3). We assert that there is a system $\{X_u\}_{u\in 2^{<\omega}}$ of sets $X_u \in \mathbb{P}$ satisfying the following:

1°. $X_\Lambda \subseteq \mathbf{H}$.

2°. For all n and $u \in 2^n$: $X_u \in \mathscr{D}_n$.[2]

[2] Here 2^n is the set of all sequences $u \in 2^{<\omega}$ of length $\mathtt{lh}\, u = n$.

3°. $X_{u^\frown i} \subseteq X_u$ and $X_{u^\frown 0} \cap X_{u^\frown 1} = \varnothing$ for all $u \in 2^{<\omega}$ and $i = 0, 1$.

4°. If $u \neq v \in 2^n$, then $X_u \times X_v \in \mathscr{D}_2^n$.

It follows from **2°** and **3°** that the set $\{X_{a \restriction m} : m \in \mathbb{N}\}$ is $\mathbb{P}$-generic over $\mathfrak{M}$ for every $a \in 2^{\mathbb{N}}$; therefore, $\bigcap_m X_{a \restriction m}$ is a singleton by Theorem A.5.4. Let x_a be its only element. The map $a \mapsto x_a$ is 1-to-1 still by **3°**.

Note that the map $a \mapsto x_a$ is continuous. Indeed let d be the Polish distance on $\mathbb{N}^{\mathbb{N}}$ defined so that $d(a,b) = \frac{1}{n+1}$ for all $a \neq b$ in $\mathbb{N}^{\mathbb{N}}$, where n is the least number such that $a(n) \neq b(n)$. If $k \in \mathbb{N}$, then the set $D_k = \{X \in \mathbb{P} : \mathtt{diam}\, X < k^{-1}\}$ is obviously a dense subset of $\mathbb{P}$ coded in $\mathfrak{M}$; therefore, D_k coincides with one of the sets $\mathscr{D}_n$. Now it follows by **2°** that $\mathtt{diam}\, X_u < k^{-1}$ for all $u \in 2^n$. Using this observation, the continuity of the map is an easy exercise.

In addition, by **4°** and Lemma 10.1.4, every pair of the form $\langle x_a, x_b \rangle$, $a \neq b$, is $\mathbb{P}^2$-generic over $\mathfrak{M}$; therefore, $x_a \not\mathrel{\mathsf{E}} x_b$ and $x_a \neq x_b$ hold whenever $a \neq b$, and hence $Y = \{x_a : a \in 2^{\mathbb{N}}\}$ is a perfect E-inequivalent set.

The construction of such a system of sets X_u does not cause much trouble. Let X_Λ be any set in $\mathscr{D}_0$ such that $X_\Lambda \subseteq \mathbf{H}$. Suppose that $n \in \mathbb{N}$ and sets $X_u \in \mathbb{P}$ are defined for all $u \in 2^n$ so that conditions **1°**–**4°** hold in the domain $\leq n$.

Step 1. For each $u \in 2^n$, do the following. Take a pair of disjoint non-empty Σ^1_1 sets $X', X'' \subseteq X_u$. They obviously belong to $\mathbb{P}$. Shrink them appropriately to satisfy **2°**, and let $X'_{u^\frown 0}$ and $X'_{u^\frown 1}$ be the sets obtained; they still belong to $\mathbb{P}$.

Step 2. Consider any pair of sequences $u^\frown i \neq v^\frown j$ in 2^{n+1}. There is a set of the form $Y \times Z \in \mathscr{D}_2^{n+1}$ such that $Y \subseteq X'_{u^\frown 0}$ and $Z \subseteq X'_{u^\frown 1}$. Let Y, Z be the "new" sets $X'_{u^\frown 0}, X'_{u^\frown 1}$. Consider all pairs $u^\frown i \neq v^\frown j$ in 2^{n+1} consecutively in this manner. After this is finished, the sets $X_{u^\frown i}$ obtained satisfy **1°**–**4°** in the domain $\leq n+1$.

This accomplishes the inductive step in the construction of a system of sets X_u as required. □ (Theorems 10.1.1 and 5.7.1)

10.3. Structural and chaotic domains

Theorem 10.1.1 opens a list of similar applications of the Gandy–Harrington forcing (or topology). A common feature of them is a partition of the domain of the equivalence relation considered into two domains: a *chaotic* domain, like the set $\mathbf{H}$ in the proof of the theorem, usually Σ^1_1; and a *structural* domain, like the set $\mathbf{S}$, usually Π^1_1. The partition depends on a given equivalence relation E. If the chaotic domain $\mathbf{H}$ is non-empty, a suitable splitting construction within $\mathbf{H}$ leads to a set, say Y, such that the restricted equivalence relation $\mathsf{E} \restriction Y$ is non-trivial in one way or another, for instance, $\sim_{\mathrm{B}}$-equivalent to a certain given equivalence relation, such as $\Delta_{2^{\mathbb{N}}}$ in Theorem 10.1.1. If the chaotic domain $\mathbf{H}$ is empty, then the structural domain $\mathbf{S}$ is equal to the whole domain considered. Then it turns out that on the contrary E is rather trivial, for instance, $\leq_{\mathrm{B}}$-reducible to a more elementary equivalence relation, such as $\Delta_{\mathbb{N}}$ in Theorem 10.1.1. The proof of the 2nd dichotomy theorem is a more complicated example of this scheme.

10.4. 2nd dichotomy theorem

Theorem 5.7.2, or the 2nd dichotomy theorem of Harrington, Kechris, and Louveau [**HKL90**], will be established in the following form:

THEOREM 10.4.1. *Suppose that* E *is a Borel equivalence relation. Then* either $\mathsf{E} \le_{\mathrm{B}} \Delta_{2^{\mathbb{N}}}$ or $\mathsf{E}_0 \sqsubseteq_{\mathrm{C}} \mathsf{E}$.

Recall that the condition $\mathsf{E} \le_{\mathrm{B}} \Delta_{2^{\mathbb{N}}}$ characterizes smooth equivalence relations (see Section 7.1). Therefore the "either" condition can be formulated as E is smooth.

PROOF. [3] Suppose, as usual, that E is a lightface Δ^1_1 equivalence relation on $\mathbb{N}^{\mathbb{N}}$. Similar to Theorem 10.1.1, the proof employs the Gandy–Harrington forcing, but is considerably more complicated. Consider an auxiliary equivalence relation

$$x \mathbin{\widehat{\mathsf{E}}} y \quad \text{iff} \quad x, y \in \mathbb{N}^{\mathbb{N}} \text{ belong to the same } \mathsf{E}\text{-invariant } \Delta^1_1 \text{ sets.}^4$$

Easily, $x \mathsf{E} y \Longrightarrow x \mathbin{\widehat{\mathsf{E}}} y$, or in brief $\mathsf{E} \subseteq \widehat{\mathsf{E}}$. In fact it follows from the next lemma that $\widehat{\mathsf{E}}$ is equal to the closure of E in the Gandy–Harrington topology.

LEMMA 10.4.2. *If* F *is a* Σ^1_1 *equivalence relation on* $\mathbb{N}^{\mathbb{N}}$ *and* $X, Y \subseteq \mathbb{N}^{\mathbb{N}}$ *are disjoint* F*-invariant* Σ^1_1 *sets, then there exists an* F*-invariant* Δ^1_1 *set* X' *that separates* X *from* Y.

PROOF. By Separation (Theorem 2.3.2), for every Σ^1_1 set A with $A \cap Y = \varnothing$, there is a Δ^1_1 set A' with $A \subseteq A'$ and $A' \cap Y = \varnothing$. Note that in this case even $[A']_{\mathsf{F}} \cap Y = \varnothing$ because Y is F-invariant. It follows that that there is a sequence

$$X = A_0 \subseteq A'_0 \subseteq A_1 \subseteq A'_1 \subseteq \cdots,$$

where all sets A'_i are Δ^1_1 sets; accordingly, all sets $A_{i+1} = [A'_i]_{\mathsf{F}}$ are Σ^1_1 sets, and $A_i \cap Y = \varnothing$. Then $X' = \bigcup_n A_n = \bigcup_n A'_n$ is an F-invariant Borel set that separates X from Y. To ensure that X' is Δ^1_1 we have to maintain the choice of sets A_n in an effective manner.

Let $U \subseteq \mathbb{N} \times \mathbb{N}^{\mathbb{N}}$ be a "good" universal Σ^1_1 set. (We make use of Theorem 2.6.2.) Then there is a recursive function $h : \mathbb{N} \to \mathbb{N}$ such that $[U_n]_{\mathsf{F}} = U_{h(n)}$ for each n. Moreover, applying Proposition 2.6.3 (to the complement of U as a "good" universal Π^1_1 set, and with a code for Y fixed), we obtain a pair of recursive functions $f, g : \mathbb{N} \to \mathbb{N}$ such that for every n, if $U_n \cap Y = \varnothing$, then $U_{f(n)}, U_{g(n)}$ are complementary Σ^1_1 sets (hence, either of them is Δ^1_1) containing, resp., U_n and Y. A suitable iteration of h and f, g allows us to define a sequence $X = A_0 \subseteq A'_0 \subseteq A_1 \subseteq A'_1 \subseteq \cdots$ as above effectively enough for the union of those sets to be Δ^1_1. □ (Lemma)

LEMMA 10.4.3. $\widehat{\mathsf{E}}$ *is a* Σ^1_1 *relation.*

PROOF. Consider sets $\mathtt{Cod}(\Delta^1_1) \subseteq \mathbb{N}$ and $W, W' \subseteq \mathbb{N} \times \mathbb{N}^{\mathbb{N}}$ as in Theorem 2.8.1. The formula $\mathtt{inv}(e)$ saying that $e \in \mathtt{Cod}(\Delta^1_1)$ and the set $(W)_e = (W')_e$ is E-invariant, that is,

$$e \in \mathtt{Cod}(\Delta^1_1) \wedge \forall a, b\, (W'(e, a) \wedge a \mathsf{E} b \Longrightarrow W(e, b)),$$

is obviously a Π^1_1 formula. On the other hand, $x \mathbin{\widehat{\mathsf{E}}} y$ iff

$$\forall e\, (\mathtt{inv}(e) \Longrightarrow [W(e, x) \Longrightarrow W'(e, y)] \wedge [W(e, y) \Longrightarrow W'(e, x)]). \quad \square \text{ (Lemma)}$$

[3] The proof will be completed in Section 10.7.

[4] Recall that a set X is E-*invariant* iff $X = [X]_{\mathsf{E}}$.

Let us return to the proof of Theorem 10.4.1. We have two cases.

Case 1. $\mathsf{E} = \widehat{\mathsf{E}}$; that is, the equivalence relation E as a set is closed in the sense of the Gandy–Harrington topology. In other words, it is assumed that the *structural domain* $\mathbf{S} = \{x \in \mathbb{N}^{\mathbb{N}} : [x]_{\mathsf{E}} = [x]_{\widehat{\mathsf{E}}}\}$ is equal to $\mathbb{N}^{\mathbb{N}}$.

The next lemma shows that in this assumption we obtain the "either" case in Theorem 10.4.1.

LEMMA 10.4.4. *If* $\mathsf{E} = \widehat{\mathsf{E}}$*, then there is a* Δ^1_1 *reduction of* E *to* $\Delta_{2^{\mathbb{N}}}$.

PROOF. Let $\mathtt{Cod}(\Delta^1_1) \subseteq \mathbb{N}$ and $W, W' \subseteq \mathbb{N} \times \mathbb{N}^{\mathbb{N}}$ be as in Theorem 2.8.1. By Kreisel Selection (Theorem 2.4.5) there is a Δ^1_1 function $\varphi : X^2 \to \mathtt{Cod}(\Delta^1_1)$ such that $(W)_{\varphi(x,y)} = (W')_{\varphi(x,y)}$ is an E-invariant Δ^1_1 set containing x but not y whenever $x, y \in X$ are E-inequivalent. Then $R = \operatorname{ran} \varphi$ is a Σ^1_1 subset of $\mathtt{Cod}(\Delta^1_1)$, hence, by Separation, there is a Δ^1_1 set N with $R \subseteq N \subseteq \mathtt{Cod}(\Delta^1_1)$. The map $\vartheta(x) = \{n \in N : x \in D_n\}$ is a Δ^1_1 reduction of E to $\Delta_{2^{\mathbb{N}}}$. □ (Lemma)

□ (Case 1)

Case 2. $\mathsf{E} \subsetneqq \widehat{\mathsf{E}}$, that is, the *chaotic domain* $\mathbf{H} = \{x : [x]_{\mathsf{E}} \subsetneqq [x]_{\widehat{\mathsf{E}}}\}$ (the union of all $\widehat{\mathsf{E}}$-classes containing more than one E-class), is non-empty.

Note that $\mathbf{H}$ is a Σ^1_1 set: indeed $\mathbf{H} = \{x : \exists y\, (x \mathrel{\widehat{\mathsf{E}}} y \wedge x \not\mathrel{\mathsf{E}} y)\}$.
The following theorem shows that Case 2 leads to the "or" case in Theorem 10.4.1.

THEOREM 10.4.5. *If* $\mathbf{H} \neq \varnothing$*, then* $\mathsf{E}_0 \sqsubseteq_{\mathrm{C}} \mathsf{E}$.

The proof of this result will be accomplished in Section 10.7. We begin with a couple of technical lemmas. The first of them says that the property $\mathsf{E} \subsetneqq \widehat{\mathsf{E}}$ holds hereditarily within the chaotic domain $\mathbf{H}$.

LEMMA 10.4.6. *If* $X \subseteq \mathbf{H}$ *is a* Σ^1_1 *set, then* $\mathsf{E} \subsetneqq \widehat{\mathsf{E}}$ *on* X.

PROOF. Suppose that $\mathsf{E} \restriction X = \widehat{\mathsf{E}} \restriction X$. Then $\mathsf{E} = \widehat{\mathsf{E}}$ on $Y = [X]_{\mathsf{E}}$ as well. (If y, $y' \in Y$, then there are $x, x' \in X$ such that $x \mathrel{\mathsf{E}} y$ and $x' \mathrel{\mathsf{E}} y'$, so that if $y \mathrel{\widehat{\mathsf{E}}} y'$, then $x \mathrel{\widehat{\mathsf{E}}} x'$ by transitivity, hence, $x \mathrel{\mathsf{E}} x'$, and $y \mathrel{\mathsf{E}} y'$ again by transitivity.) It follows that $\mathsf{E} = \widehat{\mathsf{E}}$ on an even bigger set, $Z = [X]_{\widehat{\mathsf{E}}}$. (Otherwise, the Σ^1_1 set

$$Y' = Z \smallsetminus Y = \{z : \exists x \in X\, (x \mathrel{\widehat{\mathsf{E}}} y \wedge x \not\mathrel{\mathsf{E}} y)\}$$

is non-empty and E-invariant, together with Y. Therefore, by Lemma 10.4.2 there is an E-invariant Δ^1_1 set B with $Y \subseteq B$ and $Y' \cap B = \varnothing$. This implies that no point in Y is $\widehat{\mathsf{E}}$-equivalent to a point in Y', a contradiction.) Then by definition $Z \cap \mathbf{H} = \varnothing$. □ (Lemma)

LEMMA 10.4.7. *If* $A, B \subseteq \mathbf{H}$ *are non-empty* Σ^1_1 *sets with* $A \mathrel{\mathsf{E}} B$*, then there exist non-empty* disjoint Σ^1_1 *sets* $A' \subseteq A$ *and* $B' \subseteq B$ *still satisfying* $A' \mathrel{\mathsf{E}} B'$.

Recall that $A \mathrel{\mathsf{E}} B$ means that $[A]_{\mathsf{E}} = [B]_{\mathsf{E}}$.

PROOF. We assert that there are points $a \in A$ and $b \in B$ with $a \neq b$ and $a \mathrel{\mathsf{E}} b$. (Indeed otherwise E is the equality on $X = A \cup B$. Prove that then $\mathsf{E} = \widehat{\mathsf{E}}$ on X, contrary to Lemma 10.4.6. Take any $x \neq y$ in X. Let U be a clopen set containing x but not y. Then $A = [U \cap X]_{\mathsf{E}}$ and $C = [X \smallsetminus U]_{\mathsf{E}}$ are two disjoint E-invariant Σ^1_1 sets containing resp. x, y. Then $x \mathrel{\widehat{\mathsf{E}}} y$ fails by Lemma 10.4.2.)

Thus, let a, b be as indicated. Let U be a clopen set containing a but not b. Put $A' = A \cap U \cap [U^{\complement}]_{\mathsf{E}}$ and $B' = B \cap U^{\complement} \cap [U]_{\mathsf{E}}$. □ (Lemma)

10.5. Restricted product forcing

In continuation of the proof of Theorem 10.4.5 (Case 2 in the proof of Theorem 10.4.1), we come back to the Gandy–Harrington forcing notion $\mathbb{P}$ and its two-dimensional copy $\mathbb{P}_2$ introduced in Section 10.1. Let us fix a countable model $\mathfrak{M}$ of $\mathbf{ZFC}^-$ chosen as in Definition 10.1.3.

Let $\mathbb{P}^2 \restriction \mathsf{E}$ be the collection of all sets of the form $X \times Y$, where $X, Y \subseteq \mathbb{N}^{\mathbb{N}}$ are non-empty Σ^1_1 sets and $X \mathsf{E} Y$ (which still means that $[X]_{\mathsf{E}} = [Y]_{\mathsf{E}}$). It is clear that $\mathbb{P}_2 \subseteq \mathbb{P}^2 \restriction \mathsf{E} \subseteq \mathbb{P}^2$. The forcing $\mathbb{P}^2 \restriction \mathsf{E}$ is not really a product of two copies of $\mathbb{P}$. However, if $X \times Z \in \mathbb{P}^2 \restriction \mathsf{E}$ and $\varnothing \neq X' \subseteq X$ is Σ^1_1, then $Z' = Z \cap [X']_{\mathsf{E}}$ is Σ^1_1 and $X' \times Z' \in \mathbb{P}^2 \restriction \mathsf{E}$. It follows that every set $G \subseteq \mathbb{P}^2 \restriction \mathsf{E}$, $\mathbb{P}^2 \restriction \mathsf{E}$-generic over $\mathfrak{M}$, still produces a pair of $\mathbb{P}$-generic sets

$$G_{\mathtt{left}} = \{\operatorname{dom} P : P \in G\} \quad \text{and} \quad G_{\mathtt{right}} = \{\operatorname{ran} P : P \in G\}.$$

Therefore, G produces a pair of $\mathbb{P}$-generic reals $x^G_{\mathtt{left}}$ and $x^G_{\mathtt{right}}$, whose names will be $\mathring{x}_{\mathtt{left}}$ and $\mathring{x}_{\mathtt{right}}$ as above.

LEMMA 10.5.1. *In the sense of the forcing $\mathbb{P}^2 \restriction \mathsf{E}$, any condition $P = X \times Z$ in $\mathbb{P}^2 \restriction \mathsf{E}$ forces $\langle \mathring{x}_{\mathtt{left}}, \mathring{x}_{\mathtt{right}} \rangle \in P$ and forces $\mathring{x}_{\mathtt{left}} \mathrel{\widehat{\mathsf{E}}} \mathring{x}_{\mathtt{right}}$, but $\mathbf{H} \times \mathbf{H}$ forces $\mathring{x}_{\mathtt{left}} \not\mathsf{E}\, \mathring{x}_{\mathtt{right}}$.*

PROOF. To show that $\langle \mathring{x}_{\mathtt{left}}, \mathring{x}_{\mathtt{right}} \rangle \in P$ is forced, argue as in the proof of Theorem A.5.4. (We leave this to the reader.) To prove that $\mathring{x}_{\mathtt{left}} \mathrel{\widehat{\mathsf{E}}} \mathring{x}_{\mathtt{right}}$ is forced, suppose otherwise. Then, by the definition of $\widehat{\mathsf{E}}$, there is a condition $P = X \times Z \in \mathbb{P}^2 \restriction \mathsf{E}$ and an E-invariant Δ^1_1 set B such that P forces $\mathring{x}_{\mathtt{left}} \in B$ but $\mathring{x}_{\mathtt{right}} \notin B$. Then easily $X \subseteq B$ but $Z \cap B = \varnothing$, a contradiction with $[X]_{\mathsf{E}} = [Z]_{\mathsf{E}}$.

To show that $\mathbf{H} \times \mathbf{H}$ forces $\mathring{x}_{\mathtt{left}} \not\mathsf{E}\, \mathring{x}_{\mathtt{right}}$ suppose toward the contrary that a condition $P = X \times Z \in \mathbb{P}^2 \restriction \mathsf{E}$ with $X \cup Z \subseteq \mathbf{H}$ forces $\mathring{x}_{\mathtt{left}} \mathsf{E}\, \mathring{x}_{\mathtt{right}}$, thus

(1) $\qquad x \mathsf{E} z$ holds for every $\mathbb{P}^2 \restriction \mathsf{E}$-generic pair $\langle x, z \rangle \in P$.

CLAIM 10.5.2. *If $x, y \in X$ are $\mathbb{P}$-generic over $\mathfrak{M}$, and $x \mathrel{\widehat{\mathsf{E}}} y$, then $x \mathsf{E} y$.*

PROOF. We assert that

(2) $\qquad x \in A \iff y \in A$ holds for each E-invariant Σ^1_1 set A.

Indeed, if, say, $x \in A$ but $y \notin A$, then by the genericity of y there is a Σ^1_1 set C with $y \in C$ and $A \cap C = \varnothing$. As A is E-invariant, Lemma 10.4.2 yields an E-invariant Δ^1_1 set B such that $C \subseteq B$ but $A \cap B = \varnothing$. Then $x \notin B$ but $y \in B$, a contradiction to $x \mathrel{\widehat{\mathsf{E}}} y$.

Let $\{\mathscr{D}_n\}_{n \in \mathbb{N}}$ be an enumeration of all dense subsets of $\mathbb{P}^2 \restriction \mathsf{E}$ which are coded in $\mathfrak{M}$. We define two sequences $P_0 \supseteq P_1 \supseteq \cdots$ and $Q_0 \supseteq Q_1 \supseteq \cdots$ of conditions $P_n = X_n \times Z_n$ and $Q_n = Y_n \times Z_n$ in $\mathbb{P}^2 \restriction \mathsf{E}$, so that $P_0 = Q_0 = P$, $x \in X_n$ and $y \in Y_n$ for all n, and finally $P_n, Q_n \in \mathscr{D}_{n-1}$ for $n \geq 1$. If this is done, then we have a real z (the only element of $\bigcap_n Z_n$) such that both $\langle x, z \rangle$ and $\langle y, z \rangle$ are $\mathbb{P}^2 \restriction \mathsf{E}$-generic; hence, $x \mathsf{E} z$ and $y \mathsf{E} z$ by (1), hence, $x \mathsf{E} y$.

Suppose that P_n and Q_n have been defined. As x is generic, there is (we leave details for the reader) a condition $P' = A \times C \in \mathscr{D}_n$ and $\subseteq P_n$ such that $x \in A$. We put $B = Y_n \cap [A]_{\mathsf{E}}$. Then $y \in B$ by (2), and clearly $[B]_{\mathsf{E}} = [C]_{\mathsf{E}} = [A]_{\mathsf{E}}$ (because $[X_n]_{\mathsf{E}} = [Z_n]_{\mathsf{E}} = [Y_n]_{\mathsf{E}}$). It follows that $B \times C \in \mathbb{P}^2 \restriction \mathsf{E}$, so there is a condition $Q' = V \times W \in \mathscr{D}_n$ such that $Q' \subseteq B \times C \subseteq Q_n$ and $y \in V$. Put $Y_{n+1} = V$, $Z_{n+1} = W$, and $X_{n+1} = A \cap [W]_{\mathsf{E}}$. □ (Claim)

It follows that $\mathsf{E} = \widehat{\mathsf{E}}$ on X. (Otherwise, $S = \{\langle x, y\rangle \in X^2 : x \,\widehat{\mathsf{E}}\, y \wedge x \not\mathsf{E}\, y\}$ is a non-empty Σ^1_1 set, and any $\mathbb{P}_2$-generic pair $\langle x, y\rangle \in S$ implies a contradiction to Claim 10.5.2. Recall that $\mathbb{P}_2$ is equal to all non-empty Σ^1_1 subsets of $(\mathbb{N}^{\mathbb{N}})^2$.) But this implies $X \not\subseteq \mathbf{H}$ by Lemma 10.4.6, a contradiction. □ (Lemma 10.5.1)

10.6. Splitting system

The conclusion of Theorem 10.4.5, that is $\mathsf{E}_0 \sqsubseteq_{\mathsf{C}} \mathsf{E}$, means that E_0 has a continuous "copy" of the form $\mathsf{E} \restriction X$, X being a closed set in $\mathbb{N}^{\mathbb{N}}$. To obtain such a set, we define a splitting system of sets in $\mathbb{P}$ satisfying certain requirements similar to those in Section 10.2 but more complicated.

Let us fix the enumerations $\{\mathscr{D}_n\}_{n\in\mathbb{N}}$, $\{\mathscr{D}_2^n\}_{n\in\mathbb{N}}$, $\{\mathscr{D}_n^*\}_{n\in\mathbb{N}}$ of all open dense subsets of $\mathbb{P}$, $\mathbb{P}_2$, $\mathbb{P}^2 \restriction \mathsf{E}$, respectively, which are coded in the model $\mathfrak{M}$ fixed above. We assume that $\mathscr{D}_{n+1} \subseteq \mathscr{D}_n$, $\mathscr{D}_2^{n+1} \subseteq \mathscr{D}_2^n$, and $\mathscr{D}_{n+1}^* \subseteq \mathscr{D}_n^*$.

If $u, v \in 2^m$ (binary sequences of length m) have the form $u = 0^k{}^\frown 0^\frown w$ and $v = 0^k{}^\frown 1^\frown w$ for some $k < m$ and $w \in 2^{m-k-1}$, then we call $\langle u, v\rangle$ a *crucial pair*. It can be proved by induction on m that 2^m is a connected tree (*i.e.*, a connected graph without cycles) of crucial pairs, with sequences beginning with 1 as the endpoints of the graph.

We define a system of sets $X_u \in \mathbb{P}$ ($u \in 2^{<\omega}$) and $\mathsf{R}_{uv} \in \mathbb{P}_2$, $\langle u, v\rangle$ being a crucial pair, so that the following requirements are satisfied:

1°. $X_\Lambda \subseteq \mathbf{H}$.

2°. $X_u \in \mathscr{D}_n$ for all n and $u \in 2^n$.

3°. $X_{u^\frown 0} \cap X_{u^\frown 1} = \varnothing$ and $X_{u^\frown i} \subseteq X_u$ for all $u \in 2^{<\omega}$ and $i = 0, 1$.

4°. $\mathsf{R}_{uv} \in \mathscr{D}_2^n$ for all crucial pairs $\langle u, v\rangle$ in 2^n.

5°. $\mathsf{R}_{uv} \subseteq \mathsf{E}$ and $X_u \,\mathsf{R}_{uv}\, X_v$ for all crucial pairs $\langle u, v\rangle$ in 2^n.

6°. $\mathsf{R}_{u^\frown i, v^\frown i} \subseteq \mathsf{R}_{uv}$.

7°. If $u, v \in 2^n$ and $u(n-1) \neq v(n-1)$, then $X_u \times X_v \in \mathscr{D}_n^*$.

Note that **5°** implies that $X_u \,\mathsf{E}\, X_v$ for every crucial pair $\langle u, v\rangle$, hence, also for every pair in 2^n because every $u, v \in 2^n$ are connected by a unique chain of crucial pairs. It follows that $X_u \times X_v \in \mathbb{P}^2 \restriction \mathsf{E}$ for all pairs of $u, v \in 2^n$, for all n.

Assume that such a system has been defined. Then for every $a \in 2^{\mathbb{N}}$, the sequence $\{X_{a\restriction n}\}_{n\in\mathbb{N}}$ is $\mathbb{P}$-generic over $\mathfrak{M}$ by **2°**; therefore, $\bigcap_n X_{a\restriction n} = \{x_a\}$, where x_a is $\mathbb{P}$-generic, and the map $a \mapsto x_a$ is continuous and 1-to-1 by the same reasons as those for the simpler splitting system in Section 10.2.

Suppose that $a, b \in 2^{\mathbb{N}}$. If $a \not\mathsf{E}_0\, b$, then, by **7°**, $\langle x_a, x_b\rangle$ is a $\mathbb{P}^2 \restriction \mathsf{E}$-generic pair, hence, $x_a \not\mathsf{E}\, x_b$ by Lemma 10.5.1. Now suppose that $a \,\mathsf{E}_0\, b$, prove that then $x_a \,\mathsf{E}\, x_b$. We can suppose that $a = w^\frown 0^\frown c$ and $b = w^\frown 1^\frown c$, where $w \in 2^{<\omega}$ and $c \in 2^{\mathbb{N}}$ (indeed if $a \,\mathsf{E}_0\, b$, then a, b can be connected by a finite chain of such special pairs).

Then the pair $\langle x_a, x_b\rangle$ is $\mathbb{P}_2$-generic, actually, the only member of the intersection $\bigcap_n \mathsf{R}_{w^\frown 0^\frown (c\restriction n), w^\frown 1^\frown (c\restriction n)}$ by **4°** and **5°**, in particular, $x_a \mathsf{E} x_b$ because we have $R_{uv} \subseteq \mathsf{E}$ for all u, v.

Thus we have a continuous 1-to-1 reduction of E_0 to E.

□ (Theorem 10.4.5 modulo the construction of a splitting system)

10.7. Construction of a splitting system

Thus it remains to define a splitting system of sets satisfying **1°–7°**.

Let X_Λ be any set in $\mathscr{D}_0$ such that $X_\Lambda \subseteq \mathbf{H}$.

Now suppose that X_s and R_{st} have been defined for all $s \in 2^n$ and all crucial pairs in 2^n, and extend the construction on 2^{n+1}. Temporarily, define $X_{s^\frown i} = X_s$ and $\mathsf{R}_{s^\frown i, t^\frown i} = \mathsf{R}_{st}$. This leaves $\mathsf{R}_{0^n \frown 0, 0^n \frown 1}$ still undefined, so we separately put $\mathsf{R}_{0^n \frown 0, 0^n \frown 1} = \mathsf{E} \cap (X_{0^n} \times X_{0^n})$. Note that the system of sets X_u and relations R_{uv} defined this way at level $n+1$ satisfies all requirements of **1°–7°** except for the requirements of membership in the dense sets in **2°**, **4°**, and **7°**. Say in this case that the system is "coherent". It remains to produce a still "coherent" system of smaller sets and relations that also satisfies the membership in the dense sets. This will be achieved in several steps.

Step 1. Achieve that $X_u \in \mathscr{D}_{n+1}$ for every $u \in 2^{n+1}$. Take any particular $u_0 \in 2^{n+1}$. There is, by the density, a set $X'_{u_0} \in \mathscr{D}_{n+1}$ such that $X'_{u_0} \subseteq X_{u_0}$. Suppose that $\langle u_0, v\rangle$ is a crucial pair. Put $\mathsf{R}'_{u_0,v} = \{\langle x,y\rangle \in \mathsf{R}_{u_0,v} : x \in X'_{u_0}\}$ and $X'_v = \operatorname{ran} \mathsf{R}'_{u_0,v}$. This shows how the change spreads along the whole set 2^{n+1} viewed as the tree of crucial pairs. Finally, we obtain a coherent system with the additional requirement that $X'_{u_0} \in \mathscr{D}_{n+1}$. Do this consecutively for all $u_0 \in 2^{n+1}$. The total result (we re-denote it as still X_u and R_{uv}) is a "coherent" system with $X_u \in \mathscr{D}_{n+1}$ for all u. Note that still $X_{0^n \frown 0} = X_{0^n \frown 1}$ and

$$(*) \qquad \mathsf{R}_{0^n \frown 0, 0^n \frown 1} = \mathsf{E} \cap (X_{0^n \frown 0} \times X_{0^n \frown 1}).$$

Step 2. Achieve that $X_{s^\frown 0} \times X_{t^\frown 1} \in \mathscr{D}^*_{n+1}$ for all $s, t \in 2^{n+1}$. Consider a pair of $u_0 = s_0{}^\frown 0$ and $v_0 = t_0{}^\frown 1$ in 2^{n+1}. By the density there is a set $X'_{u_0} \times X'_{v_0} \in \mathscr{D}^*_{n+1}$ and $\subseteq X_{u_0} \times X_{v_0}$. By definition we have $X'_{u_0} \mathsf{E} X'_{v_0}$, but, due to Lemma 10.4.7 we can maintain that $X'_{u_0} \cap X'_{v_0} = \varnothing$. The two "shockwaves", from the changes at nodes u_0 and v_0, as in Step 1, meet only at the pair $0^m{}^\frown 0, 0^m{}^\frown 1$, where the new sets satisfy $X'_{0^m \frown 0} \mathsf{E} X'_{0^m \frown 1}$ just because E-equivalence is everywhere preserved though the changes. Now, in view of $*$, we can define $\mathsf{R}'_{0^n \frown 0, 0^n \frown 1} = \mathsf{E} \cap (X'_{0^n \frown 0} \times X'_{0^n \frown 1})$, preserving condition $*$ as well. After all pairs are considered, we will be left with a coherent system of sets and relations, redenoted as X_u and R_{uv}, which satisfies the $\mathscr{D}_{n+1}$-requirement in **2°** and the $\mathscr{D}^*_{n+1}$-requirement in **7°**.

Step 3. Achieve that $\mathsf{R}_{uv} \in \mathscr{D}_2^{n+1}$ for all crucial pairs at level $n+1$, and also that $X'_{0^n \frown 0} \cap X'_{0^n \frown 1} = \varnothing$. Consider any crucial pair $\langle u_0, v_0\rangle$. If this is not the pair $\langle 0^n{}^\frown 0, 0^n{}^\frown 1\rangle$, then let $\mathsf{R}'_{u_0 v_0} \subseteq \mathsf{R}_{u_0 v_0}$ be any set in $\mathscr{D}_2^{n+1}$. If this is $u_0 = 0^n{}^\frown 0$ and $v_0 = 0^n{}^\frown 1$, then first we choose (Lemma 10.4.7) disjoint non-empty Σ^1_1 sets $U \subseteq X_{0^n \frown 0}$ and $V \subseteq X_{0^n \frown 1}$ still with $U \mathsf{E} V$, and only then a set $\mathsf{R}'_{u_0 v_0} \subseteq \mathsf{E} \cap (U \times V)$ that belongs to $\mathscr{D}_2^{n+1}$. In both cases, put $X'_{u_0} = \operatorname{dom} \mathsf{R}'_{u_0 v_0}$ and $X'_{v_0} = \operatorname{ran} \mathsf{R}'_{u_0 v_0}$. It remains to spread the changes, along the chain of crucial pairs, to the left of u_0 and to the right of v_0, exactly as in Step 1.

Executing such a reduction for all crucial pairs $\langle u_0, v_0\rangle$ at level $n+1$ in a one by one manner, we end up with a system of sets fully satisfying **1°–7°**.

□ (Theorems 10.4.5, 10.4.1, 5.7.2)

We add a couple of useful corollaries to the results above. The first of them can be called *effectiveness of the notion of smoothness*. Indeed, it says that whenever we know that a Δ^1_1 equivalence relation E is smooth, a Borel reduction ϑ that witnesses the smoothness is effectively defined by $\vartheta(x) = \{n : x \in X_n\}$, where $\{X_n\}_{n\in\mathbb{N}}$ is a fixed enumeration of all E-invariant Δ^1_1 sets. The generalization for $\Delta^1_1(p)$ equivalence relations for any parameter p is obvious.

Corollary 10.7.1. *Suppose that* E *is a smooth* Δ^1_1 *equivalence relation on* $\mathbb{N}^{\mathbb{N}}$. *Then for every* $x, y \in \mathbb{N}^{\mathbb{N}}$, $x\mathsf{E}y$ *holds if and only if the equivalence* $x \in X \iff y \in X$ *is true for all* E*-invariant* Δ^1_1 *sets* $X \subseteq \mathbb{N}^{\mathbb{N}}$.

Proof. The "only if" direction is obvious. Prove the "if" direction. Suppose toward the contrary that $x \in X \iff y \in X$ holds for all E-invariant Δ^1_1 sets $X \subseteq \mathbb{N}^{\mathbb{N}}$; in other words, $x \,\widehat{\mathsf{E}}\, y$, but $x \not\mathsf{E}\, y$. Then $\mathsf{E} \subsetneqq \widehat{\mathsf{E}}$, and hence $\mathbf{H} \neq \varnothing$. It follows that $\mathsf{E}_0 \leq_{\mathrm{B}} \mathsf{E}$ by Theorem 10.4.5; therefore, E_0 is smooth together with E, a contradiction since E_0 is not smooth by Proposition 7.2.1(v). □

The next result shows that a "weak" smoothness via maps more complicated than Borel, implies Borel smoothness.

Corollary 10.7.2. *Suppose that* E *is a Borel equivalence relation on* $\mathbb{N}^{\mathbb{N}}$, *and* $\vartheta : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ *is a Baire measurable reduction of* E *to* $\Delta_{\mathbb{N}^{\mathbb{N}}}$. *Then* E *is smooth.*

Proof. Otherwise $\mathsf{E}_0 \leq_{\mathrm{B}} \mathsf{E}$ by Theorem 5.7.2, and hence there is a Baire measurable reduction $\varphi : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ of E_0 to $\Delta_{2^{\mathbb{N}}}$. Following an old argument of Sierpiński, define $X_= = \{\langle a, b\rangle \in (2^{\mathbb{N}})^2 : \varphi(a) = \varphi(b)\}$ and

$$X_< = \{\langle a, b\rangle \in (2^{\mathbb{N}})^2 : \varphi(a) < \varphi(b)\}, \quad X_> = \{\langle a, b\rangle \in (2^{\mathbb{N}})^2 : \varphi(a) > \varphi(b)\},$$

where $<$ is the lexicographic linear order on $2^{\mathbb{N}}$. These three sets are E_0-invariant *w.r.t.* either argument, and have the Baire property, hence each of them is either meager or comeager in $(2^{\mathbb{N}})^2$ (see Lemma 4.4.3). Moreover, $(2^{\mathbb{N}})^2 = X_= \bigcup X_< \cup X_>$ is a disjoint partition. Therefore, exactly one of the three sets is comeager. But the symmetry $\langle a, b\rangle \mapsto \langle b, a\rangle$ moves $X_<$ onto $X_>$. It follows that only $X_=$ can be comeager. Yet this is impossible by the Ulam–Kuratowski theorem since $X_=$ is a set with countable (hence meager) sections. □

Question 10.7.3. Is there anything analogous to Corollary 10.7.1 for various other classes of equivalence relations? For instance, suppose that E is Δ^1_1 and $\mathsf{E} \leq_{\mathrm{B}} \mathsf{E}_0$. Can one effectively manufacture a Borel reduction of E to E_0?

And does Corollary 10.7.2 admit any generalization? For instance, suppose that E is a Borel equivalence relation and ϑ is a Baire measurable reduction of E to E_0. Is it necessarily true that then $\mathsf{E} \leq_{\mathrm{B}} \mathsf{E}_0$, *i.e.*, there is a Borel reduction? □

10.8. The ideal of E_0-small sets

In **[Zap04]** Zapletal considers the ideal $\mathscr{I}_{\mathsf{E}_0}$ (this time an ideal on $2^{\mathbb{N}}$) of all Borel sets $X \subseteq 2^{\mathbb{N}}$ such that $\mathsf{E}_0 \restriction X <_{\mathrm{B}} \mathsf{E}_0$. Sets in $\mathscr{I}_{\mathsf{E}_0}$ are called E_0*-small*.

Lemma 10.8.1. *Let* $X \subseteq 2^{\mathbb{N}}$ *be a Borel set. Then* X *belongs to* $\mathscr{I}_{\mathsf{E}_0}$ *iff* $\mathsf{E}_0 \restriction X$ *is smooth iff* $\mathsf{E}_0 \restriction X$ *admits a Borel transversal.*

PROOF. If the restricted equivalence relation $\mathsf{E}_0 \restriction X$ admits a Borel transversal, then it is smooth by Proposition 7.2.1(i) and hence X belongs to $\mathscr{I}_{\mathsf{E}_0}$ by the above. To prove the converse make use of Proposition 7.2.1(iii). □

Applying Corollary 7.3.2, we obtain:

COROLLARY 10.8.2. *$\mathscr{I}_{\mathsf{E}_0}$ is a σ-ideal, that is, it is closed under countable unions.* □

The next theorem gives a necessary and sufficient condition for a Borel set $X \subseteq 2^{\mathbb{N}}$ in the class Δ^1_1 to belong to $\mathscr{I}_{\mathsf{E}_0}$. Recall that a set X is pairwise E-inequivalent iff $x \not\mathsf{E}\, y$ holds for all $x \neq y$ in X.

THEOREM 10.8.3. *Suppose that $X \subseteq 2^{\mathbb{N}}$ is a Δ^1_1 set. Then $X \in \mathscr{I}_{\mathsf{E}_0}$ iff X is covered by the union $\mathbb{S}$ of all pairwise E_0-inequivalent Δ^1_1 sets. Generally, if $p \in 2^{\mathbb{N}}$, then every $\Delta^1_1(p)$ set $X \subseteq 2^{\mathbb{N}}$ belongs to $\mathscr{I}_{\mathsf{E}_0}$ iff X is covered by the union of all pairwise E_0-inequivalent $\Delta^1_1(p)$ sets.*

PROOF. We consider only the parameter-free case; the relativization to an arbitrary parameter $p \in 2^{\mathbb{N}}$ is obvious.

The "if" claim. This is easy. It is quite clear that $\mathsf{E}_0 \restriction Y$ is smooth whenever Y is a Borel pairwise E_0-inequivalent Δ^1_1 set. However, countable unions preserve smoothness by Corollary 7.3.2.

The "only if" claim. Suppose that X is not covered by the union $\mathbb{S}$ of all pairwise E_0-inequivalent Δ^1_1 sets. As in the proofs of the 1st and 2nd dichotomy theorems above, $\mathbb{S}$ is a Π^1_1 set, and hence $\mathbf{H} = 2^{\mathbb{N}} \smallsetminus \mathbb{S}$ is a Σ^1_1 set. Note that $\mathbb{S}$ and $\mathbf{H}$ are resp. the structural and the chaotic domain in terms of the scheme outlined in Section 10.3. Moreover, as $X \not\subseteq \mathbb{S}$, the Σ^1_1 set $A = X \cap \mathbf{H} = X \smallsetminus \mathbb{S}$ is non-empty. Obviously A consists of all points $x \in 2^{\mathbb{N}}$ that belong to no pairwise E_0-inequivalent Δ^1_1 set.

The key property of A is that it does not intersect pairwise E_0-inequivalent Σ^1_1 sets. (To prove this, one has to establish that every pairwise E_0-inequivalent Σ^1_1 set can be covered by a pairwise E_0-inequivalent Δ^1_1 set.) It follows that

10.8.4. *Every non-empty Σ^1_1 set $Y \subseteq A$ is not pairwise E_0-inequivalent; that is it contains a pair of points $x \neq y$ with $x \,\mathsf{E}_0\, y$.*

The following notation will be used in the proof. Suppose that $x \in 2^{\mathbb{N}}$ and $n \in \mathbb{N}$. Let $x \restriction_{>n}$, resp., $x \restriction_{\geq n}$ denote the restriction of x (as a map $\mathbb{N} \to 2$) to the domain (n, ∞), resp., $[n, \infty)$. Thus $x \restriction_{>n} \in 2^{(n,\infty)}$ and $x \restriction_{\geq n} \in 2^{[n,\infty)}$. If $X \subseteq 2^{\mathbb{N}}$, then put $X \restriction_{>n} = \{x \restriction_{>n} : x \in X\}$ and $X \restriction_{\geq n} = \{x \restriction_{\geq n} : x \in X\}$.

We make use of a splitting scheme similar to those used in connection with the SILVER and SACKS forcing notions. Let us define sequences $u^0_n \neq u^1_n \in 2^{<\omega}$ ($n \in \mathbb{N}$) such that $\mathtt{lh}\, u^0_n = \mathtt{lh}\, u^1_n$ for all n, and also a system of non-empty Σ^1_1 sets $X_s \subseteq A$ ($s \in 2^{<\omega}$) satisfying the following conditions:

(a) $X_\Lambda \subseteq A$.

(b) A condition in terms of the Gandy–Harrington forcing, similar to **2°** in Section 10.2 or **2°** in Section 10.6, such that, as a consequence, $\bigcap_n X_{a \restriction n} \neq \varnothing$ for every $a \in 2^{\mathbb{N}}$.

(c) $X_{s^\frown i} \subseteq X_s$ and $X_{s^\frown 0} \cap X_{s^\frown 1} = \varnothing$ for all $s \in 2^{<\omega}$ and $i = 0, 1$.

(d) If $s \in 2^n$, then $X_s \subseteq \mathscr{O}_{w_s}$, where $w_s = u_0^{s(0)} {}^\frown u_1^{s(1)} {}^\frown \dots {}^\frown u_{n-1}^{s(n-1)}$ and $\mathscr{O}_w = \{a \in 2^{\mathbb{N}} : w \subset a\}$ for $w \in 2^{<\omega}$.

(e) If $s, t \in 2^n$, then $X_s \restriction_{\geq \ell_n} = X_t \restriction_{\geq \ell_n}$, where $\ell_n = \mathtt{lh}\, u_0^i + \dots + \mathtt{lh}\, u_{n-1}^i$.

When this construction is accomplished, define $\vartheta(a) = u_0^{a(0)} {}^\frown u_1^{a(1)} {}^\frown \dots$ for every $a \in 2^{\mathbb{N}}$; thus $\vartheta : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is a continuous 1-to-1 map. Moreover, as $u_n^0 \neq u_n^1$, $\forall n$, the map ϑ satisfies $a \mathsf{E}_0 b \iff \vartheta(a) \mathsf{E}_0 \vartheta(b)$, hence ϑ is a reduction of E_0 to E_0. Finally, the set $Y = \mathtt{ran}\, \vartheta = \bigcap_n \bigcup_{s \in 2^n} X_s$ satisfies $Y \subseteq X_\Lambda \subseteq A \subseteq X$; therefore, ϑ witnesses $\mathsf{E}_0 \leq_{\mathrm{B}} \mathsf{E}_0 \restriction Y \leq_{\mathrm{B}} \mathsf{E}_0 \restriction X$, thus $X \notin \mathscr{I}_{\mathsf{E}_0}$.

Let us carry out the construction of sequences u_n^i and sets X_s.

To begin with, choose a non-empty Σ_1^1 set $X_\Lambda \subseteq A$ such that (b) is satisfied, that is, X_λ belongs to $\mathscr{D}_0$, where $\{\mathscr{D}_n\}_{n \in \mathbb{N}}$ is a fixed enumeration of certain dense sets, as in Section 7.2. By definition $w_\Lambda = \Lambda$, therefore (d) is satisfied.

Now suppose that $n \in \mathbb{N}$ and non-empty Σ_1^1 sets $X_s \in \mathbb{P}$ are defined for all $s \in 2^k$, $k \leq n$, and sequences u_k^i are defined, too, for $k < n$, so that conditions (a)–(e) hold in the domain $\leq n$. Define $\ell_n = \mathtt{lh}\, u_0^i + \dots + \mathtt{lh}\, u_{n-1}^i$ as in (e) and $w_s \in 2^{\ell_n}$ for all $s \in 2^n$ as in (d).

To extend the construction to the next level, choose an arbitrary $\sigma \in 2^n$, for instance, σ can be the sequence of n zeros. It follows from 10.8.4 that there exist points $x_0 \neq x_1$ in X_σ such that $x_0 \mathsf{E}_0 x_1$. Then there is a number $\ell_n > \ell_{n-1}$ such that $x_0 \restriction_{\geq \ell_n} = x_1 \restriction_{\geq \ell_n}$. On the other hand $x_0 \restriction_{< \ell_{n-1}} = x_1 \restriction_{< \ell_{n-1}} = w_\sigma$ by (d). Therefore there exist sequences $u_n^0 \neq u_n^1$ in $2^{\ell_n - \ell_{n-1}}$ such that $w_\sigma {}^\frown u_n^0 \subset x_0$ and $w_\sigma {}^\frown u_n^1 \subset x_1$. Then the set Z of all points $z \in 2^{[\ell_n, \infty)}$ such that

$$\exists\, y_0, y_1 \in X_\sigma\, (y_0 \restriction_{\geq \ell_n} = y_1 \restriction_{\geq \ell_n} = z \wedge w_\sigma {}^\frown u_n^0 \subset y_0 \wedge w_\sigma {}^\frown u_n^1 \subset y_1)$$

is Σ_1^1 and non-empty: it contains the element $x_0 \restriction_{\geq \ell_n} = x_1 \restriction_{\geq \ell_n}$. In addition $Z \subseteq X_s \restriction_{\geq \ell_n}$ for all $s \in 2^n$ by (e). It follows that the family of sets

$$X'_{s^\frown i} = \{x \in X_s : x \restriction_{\geq \ell_n} \in Z \wedge w_s {}^\frown u_n^i \subset x\} \quad \text{for } s \in 2^n \text{ and } i = 0, 1$$

satisfies (c), (d), and (e). To fulfill (b), choose an arbitrary sequence $\tau = \sigma^\frown i \in 2^{n+1}$. By the density of $\mathscr{D}_{n+1}$, there is a non-empty Σ_1^1 set $Y \subseteq X'_\tau \restriction_{\geq \ell_{n+1}}$ such that for every $t \in 2^{n+1}$ there exists a set $X_t \in \mathscr{D}_{n+1}$, $X_t \subseteq X'_t$ satisfying $Y = X_t \restriction_{\geq \ell_{n+1}}$. The system of those sets X_t, $t \in 2^{n+1}$, still satisfies conditions (c), (d), and (e), and it satisfies (b) as well.

This accomplishes the inductive step in the construction of a system of sets and sequences satisfying requirements (a)–(e). □ (Theorem 10.8.3)

10.9. A forcing notion associated with E_0

Now it is quite natural to consider the collection $\mathbb{P}_{\mathsf{E}_0}$ of all E_0-*large* (that is, non-E_0-small) Borel sets $X \subseteq 2^{\mathbb{N}}$ as a forcing notion. Thus $\mathbb{P}_{\mathsf{E}_0}$ consists of all Borel sets $X \subseteq 2^{\mathbb{N}}$ such that $\mathsf{E}_0 \sim_{\mathrm{B}} \mathsf{E}_0 \restriction X$. Forcings like $\mathbb{P}_{\mathsf{E}_0}$ (that is, those defined in the form of the collection of all Borel sets X such that a given Borel equivalence relation E satisfies $\mathsf{E} \sim_{\mathrm{B}} \mathsf{E} \restriction X$) are still works in progress, their applications not yet established.

LEMMA 10.9.1. *A Borel set* $X \subseteq 2^{\mathbb{N}}$ *belongs to* $\mathbb{P}_{\mathsf{E}_0}$ *iff* $\mathsf{E}_0 \sqsubseteq_{\mathrm{C}} \mathsf{E}_0 \restriction X$.

PROOF. If $X \in \mathbb{P}_{\mathsf{E}_0}$, then $\mathsf{E}_0 \sqsubseteq_{\mathrm{C}} \mathsf{E}_0 \restriction X$ by Theorem 10.4.1. On the other hand if $\mathsf{E}_0 \sqsubseteq_{\mathrm{C}} \mathsf{E}_0 \restriction X$, then $\mathsf{E}_0 \restriction X$ is not smooth since E_0 itself is not smooth by Proposition 7.2.1(v). □

Note that every set $X \in \mathbb{P}_{\mathsf{E}_0}$ contains a closed subset $Y \subseteq X$ also in $\mathbb{P}_{\mathsf{E}_0}$ by Theorem 10.4.1. (Apply the theorem for $\mathsf{E} = \mathsf{E}_0 \restriction X$. As $\mathsf{E}_0 \restriction X$ is not smooth, we have $\mathsf{E}_0 \sqsubseteq_{\mathrm{C}} \mathsf{E}_0 \restriction X$, by a continuous reduction ϑ. Take as Y the full image of ϑ. Y is compact, hence closed.) Such sets Y can be chosen in a special family.

DEFINITION 10.9.2. Suppose that two binary sequences $u_n^0 \neq u_n^1 \in 2^{<\omega}$ of equal length $\mathtt{lh}\, u_n^0 = \mathtt{lh}\, u_n^1 \geq 1$ are chosen for each n. Define $\vartheta(a) = u_0^{a(0)} {}^\frown u_1^{a(1)} {}^\frown \dots$ for all $a \in 2^{\mathbb{N}}$. Then ϑ is a continuous injection $2^{\mathbb{N}} \to 2^{\mathbb{N}}$, $Y = \mathtt{ran}\, \vartheta$ is a closed set in $2^{\mathbb{N}}$, and ϑ is a reduction of E_0 to $\mathsf{E}_0 \restriction Y$; therefore, $\mathsf{E}_0 \sqsubseteq_{\mathrm{C}} \mathsf{E}_0 \restriction Y$ and $Y \in \mathbb{P}_{\mathsf{E}_0}$.
Let $\mathbb{P}'_{\mathsf{E}_0}$ denote the collection of all sets Y definable in such a form. □

THEOREM 10.9.3 (ZAPLETAL [**Zap04**]). *$\mathbb{P}'_{\mathsf{E}_0}$ is a dense subset of $\mathbb{P}_{\mathsf{E}_0}$: for every $X \in \mathbb{P}_{\mathsf{E}_0}$ there exists $Y \in \mathbb{P}'_{\mathsf{E}_0}$ such that $Y \subseteq X$. In addition, $\mathbb{P}_{\mathsf{E}_0}$ forces that the "old" continuum $\mathfrak{c}$ remains uncountable.*

PROOF. Suppose that X is Δ^1_1. (As usual the case when X is $\Delta^1_1(p)$ for some $p \in 2^{\mathbb{N}}$ does not differ in any detail.) Obviously, $X \notin \mathscr{I}_{\mathsf{E}_0}$. Then $X \not\subseteq \mathbb{S}$ by Theorem 10.8.3, and hence the set $A = X \setminus \mathbb{S}$ is non-empty. Therefore, the splitting construction as in the proof of Theorem 10.8.3 can be carried out in the domain A. It yields a closed set $Y \subseteq X$ which belongs to $\mathbb{P}'_{\mathsf{E}_0}$.

As for the additional claim, it suffices to prove the same result for the subforcing $\mathbb{P}'_{\mathsf{E}_0}$. Given a sequence of dense sets $\mathscr{D}_n \subseteq \mathbb{P}'_{\mathsf{E}_0}$, we carry out a splitting construction as in the proof of Theorem 10.8.3, with the following amendments. First, each set X_s belongs to $\mathscr{D}_{\mathtt{lh}\, s}$, hence to $\mathbb{P}'_{\mathsf{E}_0}$, and therefore is a closed set in $2^{\mathbb{N}}$. Second, condition (b) is abolished, of course. That every set $X \in \mathbb{P}'_{\mathsf{E}_0}$ satisfies the key condition 10.8.4 (that is, it contains a pair of points $x \neq y$ with $x \,\mathsf{E}_0\, y$) is obvious. □

The following results on $\mathbb{P}_{\mathsf{E}_0}$ resemble some known properties of the SILVER and SACKS forcings.

EXERCISE 10.9.4 (ZAPLETAL). For those interested in forcing, prove that $\mathbb{P}_{\mathsf{E}_0}$, similar to the SACKS forcing, produces reals of minimal degree.

The proof resembles known arguments, but in addition the following is applied: if $X \in \mathbb{P}_{\mathsf{E}_0}$ and $f : X \to 2^{\mathbb{N}}$ is a Borel E_0-invariant map (that is, $x \,\mathsf{E}_0\, y \Longrightarrow f(x) = f(y)$), then f is constant on a set $Y \in \mathbb{P}_{\mathsf{E}_0}$, $Y \subseteq X$. Indeed, suppose, for the sake of brevity, that $X = 2^{\mathbb{N}}$. For every n, the set $Y_n^0 = \{a : f(a)(n) = 0\}$ is Borel and E_0-invariant. It follows that Y_n^0 is either meager or comeager. Define $b \in 2^{\mathbb{N}}$ so that $b(n) = 0$ iff Y_n^0 is comeager. Then the set $D = \{a : f(a) = b\}$ is comeager. A splitting construction as in the proof of Theorem 10.8.3 yields a set $Y \in \mathbb{P}_{\mathsf{E}_0}$, $Y \subseteq D$. □

There is another similarity between $\mathbb{P}_{\mathsf{E}_0}$ and the SACKS forcing:

EXERCISE 10.9.5. Let $\mathbb{S}$ denote the set of all closed uncountable sets $X \subseteq 2^{\mathbb{N}}$ — the SACKS forcing. Prove that a closed Δ^1_1 set $X \subseteq 2^{\mathbb{N}}$ belongs to $\mathbb{S}$ iff $X \not\subseteq \Delta^1_1$; that is, iff X contains at least one point not in Δ^1_1. In this case, the chaotic domain, as outlined in Section 10.3, consists of all non-Δ^1_1 points $x \in 2^{\mathbb{N}}$. □

CHAPTER 11

Ideal $\mathscr{I}_1$ and the equivalence relation E_1

By definition the ideal $\mathsf{Fin} \times 0 = \mathscr{I}_1$ consists of all sets $x \subseteq \mathscr{P}(\mathbb{N} \times \mathbb{N})$ such that all, except for finitely many, cross-sections $(x)_n = \{k : \langle n, k \rangle \in x\}$ are empty. The ideal $\mathscr{I}_1$ naturally defines the equivalence relation $\mathsf{E}_1 = \mathsf{E}_{\mathscr{I}_1}$ on $\mathscr{P}(\mathbb{N} \times \mathbb{N})$ so that $x \,\mathsf{E}_1\, y$ iff $x \,\Delta\, y \in \mathscr{I}_1$. We can as well consider E_1 to be an equivalence relation on $(2^{\mathbb{N}})^{\mathbb{N}}$, or even on $\mathbb{X}^{\mathbb{N}}$ for an arbitrary uncountable Polish space $\mathbb{X}$, defined so that $x \,\mathsf{E}_1\, y$ iff $x(k) = y(k)$ for all but finite k.

This chapter contains proofs of some key results related to $\mathscr{I}_1$ and E_1. First of all we prove a theorem, due to KECHRIS, that there exist only three up to isomorphism types of ideals on $\mathbb{N}$ Borel reducible to $\mathscr{I}_1$, two of them being Fin and $\mathscr{I}_1$ itself. We also present several theorems related to E_1, the most important of them being Theorem 5.7.3 of KECHRIS and LOUVEAU, or the 3rd dichotomy theorem. In addition, Section 11.2 contains several results that characterize E_1 in terms of hypersmoothness and essential countability. Another important property of E_1 is presented in Section 11.8: E_1 is not Borel reducible to a Borel action of a Polish group. Finally, a forcing notion associated with E_1 is considered in Section 11.7.

11.1. Ideals below $\mathscr{I}_1$

Recall that $\mathscr{I} \cong \mathscr{J}$ means the isomorphism of ideals $\mathscr{I}, \mathscr{J}$ via a bijection between the underlying sets. If the underlying sets are countable, then that is a type of Borel isomorphism. The ideal $\mathsf{Fin} \oplus \mathscr{P}(\mathbb{N})$ (the disjoint sum in the sense of Section 3.7) in the next theorem is isomorphic to the ideal $\mathsf{Fin}_{\text{EVEN}} = \{x \subseteq \mathbb{N} : x \cap 2\mathbb{N} \in \mathsf{Fin}\}$, where $2\mathbb{N}$ = all even numbers.[1]

THEOREM 11.1.1 (KECHRIS [**Kec98**]). *If $\mathscr{I}$ is a Borel (non-trivial) ideal on $\mathbb{N}$ and $\mathsf{E}_{\mathscr{I}} \le_{\mathrm{B}} \mathsf{E}_1$, then $\mathscr{I}$ is isomorphic (via a bijection between the underlying sets) to one of the following three ideals:* $\mathscr{I}_1$, Fin, $\mathsf{Fin} \oplus \mathscr{P}(\mathbb{N})$.

Thus there exist only three different ideals on $\mathbb{N}$ Borel reducible to $\mathscr{I}_1$: they are Fin, the disjoint sum $\mathsf{Fin} \oplus \mathscr{P}(\mathbb{N})$, and $\mathscr{I}_1$ itself.

PROOF. We begin with another version of the method used in the proof of Theorem 3.2.1. Suppose that $\{\mathscr{B}_k\}_{k \in \mathbb{N}}$ is a fixed system of Borel subsets of $\mathscr{P}(\mathbb{N})$. (It will be specified later.) Then there exists an increasing sequence of integers $0 = n_0 < n_1 < n_2 < \cdots$ and sets $s_k \subseteq [n_k, n_{k+1})$ such that

(A) every set $x \subseteq \mathbb{N}$ with $\forall^\infty k \,(x \cap [n_k, n_{k+1}) = s_k)$ is "generic";[2]

---

[1] Ideals isomorphic to any of $\mathscr{I}, \mathscr{I} \oplus \mathscr{P}(\mathbb{N})$ were called *trivial variations* of $\mathscr{I}$ in [**Kec98**].

[2] We mean Cohen-generic over a certain countable family of dense open subsets of $\mathscr{P}(\mathbb{N})$ that depends on the choice of the family of sets $\mathscr{B}_k$.

(B) if $k' \geq k$ and $u \subseteq [0, n_{k'})$, then $u \cup s_{k'}$ decides $\mathscr{B}_k$ in the sense that either all "generic" $x \in \mathscr{P}(\mathbb{N})$ with $x \cap [0, n_{k'+1}) = u \cup s_{k'}$ belong to $\mathscr{B}_k$ or all "generic" x with $x \cap [0, n_{k'+1}) = u \cup s_{k'}$ do not belong to $\mathscr{B}_k$.

Now put $\mathscr{D}_0 = \{x \cup S_1 : x \subseteq Z_0\}$ and $\mathscr{D}_1 = \{x \cup S_0 : x \subseteq Z_1\}$, where

$S_0 = \bigcup_k s_{2k} \subseteq Z_0 = \bigcup_k [n_{2k}, n_{2k+1})$, $\quad S_1 = \bigcup_k s_{2k+1} \subseteq Z_1 = \bigcup_k [n_{2k+1}, n_{2k+2})$.

Clearly, every $x \in \mathscr{D}_0 \cup \mathscr{D}_1$ is "generic" by (A), hence it follows from (B) that

(C) each $\mathscr{B}_k$ is clopen on both $\mathscr{D}_0$ and $\mathscr{D}_1$.

As $\mathscr{I} \leq_{\mathrm{B}} \mathscr{I}_1$, it follows from Lemma 5.3.1 (and the trivial fact that $\mathscr{I}_1 \oplus \mathscr{I}_1 \cong \mathscr{I}_1$) that there exists a *continuous* reduction $\vartheta : \mathscr{P}(\mathbb{N}) \to \mathscr{P}(\mathbb{N} \times \mathbb{N})$ of $\mathscr{I}$ to $\mathscr{I}_1$. Thus $\mathsf{E}_{\mathscr{I}}$ is the union of an increasing sequence of (topologically) closed equivalence relations $\mathsf{R}_m \subseteq \mathsf{E}_{\mathscr{I}}$ just because $\mathscr{I}_1$ admits such a form. We now require that $\{\mathscr{B}_k\}$ includes all sets $B^m_l = \{x \in \mathscr{P}(\mathbb{N}) : \forall\, s \subseteq [0, l)\; x \,\mathsf{R}_m\, (x \,\triangle\, s)\}$. Then by (C) and the compactness of $\mathscr{D}_i$ for every l there is $m(l) \geq l$ satisfying

(D) $\forall\, x \in \mathscr{D}_0 \cup \mathscr{D}_1\, \forall\, s \subseteq [0, l)\; (x \,\mathsf{R}_{m(l)}\, (x \,\triangle\, s))$.

To prove the theorem, it suffices to obtain a sequence $x_0 \subseteq x_1 \subseteq x_2 \subseteq \cdots$ of sets $x_k \in \mathscr{I}$ with $\mathscr{I} = \bigcup_n \mathscr{P}(x_n)$: that in this case $\mathscr{I}$ is as required is an easy exercise. As every topologically closed ideal is easily $\mathscr{P}(x)$ for some $x \subseteq \mathbb{N}$, it suffices to show that $\mathscr{I}$ is a union of a countable sequence of closed subideals. It suffices to demonstrate this fact separately for $\mathscr{I} \restriction Z_0$ and $\mathscr{I} \restriction Z_1$. Prove that $\mathscr{I} \restriction Z_0$ is a countable union of closed subideals, ending the proof of the theorem.

If $m \in \mathbb{N}$ and $s \subseteq u \subseteq Z_0$ are finite, then let

$$I^m_{us} = \{A \subseteq Z_0 : \forall\, x \in \mathscr{D}_0\, (x \cap u = s \Longrightarrow (x \cup (A \smallsetminus u))\, \mathsf{R}_m\, x)\}.$$

LEMMA 11.1.2. *Sets I^m_{us} are closed topologically and under $\cup$, and $I^m_{us} \subseteq \mathscr{I}$.*

PROOF. I^m_{us} are topologically closed because R_m are also.

Suppose that $A, B \in I^m_{us}$. To prove that $A \cup B \in I^m_{us}$, let $x \in \mathscr{D}_0$ satisfy $x \cap u = s$. Then $x' = x \cup (A \smallsetminus u) \in \mathscr{D}_0$ satisfies $x' \cap u = s$, too, hence, as $B \in I^m_{us}$, we have $(x' \cup (B \smallsetminus u))\, \mathsf{R}_m\, x'$, thus, $(x \cup ((A \cup B) \smallsetminus u))\, \mathsf{R}_m\, x'$. However $x' \,\mathsf{R}_m\, x$ just because $A \in I^m_{us}$. It remains to recall that R_m is an equivalence relation.

To prove that every set $A \in I^m_{us}$ belongs to $\mathscr{I}$ take $x = s \cup S_1$. Then we have $x \cup (A \smallsetminus u)\, \mathsf{R}_m\, x$, thus, $A \in \mathscr{I}$ as s is finite and $\mathsf{R}_m \subseteq \mathsf{E}_{\mathscr{I}}$. □ (Lemma)

LEMMA 11.1.3. $\mathscr{I} \restriction Z_0 = \bigcup_{m,u,s} I^m_{us}$.

PROOF. Let $A \in \mathscr{I}$, $A \subseteq Z_0$. The sets $Q_m = \{x \in \mathscr{D}_0 : (x \cup A)\, \mathsf{R}_m\, x\}$ are closed and satisfy $\mathscr{D}_0 = \bigcup_m Q_m$. It follows that one of them has a non-empty interior in $\mathscr{D}_0$; thus, there exist finite sets $s \subseteq u \subseteq Z_0$ and some m_0 with

$$\forall\, x \in \mathscr{D}_0\, (x \cap u = s \Longrightarrow (x \cup A)\, \mathsf{R}_{m_0}\, x).$$

This is not exactly what we need. However, by (D), there exists a number $m = \mathtt{max}\{m_0, m(\sup u)\}$ big enough for

$$\forall\, x \in \mathscr{D}_0 : (x \cup A)\, \mathsf{R}_m\, (x \cup (A \smallsetminus u)).$$

It follows that $A \in I^m_{su}$, as required. □ (Lemma)

Let J^m_{su} be the hereditary hull of I^m_{su} (all subsets of sets in I^m_{su}). It follows from Lemma 11.1.2 that every J^m_{su} is a topologically closed subideal of $\mathscr{I} \restriction Z_0$; however, $\mathscr{I} \restriction Z_0$ is the union of those ideals by Lemma 11.1.3, as required. □

COROLLARY 11.1.4. *The equivalence relations* E_2 *and* E_3 *are Borel irreducible to* E_1. *It follows that they are Borel irreducible to* E_0, *and hence* $\mathsf{E}_0 <_{\mathrm{B}} \mathsf{E}_2$ *and* $\mathsf{E}_0 <_{\mathrm{B}} \mathsf{E}_3$.

PROOF. It is quite clear that neither $\mathscr{I}_2$ nor $\mathscr{I}_3$ belong to the types of ideals mentioned in Theorem 11.1.1. □

That $\mathsf{E}_0 <_{\mathrm{B}} \mathsf{E}_1$ strictly, and even that E_1 is not essentially countable (formally $\mathsf{E}_1 \not\leq_{\mathrm{B}} \mathsf{E}_\infty$), will be established by Lemma 11.2.3 below.

11.2. E_1: hypersmoothness and non-countability

Recall that a hypersmooth equivalence relation is a countable increasing union of Borel smooth equivalence relations. This section contains several results on the relationships between hypersmooth and countable equivalence relations. First of all prove that E_1 is universal in the class of hypersmooth equivalence relations.

LEMMA 11.2.1. *For a Borel equivalence relation* E *to be hypersmooth, it is necessary and sufficient that* $\mathsf{E} \leq_{\mathrm{B}} \mathsf{E}_1$.

PROOF. Let $\mathbb{X}$ be the domain of E. Assume that E is hypersmooth, *i.e.*, $\mathsf{E} = \bigcup_n \mathsf{F}_n$, where $x \mathrel{\mathsf{F}_n} y$ iff $\vartheta_n(x) = \vartheta_n(y)$, each $\vartheta_n : \mathbb{X} \to 2^{\mathbb{N}}$ is Borel, and $\mathsf{F}_n \subseteq \mathsf{F}_{n+1}$, $\forall n$. Then $\vartheta(x) = \{\vartheta_n(x)\}_{n\in\mathbb{N}}$ witnesses $\mathsf{E} \leq_{\mathrm{B}} \mathsf{E}_1$. Conversely, if $\vartheta : \mathbb{X} \to (2^{\mathbb{N}})^{\mathbb{N}}$ is a Borel reduction of E to E_1, then the sequence of equivalence relations $x \mathrel{\mathsf{F}_n} y$ iff $\vartheta(x) \restriction_{\geq n} = \vartheta(y) \restriction_{\geq n}$ witnesses that E is hypersmooth.

Here $\vartheta(x) \restriction_{\geq n}$ is the restriction of the $\mathbb{N}$-sequence $\vartheta(x) \in (2^{\mathbb{N}})^{\mathbb{N}}$ to the set $[n, \infty) = \{k \in \mathbb{N} : k \geq n\}$. □

COROLLARY 11.2.2. $\mathsf{E}_\infty \not\leq_{\mathrm{B}} \mathsf{E}_1$.

PROOF. Otherwise E_∞ is a hypersmooth equivalence relation by Lemma 11.2.1. But E_∞ is countable as well. It follows that $\mathsf{E}_\infty \leq_{\mathrm{B}} \mathsf{E}_0$ by Theorem 8.1.1. This contradicts Theorem 7.5.1. □

The following result is given in [**KL97**] with a reference to earlier papers.

LEMMA 11.2.3. (i) E_1 *is not essentially countable, that is, it is not Borel reducible to a countable* (with at most countable classes) *Borel equivalence relation.*

(ii) $\mathsf{E}_0 <_{\mathrm{B}} \mathsf{E}_1$, *in other words,* $\mathrm{Fin} <_{\mathrm{B}} \mathscr{I}_1$.

PROOF. (i) (A version of the argument in [**KL97**], 1.4 and 1.5.) Suppose that F is a Borel countable equivalence relation on, say, $\mathbb{N}^{\mathbb{N}}$, and $\vartheta : (2^{\mathbb{N}})^{\mathbb{N}} \to \mathbb{X}$ is a Borel map satisfying $x \mathrel{\mathsf{E}_1} y \Longrightarrow \vartheta(x) \mathrel{\mathsf{F}} \vartheta(y)$. Then ϑ is continuous on a dense $\mathbf{G}_\delta$ set $D \subseteq (2^{\mathbb{N}})^{\mathbb{N}}$. Our goal is to show that ϑ is not a reduction. We begin with a few definitions. Let us fix a countable transitive model $\mathfrak{M}$ of $\mathbf{ZFC}^-$ chosen as in Definition 10.1.3 and containing codes for D, $\vartheta \restriction D$, $\mathbb{X}$ (in the sense of Appendix A.2).

We are going to define, for every k, a pair of points $a_k \neq b_k \in 2^{\mathbb{N}}$, a number $\ell(k)$ and a tuple $\tau_k \in (2^{\mathbb{N}})^{\ell(k)}$ such that

(a) both $x = \langle a_0\rangle^\frown\tau_0{}^\frown\langle a_1\rangle^\frown\tau_1{}^\frown\dots$ and $y = \langle b_0\rangle^\frown\tau_0{}^\frown\langle b_1\rangle^\frown\tau_1{}^\frown\dots$ are elements of $(2^{\mathbb{N}})^{\mathbb{N}}$ Cohen-generic over $\mathfrak{M}$;

(b) for every k, the finite sequence

$$\zeta_k = \langle a_0, b_0\rangle^\frown \tau_0{}^\frown \langle a_1, b_1\rangle^\frown \tau_1{}^\frown \cdots {}^\frown \langle a_k, b_k\rangle^\frown \tau_k$$

is Cohen-generic over $\mathfrak{M}$, hence so are the subsequences

$$\xi_k = \langle a_0\rangle^\frown \tau_0{}^\frown \cdots {}^\frown \langle a_k\rangle^\frown \tau_k \quad \text{and} \quad \eta_k = \langle b_0\rangle^\frown \tau_0{}^\frown \cdots {}^\frown \langle b_k\rangle^\frown \tau_k\,;$$

(c) for every k and every $z \in (2^{\mathbb{N}})^{\mathbb{N}}$ such that $\zeta_k{}^\frown z$ is generic over $\mathfrak{M}$ we have $\vartheta(\xi_k{}^\frown z) = \vartheta(\eta_k{}^\frown z)$.

If this is done, then by (b) choose for every k a point $z_k \in (2^{\mathbb{N}})^{\mathbb{N}}$ Cohen-generic over $\mathfrak{M}[\zeta_k]$. Then $\zeta_k{}^\frown z_k$ is Cohen-generic over $\mathfrak{M}$ by the product forcing theorem. It follows by (c) that $\vartheta(x_k) = \vartheta(y_k)$, where $x_k = \xi_k{}^\frown z_k$ and $y_k = \eta_k{}^\frown z_k$. Note that $x_k \to x$ and $y_k \to y$ in $(2^{\mathbb{N}})^{\mathbb{N}}$ with $k \to \infty$, and on the other hand, all of x_k, x, y_k, y belong to D because of the genericity. It follows that $\vartheta(x) = \vartheta(y)$ by the choice of D. However, obviously, $\neg\, x \,\mathsf{E}_1\, y$, so that ϑ is not a reduction, as required.

To define a_0, b_0, τ_0 note that there exists a perfect set $X \subseteq 2^{\mathbb{N}}$ and a point $z \in (2^{\mathbb{N}})^{\mathbb{N}}$ such that $\langle a, b\rangle^\frown z$ is Cohen-generic over $\mathfrak{M}$ for every two $a \neq b \in X$. (Indeed let $\langle w, z\rangle \in 2^{2^{<\omega}} \times (2^{\mathbb{N}})^{\mathbb{N}}$ be Cohen-generic over $\mathfrak{M}$. Put $X = \{w_a : a \in 2^{\mathbb{N}}\}$, where $w_a \in 2^{\mathbb{N}}$ is defined by $w_a(k) = w(a \restriction k)$, $\forall\, k$.) In particular, $\langle a\rangle^\frown z$ is Cohen-generic over $\mathfrak{M}$ for every $a \in X$. However, all points of the form $\langle a\rangle^\frown z$ are pairwise E_1-equivalent. Thus ϑ sends all of them into one and the same F-class, which is a countable set by the choice of F. It follows that there is a pair of $a \neq b$ in X such that $\vartheta(\langle a\rangle^\frown z) \neq \vartheta(\langle b\rangle^\frown z)$. This equality is a property of the generic point $\langle a, b\rangle^\frown z$; hence, it is forced in the sense that there is a number ℓ such that $\vartheta(\langle a\rangle^\frown \hat z) = \vartheta(\langle b\rangle^\frown \hat z)$ whenever $z \in (2^{\mathbb{N}})^{\mathbb{N}}$, $\langle a, b\rangle^\frown \hat z$ is Cohen-generic over $\mathfrak{M}$, and $\hat z \restriction \ell = z \restriction \ell$. Put $a_0 = a$, $b_0 = b$, $\tau_0 = z \restriction \ell$.

The induction step is carried out by a similar argument. For instance to define a_1, b_1, τ_1 we find points $a' \neq b' \in 2^{\mathbb{N}}$ and $z' \in (2^{\mathbb{N}})^{\mathbb{N}}$ such that $\langle a', b'\rangle^\frown z'$ is Cohen-generic over $\mathfrak{M}[a_0, b_0, z]$ and $\vartheta(\langle a_0\rangle^\frown \tau_0{}^\frown \langle a'\rangle^\frown z') = \vartheta(\langle a_0\rangle^\frown \tau_0{}^\frown \langle b'\rangle^\frown z')$. Yet we have $\vartheta(\langle a_0\rangle^\frown \tau_0{}^\frown \langle b'\rangle^\frown z') = \vartheta(\langle b_0\rangle^\frown \tau_0{}^\frown \langle b'\rangle^\frown z')$ by the choice of ℓ (take $\hat z = \tau_0{}^\frown \langle b'\rangle^\frown z'$). Thus $\vartheta(\langle a_0\rangle^\frown \tau_0{}^\frown \langle a'\rangle^\frown z') = \vartheta(\langle b_0\rangle^\frown \tau_0{}^\frown \langle b'\rangle^\frown z')$. It follows that there is a number ℓ' satisfying $\vartheta(\langle a_0\rangle^\frown \tau_0{}^\frown \langle a'\rangle^\frown \hat z) = \vartheta(\langle b_0\rangle^\frown \tau_0{}^\frown \langle b'\rangle^\frown \hat z)$ for every $\hat z \in (2^{\mathbb{N}})^{\mathbb{N}}$ such that $\langle a_0, b_0\rangle^\frown \tau_0{}^\frown \langle a', b'\rangle^\frown \hat z$ is Cohen-generic over $\mathfrak{M}$ and $\hat z \restriction \ell' = z' \restriction \ell'$. Put $a_1 = a'$, $b_1 = b'$, $\tau_1 = z' \restriction \ell'$.

(ii) That $\mathsf{E}_0 \leq_{\mathrm{B}} \mathsf{E}_1$ is witnessed by the map $f(x) = \{\langle 0, n\rangle : n \in x\}$. □

While E_1 is not countable, the conjunction of hypersmoothness and countability characterizes the class of hyperfinite equivalence relations, considered essentially much more elementary from the point of view of Borel reducibility.

11.3. 3rd dichotomy

In accordance with Corollary 5.2.2, Theorem 5.7.3 is a consequence of the next slightly more concrete theorem:

THEOREM 11.3.1. *Suppose that $D \subseteq (2^{\mathbb{N}})^{\mathbb{N}}$ is a Σ^1_1 set. Then* either $\mathsf{E}_1 \restriction D \leq_{\mathrm{B}} \mathsf{E}_0$ or $\mathsf{E}_1 \sqsubseteq_{\mathrm{C}} \mathsf{E}_1 \restriction D$, *and hence* $\mathsf{E}_1 \restriction D \sim_{\mathrm{B}} \mathsf{E}_1$.

Recall that $\mathsf{E} \sqsubseteq_{\mathrm{C}} \mathsf{F}$ means the existence of a continuous (hence, Borel) embedding, that is, a 1-to-1 reduction, of E into F.

PROOF. As usual, we may assume that D is a lightface Σ^1_1 subset of $(2^{\mathbb{N}})^{\mathbb{N}}$. The idea behind the proof is to show that the set D is either small enough for $\mathsf{E}_1 \restriction D$ to be Borel reducible to E_0, or otherwise it is big enough to contain a closed subset X such that $\mathsf{E}_1 \restriction X$ is Borel isomorphic to E_1.

DEFINITION 11.3.2. Relations $\prec$ and $\preccurlyeq$ will denote the inverse order relations on $\mathbb{N}$, *i.e.*, $m \preccurlyeq n$ iff $n \le m$, and $m \prec n$ iff $n < m$. If $x \in (2^{\mathbb{N}})^{\mathbb{N}}$, then $x \restriction_{\preccurlyeq n}$ denotes the restriction of x (a function defined on $\mathbb{N}$) to the domain $\preccurlyeq n$, *i.e.*, $[n, \infty)$. If $X \subseteq (2^{\mathbb{N}})^{\mathbb{N}}$, then let $X \restriction_{\preccurlyeq n} = \{x \restriction_{\preccurlyeq n} : x \in X\}$. Define $x \restriction_{\prec n}$ and $X \restriction_{\prec n}$ similarly. In particular, $(2^{\mathbb{N}})^{\mathbb{N}} \restriction_{\preccurlyeq n} = (2^{\mathbb{N}})^{\preccurlyeq n} = (2^{\mathbb{N}})^{[n,\infty)}$.

For any sequence $x \in (2^{\mathbb{N}})^{\preccurlyeq n}$, let $\mathsf{dep}\, x$ (the *depth* of x) be the number (finite or ∞) of elements of the set $\nabla(x) = \{j \preccurlyeq n : x(j) \notin \Delta^1_1(x \restriction_{\prec j})\}$. □

In continuation of the proof of the theorem, we prove

LEMMA 11.3.3. $\mathbb{S} = \{x \in (2^{\mathbb{N}})^{\mathbb{N}} : \mathsf{dep}\, x < +\infty\}$ *is a* Π^1_1 *set.*

PROOF. The relation $\mathsf{dep}\, x \le d$ (of two variables, d running over $\mathbb{N}$) is Π^1_1 since the background formula $x \in \Delta^1_1(y)$ is Π^1_1 by Corollary 2.8.5. □

We have two cases:

Case 1. All $x \in D$ satisfy $\mathsf{dep}\, x < +\infty$. In other words, it is assumed that D, the Σ^1_1 set considered, is a subset of the the structural domain (see Section 10.3) $\mathbb{S} = \{x \in (2^{\mathbb{N}})^{\mathbb{N}} : \mathsf{dep}\, x < +\infty\}$.

Case 2. There exist points $x \in D$ with $\mathsf{dep}\, x = \infty$. Thus the chaotic domain $\mathbf{H} = (2^{\mathbb{N}})^{\mathbb{N}} \setminus \mathbb{S}$ here consists of all points $x \in (2^{\mathbb{N}})^{\mathbb{N}}$ such that $x(j) \notin \Delta^1_1(x \restriction_{\prec j})$ for infinitely many $j \in \mathbb{N}$, and the Case 2 assumption is that D has a non-empty intersection with $\mathbf{H}$.

It will be proved below in this chapter that $\mathsf{E}_1 \restriction D \sim_{\mathrm{B}} \mathsf{E}_0$ and therefore $\mathsf{E}_1 \restriction D <_{\mathrm{B}} \mathsf{E}_1$ in Case 1, but $\mathsf{E}_1 \sqsubseteq_{\mathrm{B}} \mathsf{E}_1 \restriction D$ in Case 2.

REMARK 11.3.4. Our goal here, that is, the separation of Σ^1_1 sets $D \subseteq (2^{\mathbb{N}})^{\mathbb{N}}$ such that $\mathsf{E}_1 \restriction D <_{\mathrm{B}} \mathsf{E}_1$ from those satisfying $\mathsf{E}_1 \restriction D \sim_{\mathrm{B}} \mathsf{E}_1$, is quite similar to the content of Section 10.8, where Δ^1_1 sets $X \subseteq 2^{\mathbb{N}}$ such that $\mathsf{E}_0 \restriction X <_{\mathrm{B}} \mathsf{E}_0$ (*i.e.*, sets in $\mathscr{I}_{\mathsf{E}_0}$) were separated from those satisfying $\mathsf{E}_0 \restriction X \sim_{\mathrm{B}} \mathsf{E}_0$ by Theorem 10.8.3. Our results (Theorem 10.8.3 and Theorem 11.5.1 below) can be summarized as follows: if $X \subseteq 2^{\mathbb{N}}$ and $D \subseteq (2^{\mathbb{N}})^{\mathbb{N}}$ are Δ^1_1 sets, then

$$\mathsf{E}_0 \restriction X \sim_{\mathrm{B}} \mathsf{E}_0 \quad \text{iff} \quad \exists a \in X\ (a \text{ belongs to no pairwise } \mathsf{E}_0\text{-inequivalent } \Delta^1_1 \text{ set});$$

$$\mathsf{E}_1 \restriction D \sim_{\mathrm{B}} \mathsf{E}_1 \quad \text{iff} \quad \exists x \in D\ (x(j) \notin \Delta^1_1(x \restriction_{\prec j}) \text{ for infinitely many } j \in \mathbb{N}).$$

The right-hand sides of the two displayed equivalences do not seem similar at first glance. Yet a closer inspection shows some similarities.

Indeed assume that $a \in 2^{\mathbb{N}}$ does not belong to a pairwise E_0-inequivalent Δ^1_1 set. Then every Δ^1_1 set $A \subseteq 2^{\mathbb{N}}$ containing a also contains, for each n, another point $a' \in C$ such that $a \restriction_{\prec n} = a' \restriction_{\prec n}$ (that is, $a(k) = a'(k)$ for all $k \ge n$) but $a(n) \ne a'(n)$. Thus a admits infinitely many branching points, in the sense of the order $\prec$, in every Δ^1_1 set containing a.

Now assume that $x \in (2^{\mathbb{N}})^{\mathbb{N}}$ satisfies $x(j) \notin \Delta^1_1(x \restriction_{\prec j})$ for infinitely many indices j. Consider any Δ^1_1 set $P \subseteq (2^{\mathbb{N}})^{\mathbb{N}}$ containing x. If $j \in \mathbb{N}$ and $x(j) \notin \Delta^1_1(x \restriction_{\prec j})$, then

$$P_j(x) = \{x'(j) : x' \in P \wedge x' \restriction_{\preccurlyeq j} = x \restriction_{\preccurlyeq j}\}$$

is a $\Delta^1_1(x \restriction_{\preccurlyeq j})$ set containing an element $x(j)$ not in $\Delta^1_1(x \restriction_{\prec j})$. It follows that $P_j(x)$ is uncountable (and contains a perfect subset). In other words, x admits infinitely many uncountably branching points, in the sense of $\prec$, in every Δ^1_1 set containing x. □

11.4. Case 1

We come back to the proof of Theorem 11.3.1. Case 1 is the easier of the two cases. The following lemma proves that the Case 1 assumption implies the "either" case of Theorem 11.3.1.

LEMMA 11.4.1. *Suppose that* $D \subseteq (2^{\mathbb{N}})^{\mathbb{N}}$ *is a* Σ^1_1 *set and every* $x \in D$ *satisfies* $\mathtt{dep}\, x < \infty$. *Then* $\mathsf{E}_1 \restriction D \le_{\mathrm{B}} \mathsf{E}_0$.

PROOF. By Theorem 2.3.2 (Separation) there is a Δ^1_1 set D' such that $D \subseteq D'$ and still $D' \subseteq \mathbb{S}$. Thus it can be assumed that D is Δ^1_1. By definition for every $x \in D$, there is a number n such that $\forall\, m \preccurlyeq n\ (x(m) \in \Delta^1_1(x \restriction_{\prec m}))$. As the relation between x and n here is clearly Π^1_1, Theorem 2.4.5 (Kreisel Selection) yields a Δ^1_1 map $\nu : D \to \mathbb{N}$ such that $x(m) \in \Delta^1_1(x \restriction_{\prec m})$ holds whenever $x \in D$ and $m \preccurlyeq \nu(x)$. Define, for each $x \in D$, $f(x) \in (2^{\mathbb{N}})^{\mathbb{N}}$ as follows: $f(x) \restriction_{\preccurlyeq \nu(x)} = x \restriction_{\preccurlyeq \nu(x)}$, but $f(x)(j) = \varnothing$ for all $j < \nu(x)$. Note that $x \;\mathsf{E}_1\; f(x)$ for every $x \in D$.

The other important thing is that $\mathtt{ran}\, f \subseteq Z = \{x \in (2^{\mathbb{N}})^{\mathbb{N}} : \mathtt{dep}\, x = 0\}$, where Z is a Π^1_1 set, hence, there is a Δ^1_1 set Y with $\mathtt{ran}\, f \subseteq Y \subseteq Z$. In particular f reduces $\mathsf{E}_1 \restriction D$ to $\mathsf{E}_1 \restriction Y$. We observe that $\mathsf{E}_1 \restriction Y$ is a countable equivalence relation: all E_1-classes in Y, and even in Z, a bigger set, are at most countable. Thus, $\mathsf{E}_1 \restriction Y$ is hyperfinite by Theorem 8.1.1. □

11.5. Case 2

In continuation of the proof of Theorem 11.3.1, we prove the following theorem. It shows that the Case 2 assumption implies the "or" case of Theorem 11.3.1.

THEOREM 11.5.1. *Suppose that* $D \subseteq (2^{\mathbb{N}})^{\mathbb{N}}$ *is a* Σ^1_1 *set containing a point* $x \in D$ *with* $\mathtt{dep}\, x = \infty$, *that is, a point in* $\mathbf{H}$. *Then* $\mathsf{E}_1 \sqsubseteq_{\mathrm{C}} \mathsf{E}_1 \restriction D$.

PROOF. We apply a splitting construction, developed in [**Kan99**] for the study of "ill"-founded SACKS iterations, to get a closed set $X \subseteq D$ and a Borel reduction of E_1 to $\mathsf{E}_1 \restriction X$. The construction involves a map $\varphi : \mathbb{N} \to \mathbb{N}$ assuming infinitely many values and each of its values infinitely many times (but $\mathtt{ran}\, \varphi$ may be a proper subset of $\mathbb{N}$), and, for each $u \in 2^{<\omega}$, a non-empty Σ^1_1 subset $X_u \subseteq D$, which satisfies a quite long list of properties. First of all, if φ is already defined at least on $[0, n)$ and $u \neq v \in 2^n$, then let $\nu_\varphi[u, v] = \mathtt{min}_{\preccurlyeq}\{\varphi(k) : k < n \wedge u(k) \neq v(k)\}$. (Note that the minimum is taken in the sense of $\preccurlyeq$, hence, it is $\mathtt{max}$ in the sense of $\le$, the usual order). Separately, put $\nu_\varphi[u, u] = -1$ for any u.

Now we present the list of requirements.

1°. If $\varphi(n) \notin \{\varphi(k) : k < n\}$, then $\varphi(n) \prec \varphi(k)$ for every $k < n$.

2°. Every X_u is a non-empty Σ^1_1 subset of $D \cap \mathbf{H}$.

3°. If $u \in 2^n$, $x \in X_u$, and $k < n$, then $\varphi(k) \in \nabla(x)$.

4°. If $u, v \in 2^n$, then $X_u \restriction_{\prec \nu_\varphi[u,v]} = X_v \restriction_{\prec \nu_\varphi[u,v]}$.

5°. If $u, v \in 2^n$, then $X_u \restriction_{\preccurlyeq \nu_\varphi[u,v]} \cap X_v \restriction_{\preccurlyeq \nu_\varphi[u,v]} = \varnothing$.

6°. $X_{u^\frown i} \subseteq X_u$ for all $u \in 2^{<\omega}$ and $i = 0, 1$.

7°. For every n and $u \in 2^n$: $X_u \in \mathscr{D}_n$, where $\{\mathscr{D}_n\}_{n\in\mathbb{N}}$ is a fixed enumeration of all open dense sets $D \subseteq \mathbb{P}[(2^{\mathbb{N}})^{\mathbb{N}}]$ coded in a fixed model $\mathfrak{M}$ of **ZFC**$^-$ chosen as in Definition 10.1.3. Here $\mathbb{P}[(2^{\mathbb{N}})^{\mathbb{N}}]$ consists of all non-empty Σ^1_1 subsets of $(2^{\mathbb{N}})^{\mathbb{N}}$, the Gandy–Harrington forcing for $(2^{\mathbb{N}})^{\mathbb{N}}$. This is similar to **2°** in Section 10.2 or **2°** in Section 10.6, and therefore as a consequence, if $a \in 2^{\mathbb{N}}$, then $\bigcap_n X_{a\restriction n}$ is a singleton in $(2^{\mathbb{N}})^{\mathbb{N}}$.

Let us demonstrate how such a system of sets and a function φ accomplish Case 2. Put $X = \bigcap_n \bigcup_{u\in 2^n} X_u$. According to **7°**, for every $a \in 2^{\mathbb{N}}$ the intersection $\bigcap_n X_{a\restriction n}$ contains a single point, let it be $f(a)$. Then obviously $X = \{f(a) : a \in 2^{\mathbb{N}}\}$ and f is a continuous bijection $2^{\mathbb{N}} \xrightarrow{\text{onto}} X$ (see Section 10.2).

Put $J = \mathtt{ran}\,\varphi = \{j_m : m \in \mathbb{N}\}$, in the $<$-increasing order; $J \subseteq \mathbb{N}$ is infinite. Let $n \in \mathbb{N}$. Then $\varphi(n) = j_m$ for some (unique) m : we put $\psi(n) = m$. Thus $\psi : \mathbb{N} \xrightarrow{\text{onto}} \mathbb{N}$ and the preimage $\psi^{-1}(m) = \varphi^{-1}(j_m)$ is an infinite subset of $\mathbb{N}$ for every m. This allows us to define a parallel system of sets $Y_u \subseteq (2^{\mathbb{N}})^{\mathbb{N}}$, $u \in 2^{<\omega}$, as follows. Put $Y_\Lambda = (2^{\mathbb{N}})^{\mathbb{N}}$. Suppose that Y_u has been defined, $u \in 2^n$. Put $j = \varphi(n) = j_{\psi(n)}$. Let K be the number of all indices $k < n$ still satisfying $\varphi(k) = j$, perhaps $K = 0$. Put $Y_{u^\frown i} = \{x \in Y_u : x(j)(K) = i\}$ for $i = 0, 1$.

Each of Y_u is clearly a basic clopen set in $(2^{\mathbb{N}})^{\mathbb{N}}$, and one easily verifies that conditions **1°**–**6°**, except for **3°**, are satisfied for the sets Y_u (instead of X_u) and the map ψ (instead of φ). In particular, for every $a \in 2^{\mathbb{N}}$, $\bigcap_n Y_{a\restriction n} = \{g(a)\}$ is a singleton, and the map g is continuous and 1-to-1. (We can, of course, define g explicitly: $g(a)(m)(l) = a(n)$, where $n \in \mathbb{N}$ is chosen so that $\psi(n) = m$ and there are exactly l numbers $k < n$ with $\psi(k) = m$.) Note finally that $\{g(a) : a \in 2^{\mathbb{N}}\} = (2^{\mathbb{N}})^{\mathbb{N}}$ since by definition $Y_{u^\frown 1} \cup Y_{u^\frown 0} = Y_u$ for all u.

We conclude that the map $\vartheta(x) = f(g^{-1}(x))$ is a continuous bijection (hence, in this case, a homeomorphism by compactness) $(2^{\mathbb{N}})^{\mathbb{N}} \xrightarrow{\text{onto}} X$.

LEMMA 11.5.2. *The map ϑ is a continuous isomorphism of* E_1 *onto* $\mathsf{E}_1 \restriction X$*, and hence ϑ witnesses* $\mathsf{E}_1 \sqsubseteq_{\mathrm{C}} \mathsf{E}_1 \restriction X$.

PROOF. It suffices to check that the map ϑ satisfies the following requirement: for each $y, y' \in (2^{\mathbb{N}})^{\mathbb{N}}$ and m,

$$(*) \qquad y \restriction_{\preccurlyeq m} = y' \restriction_{\preccurlyeq m} \quad \text{iff} \quad \vartheta(y) \restriction_{\preccurlyeq j_m} = \vartheta(y') \restriction_{\preccurlyeq j_m} .$$

Indeed, suppose that $y = g(a)$ and $x = f(a) = \vartheta(y)$, and similarly $y' = g(a')$ and $x' = f(a') = \vartheta(y')$, where $a, a' \in 2^{\mathbb{N}}$. Suppose that $y \restriction_{\preccurlyeq m} = y' \restriction_{\preccurlyeq m}$. According to **5°** for ψ and the sets Y_u, we then have $m \prec \nu_\psi[a \restriction n, a' \restriction n]$ for every n. It follows, by the definition of ψ, that $j_m \prec \nu_\varphi[a \restriction n, a' \restriction n]$ for every n, hence, $X_{a\restriction n} \restriction_{\preccurlyeq j_m} = X_{a'\restriction n} \restriction_{\preccurlyeq j_m}$ for every n by **4°**. Assuming now that Polish metrics on all spaces $(2^{\mathbb{N}})^{\preccurlyeq j}$ are chosen so that $\mathtt{diam}\, Z \geq \mathtt{diam}\,(Z \restriction_{\preccurlyeq j})$ for all $Z \subseteq 2^{\mathbb{N}}$ and j, we easily obtain that $x \restriction_{\preccurlyeq j_m} = x' \restriction_{\preccurlyeq j_m}$, *i.e.*, the right-hand side of $(*)$. The inverse implication in $(*)$ is proved similarly. □ (Lemma)

□ (Theorems 11.5.1, 11.3.1, 5.7.3 modulo the construction **1°**–**7°**)

11.6. The construction

We continue the proof of Theorem 11.5.1. Recall that $D \subseteq (2^{\mathbb{N}})^{\mathbb{N}}$ is a Σ^1_1 set such that $D \cap \mathbf{H} \neq \varnothing$, i.e., $\mathtt{dep}\, x = \infty$ for some $x \in D$. As "$\mathtt{dep}\, x = +\infty$" is a Σ^1_1 relation, $D' = \{x \in D : \mathtt{dep}\, x = +\infty\}$ is still a Σ^1_1 set. Put $X_\Lambda = D'$.

Now suppose that the sets $X_u \subseteq D$ with $u \in 2^n$ have been defined and satisfy the applicable part of conditions **1°**–**7°** of Section 11.5. Let us extend the construction to the next level.

Step 1. Our first task is to choose $\varphi(n)$. Let $\{j_1 < \cdots < j_m\} = \{\varphi(k) : k < n\}$. For every $1 \le p \le m$, let N_p be the number of all $k < n$ with $\varphi(k) = j_p$.

Case 1a. If some numbers N_p are $< m$, then choose $\varphi(n)$ among j_p with the least N_p, and among them the least one.

Case 1b: $N_p \ge m$ (then actually $N_p = m$) for all $p \le m$. It follows from our assumptions, in particular **4°**, that $X_u \restriction_{\prec j_m} = X_v \restriction_{\prec j_m}$ for all $u, v \in 2^n$. Let $Y = X_u \restriction_{\prec j_m}$ for any such u. Take an arbitrary $y \in Y$. Then $\nabla(y)$ is infinite, hence, there is some $j \in \nabla(y)$ with $j \prec j_m$. Put $\varphi(n) = j$.

We have something else to do in this case. Let $X'_u = \{x \in X_u : j \in \nabla(x)\}$ for any $u \in 2^n$. Then we easily have $X'_u = \{x \in X_u : x \restriction_{\prec j_m} \in Y'\}$, where $Y' = \{y \in Y : j \in \nabla(y)\}$ is a non-empty Σ^1_1 set, so that the sets $X'_u \subseteq X_u$ are non-empty Σ^1_1. Moreover, as j_m is the $\preccurlyeq$-least in $\{\varphi(k) : k < n\}$, we can easily show that the system of sets X'_u still satisfies **4°**. This allows us to assume, without any loss of generality, that, in Case 1b, $X'_u = X_u$ for all u, or, in other words, that every $x \in \bigcup_{u \in 2^n} X_u$ satisfies $j = \varphi(n) \in \nabla(x)$. (This is true in Case 1a, of course, because then $\varphi(n) = \varphi(k)$ for some $k < n$.)

Note that this manner of choosing $\varphi(n)$ implies **1°** and also implies that φ takes infinitely many values and takes each of its values infinitely many times.

The continuation of the construction requires the following

LEMMA 11.6.1. *If $u_0 \in 2^n$ and $X' \subseteq X_{u_0}$ is a non-empty Σ^1_1 set, then there is a system of Σ^1_1 sets $\varnothing \neq X'_u \subseteq X_u$ with $X'_{u_0} = X'$ that still satisfies* **4°**.

PROOF. For any $u \in 2^n$, let $X'_u = \{x \in X_u : x \restriction_{\prec n(u)} \in X' \restriction_{\prec n(u)}\}$, where $n(u) = \nu_\varphi[u, u_0]$. In particular, this gives $X'_{u_0} = X'$, because $\nu_\varphi[u_0, u_0] = -1$. The sets X'_u are as required, via a routine verification. □ (Lemma)

Step 2. First of all put $j = \varphi(n)$ and $Y_u = X_u \restriction_{\prec j}$. (All Y_u are equal to Y in Case 1b, but the argument pretends to make no difference between 1a and 1b). Take any $u_1 \in 2^n$. By the construction all elements $x \in X_{u_1}$ satisfy $j \in \nabla(x)$, so that $x(j) \notin \Delta^1_1(x \restriction_{\prec j})$. Since X_{u_1} is a Σ^1_1 set, it follows that the set $\{x'(j) : x' \in X_{u_1} \wedge x' \restriction_{\prec j} = x \restriction_{\prec j}\}$ is not a singleton, in fact it is uncountable. Therefore, there is a number l_{u_1} having the property that the Σ^1_1 set

$$Y'_{u_1} = \{y \in Y_{u_1} : \exists x, x' \in X_{u_1} \, (x' \restriction_{\prec j} = x \restriction_{\prec j} = y \wedge l_{u_1} \in x(j) \wedge l_{u_1} \notin x'(j))\}$$

is non-empty. We now put $X' = \{x \in X_{u_1} : x \restriction_{\prec j} \in Y'_{u_1}\}$ and define Σ^1_1 sets $\varnothing \neq X'_u \subseteq X_u$ as in the lemma, in particular, $X'_{u_1} = X'$, $X'_{u_1} \restriction_{\prec j} = Y'_{u_1}$, still **4°** is satisfied, and in addition

$$(1) \quad \forall y \in X'_{u_1} \restriction_{\prec j} \, \exists x, x' \in X'_{u_1} \, (x' \restriction_{\prec j} = x \restriction_{\prec j} = y \wedge l_{u_1} \in x(j) \wedge l_{u_1} \notin x'(j)).$$

Now take a different sequence $u_2 \in 2^n$. Let $\nu = \nu_\varphi[u_1, u_2]$. If $j \prec \nu$, then $X_{u_1} \restriction_{\prec j} = X_{u_2} \restriction_{\prec j}$, so that we already have, for $l_{u_2} = l_{u_1}$, that

$$(2) \quad \forall y \in X'_{u_2} \restriction_{\prec j} \exists x, x' \in X'_{u_2} \left(x' \restriction_{\prec j} = x \restriction_{\prec j} = y \wedge l_{u_2} \in x(j) \wedge l_{u_2} \notin x'(j) \right),$$

and we can pass to some $u_3 \in 2^n$. Suppose that $\nu \preccurlyeq j$. Now things are somewhat nastier. As above, there is a number l_{u_2} such that

$$Y'_{u_2} = \{ y \in Y_{u_2} : \exists x, x' \in X_{u_2} \left(x' \restriction_{\prec j} = x \restriction_{\prec j} = y \wedge l_{u_2} \in x(j) \wedge l_{u_2} \notin x'(j) \right) \}$$

is a non-empty Σ^1_1 set, thus, we can define $X'' = \{x \in X_{u_1} : x \restriction_{\prec j} \in Y'_{u_1}\}$ and maintain the construction of Lemma 11.6.1, getting non-empty Σ^1_1 sets $X''_u \subseteq X'_u$ still satisfying **4°** and $X''_{u_2} = X''$. Therefore, we still have (2) for the set X''_{u_2}.

Yet it is most important in this case that condition (1) is preserved, that is, it still holds for the set X''_{u_1} instead of X'_{u_1}! Why is this? Indeed, according to the construction in the proof of Lemma 11.6.1, we have

$$X''_{u_1} = \{ x \in X'_{u_1} : x \restriction_{\prec \nu} \in X'' \restriction_{\prec \nu} \}.$$

Thus, although, in principle, the set X''_{u_1} is smaller than X'_{u_1}, the equality

$$\{ x \in X''_{u_1} : x \restriction_{\prec j} = y \} = \{ x \in X'_{u_1} : x \restriction_{\prec j} = y \},$$

still holds for every $y \in X''_{u_1} \restriction_{\prec j}$ simply because now we assume that $\nu \preccurlyeq j$. This implies that (1) still holds.

Iterating this construction so that each $u \in 2^n$ is eventually encountered, we obtain, in the end, a system of non-empty Σ^1_1 sets, let us call them "new" X_u. But they are subsets of the "original" X_u, still satisfying **4°**, still satisfying that $\varphi(n) \in \nabla(x)$ for each $x \in \bigcap_{u \in 2^n} X_u$, and, in addition, for every $u \in 2^n$ there is a number l_u such that $j \prec \nu_\varphi[u, v] \Longrightarrow l_u = l_v$ and

$$(3) \quad \forall y \in X_u \restriction_{\prec j} \exists x, x' \in X_u \left(x' \restriction_{\prec j} = x \restriction_{\prec j} = y \wedge l_u \in x(j) \wedge l_u \notin x'(j) \right).$$

Step 3. We define the $(n+1)$-th level of sets by $X_{u^\frown 0} = \{x \in X_u : l_u \in x(j)\}$ and $X_{u^\frown 1} = \{x \in X_u : l_u \notin x(j)\}$ for all $u \in 2^n$, where still $j = \varphi(n)$. It follows from (3) that all these Σ^1_1 sets are non-empty.

LEMMA 11.6.2. *The system of sets X_s, $s \in 2^{n+1}$, just defined satisfies* **4°**, **5°**.

PROOF. Let $s = u^\frown i$ and $t = v^\frown i'$ belong to 2^{n+1}, so that $u, v \in 2^n$ and $i, i' \in \{0, 1\}$. Let $\nu = \nu_\varphi[u, v]$ and $\nu' = \nu_\varphi[s, t]$.

Case 3a: $\nu \preccurlyeq j = \varphi(n)$. Then easily $\nu = \nu'$, so that **5°** immediately follows from **5°** at level n for X_u and X_v. As for **4°**, we have $X_s \restriction_{\prec \nu} = X_u \restriction_{\prec \nu}$ (because by definition $X_s \restriction_{\prec j} = X_u \restriction_{\prec j}$), and similarly $X_t \restriction_{\prec \nu} = X_v \restriction_{\prec \nu}$. Therefore, $X_t \restriction_{\prec \nu'} = X_s \restriction_{\prec \nu'}$ since $X_u \restriction_{\prec \nu} = X_v \restriction_{\prec \nu}$ by **4°** at level n.

Case 3b: $j \prec \nu$ and $i = i'$. Then still $\nu = \nu'$, thus we have **5°**. Further, $X_u \restriction_{\prec \nu} = X_v \restriction_{\prec \nu}$ by **4°** at level n, hence, $X_u \restriction_{\preccurlyeq j} = X_v \restriction_{\preccurlyeq j}$, hence, $l_u = l_v$ (see above). Now, assuming that, say, $i = i' = 1$ and $l_u = l_v = l$, we conclude that

$$X_s \restriction_{\prec \nu'} = \{ y \in X_u \restriction_{\prec \nu} : l \in y(j) \} = \{ y \in X_v \restriction_{\prec \nu} : l \in y(j) \} = X_t \restriction_{\prec \nu'}.$$

Case 3c: $j \prec \nu$ and $i \neq i'$, say, $i = 0$ and $i' = 1$. Now $\nu' = j$. Yet by definition $X_s \restriction_{\prec j} = X_u \restriction_{\prec j}$ and $X_t \restriction_{\prec j} = X_v \restriction_{\prec j}$, so it remains to apply **4°** for level n. As for **5°**, note that by definition $l \notin x(j)$ holds for every $x \in X_s = X_{u^\frown 0}$ while $l \in x(j)$ holds for every $x \in X_t = X_{v^\frown 1}$, where $l = l_u = l_v$. □ (Lemma)

Step 4. In addition to **4°** and **5°**, we already have **1°**, **2°**, **3°**, **6°** at level $n+1$. To achieve the remaining property **7°**, it suffices to consider, one by one, all elements $s \in 2^{n+1}$, finding, at each such a substep, a non-empty Σ^1_1 subset of X_s which is consistent with the requirements of **7°**, and then reducing all other sets X_t by Lemma 11.6.1 at level $n+1$. □ (Construction)

□ (Theorems 11.5.1, 11.3.1, 5.7.3)

11.7. A forcing notion associated with E_1

ZAPLETAL [**Zap04**] defined a forcing notion $\mathbb{P}_{\mathsf{E}_0\mathsf{E}_1}$ that consists of all $\boldsymbol{\Sigma}^1_1$ sets $X \subseteq (2^{\mathbb{N}})^{\mathbb{N}}$ such that $\mathsf{E}_1 \restriction X \sim_{\mathrm{B}} \mathsf{E}_1$. It follows from Theorem 11.3.1 that the associated ideal $\mathscr{I}_{\mathsf{E}_0\mathsf{E}_1}$ consists of all $\boldsymbol{\Sigma}^1_1$ sets $X \subseteq (2^{\mathbb{N}})^{\mathbb{N}}$ satisfying $\mathsf{E}_1 \restriction X \leq_{\mathrm{B}} \mathsf{E}_0$. Thus, for a Borel set $X \subseteq (2^{\mathbb{N}})^{\mathbb{N}}$ to be in $\mathscr{I}_{\mathsf{E}_0\mathsf{E}_1}$, it is necessary and sufficient that $\mathsf{E}_1 \restriction X$ is hyperfinite. It follows that $\mathscr{I}_{\mathsf{E}_0\mathsf{E}_1}$ is a σ-ideal by Corollary 7.3.2.

EXERCISE 11.7.1. Prove that a Σ^1_1 set $X \subseteq (2^{\mathbb{N}})^{\mathbb{N}}$ belongs to $\mathbb{P}_{\mathsf{E}_0\mathsf{E}_1}$ iff there is a point $x \in X$ such that $\mathtt{dep}\, x = \infty$, and accordingly, a Δ^1_1 set $X \subseteq (2^{\mathbb{N}})^{\mathbb{N}}$ belongs to $\mathscr{I}_{\mathsf{E}_0\mathsf{E}_1}$ iff $\mathtt{dep}\, x$ is finite for all $x \in X$, □

EXERCISE 11.7.2. Prove, using the splitting construction applied in Case 2 of the proof of Theorem 11.3.1, that if $X \in \mathbb{P}_{\mathsf{E}_0\mathsf{E}_1}$, then there exists a closed set $Y \subseteq X$, $Y \in \mathbb{P}_{\mathsf{E}_0\mathsf{E}_1}$. *Hint*: If X is Σ^1_1, then employ the splitting construction beginning with $X_\Lambda = X$. □

The forcing $\mathbb{P}_{\mathsf{E}_0\mathsf{E}_1}$ can be compared with the inverse-$\mathbb{N}$-iterated SACKS forcing studied from different standpoints in [**Kan99, Zap04**]. The latter, denoted by $\mathfrak{P}$, consists of all Borel (equivalently, all closed, which gives a dense subforcing of the "all Borel" version) sets $X \subseteq (2^{\mathbb{N}})^{\mathbb{N}}$ such that the set

$$X_n[x] = \{y(n) : y \in X \wedge y \restriction_{\prec n} = x \restriction_{\prec n}\}$$

is uncountable for every $x \in X$ and n. Clearly, $\mathbb{P} \subseteq \mathbb{P}_{\mathsf{E}_0\mathsf{E}_1}$.

EXERCISE 11.7.3. Prove that for a Δ^1_1 set $X \subseteq (2^{\mathbb{N}})^{\mathbb{N}}$ to belong to $\mathfrak{P}$, it is necessary and sufficient that for every $x \in X$ and n the set $X_n(x)$ contains an element not in $\Delta^1_1(x \restriction_{\prec n})$. □

Despite some similarities between $\mathbb{P}_{\mathsf{E}_0\mathsf{E}_1}$ and the above mentioned SACKS iteration, there is a crucial difference.

THEOREM 11.7.4 (ZAPLETAL [**Zap04**]). *$\mathbb{P}_{\mathsf{E}_0\mathsf{E}_1}$ forces the existence of a countable set $X \subseteq 2^{\mathbb{N}}$ of "old" elements which cannot be covered by any "old" set Y countable in the ground universe.*

Recall that "old" means a set in the ground universe $\mathbf{V}$.

PROOF (sketch). Let us fix a reasonable coding system for continuous functions of type $f : (2^{\mathbb{N}})^{[n,\infty)} \to 2^{\mathbb{N}}$, where $n \in \mathbb{N}$, and let f^n_p be a continuous function $f : (2^{\mathbb{N}})^{[n,\infty)} \to 2^{\mathbb{N}}$ coded by $p \in 2^{\mathbb{N}}$. It is assumed that for every n the ternary relation $f^n_p(z) = a$ (where $a, p \in 2^{\mathbb{N}}$ and $z \in (2^{\mathbb{N}})^{[n,\infty)}$) is Borel.

It is clear that $\mathbb{P}_{\mathsf{E}_0\mathsf{E}_1}$ forces a generic point $\mathbf{x} \in (2^{\mathbb{N}})^{\mathbb{N}}$. We are going to prove that for every n there exists an "old" code $p_n \in \mathbf{V} \cap 2^{\mathbb{N}}$, such that $\mathbf{x}(n) = f^n_{p_n}(\mathbf{x} \restriction_{\prec n})$ in the extension. The second part is to show that there is no countable in $\mathbf{V}$ set $Y \in \mathbf{V}$, $Y \subseteq 2^{\mathbb{N}}$ such that $\{p_n : n \in \mathbb{N}\} = X \subseteq Y$. The proof consists of several steps related rather to the structure of sets in $\mathbb{P}_{\mathsf{E}_0\mathsf{E}_1}$.

Step 1. If $X \in \mathbb{P}_{\mathsf{E}_0\mathsf{E}_1}$ is closed and a $\mathbf{G}_\delta$ set $G \subseteq X$ is dense in X, then there is a set $Y \in \mathbb{P}_{\mathsf{E}_0\mathsf{E}_1}$, $Y \subseteq G$. To get Y, assume that X is Δ^1_1 and employ the splitting construction of Section 11.5 beginning with $X_\Lambda = X$ and satisfying the extra requirement that $X_u \in U_n$ whenever $u \in 2^{n+1}$, where U_n are dense relatively open subsets of X such that $G = \bigcap_n U_n$.

Step 2. If $X \in \mathbb{P}_{\mathsf{E}_0\mathsf{E}_1}$ is closed and $P \subseteq X \times 2^{\mathbb{N}}$ is a Borel set such that $D = \operatorname{dom} P$ is comeager in X, then there is a set $Y \in \mathbb{P}_{\mathsf{E}_0\mathsf{E}_1}$, $Y \subseteq D$ and a continuous map $f : Y \to 2^{\mathbb{N}}$ such that $\langle x, f(x)\rangle \in P$ for every $x \in Y$. Indeed it is known that there is a still comeager set $G \subseteq D$ and a continuous map $f' : D' \to 2^{\mathbb{N}}$ such that $\langle x, f'(x)\rangle \in P$ for every $x \in G$. It remains to apply the result of Step 1.

Step 3. If $X \in \mathbb{P}_{\mathsf{E}_0\mathsf{E}_1}$ is closed and $n \in \mathbb{N}$, then there exist a set $Y \in \mathbb{P}_{\mathsf{E}_0\mathsf{E}_1}$, $Y \subseteq X$, and a continuous map $f : Y \restriction_{\prec n} \to 2^{\mathbb{N}}$ such that $Y_n[y] = \{f(y \restriction_{\prec n})\}$ for every $y \in Y$. To prove this, apply the result of Step 2 for the set $X \restriction_{\prec n}$ in the role of X and the set $\{\langle x \restriction_{\prec n}, x(n)\rangle : x \in X\}$ in the role of P.

We conclude that $\mathbb{P}_{\mathsf{E}_0\mathsf{E}_1}$ forces the following: "for every n there is $p \in \mathbf{V} \cap 2^{\mathbb{N}}$ such that $\mathbf{x}(n) = f^n_p(\mathbf{x} \restriction_{\prec n})$".

Step 4. Finally fix a countable set $\{q_k : k \in \mathbb{N}\} \subseteq 2^{\mathbb{N}}$ (in the ground universe $\mathbf{V}$) and show that $\mathbb{P}_{\mathsf{E}_0\mathsf{E}_1}$ forces the following: "there is n such that $\mathbf{x}(n) \notin \{f^n_{q_k}(\mathbf{x} \restriction_{\prec n}) : k \in \mathbb{N}\}$". Suppose toward the contrary that some $X \in \mathbb{P}_{\mathsf{E}_0\mathsf{E}_1}$ forces the opposite, that is, it forces $\forall n\, \exists k\, (\mathbf{x}(n) = f^n_{q_k}(\mathbf{x} \restriction_{\prec n}))$. Consider the Borel set $Z = \{x \in (2^{\mathbb{N}})^{\mathbb{N}} : \forall n\, \exists k\, (x(n) = f^n_{q_k}(x \restriction_{\prec n}))\}$. For every $x \in Z$, the E_1-class $[x]_{\mathsf{E}_1} \cap Z$ is at most countable, thus $\mathsf{E}_1 \restriction Z$ is a Borel countable equivalence relation, hence it is hyperfinite. Thus the relation $\mathsf{E}_1 \restriction Y$, where $Y = X \smallsetminus Z$, is not hyperfinite since otherwise $\mathsf{E}_1 \restriction X$ would be hyperfinite by Corollary 7.3.2, contrary to the choice of X. It follows that $\mathsf{E}_1 \restriction Y \sim_{\mathrm{B}} \mathsf{E}_1$ by Theorem 5.7.3.

Thus $Y \in \mathbb{P}_{\mathsf{E}_0\mathsf{E}_1}$.

By definition $Y = \bigcup_n Y_n$, where $Y_n = \{y \in Y : \forall k\, (x(n) \neq f^n_{q_k}(x \restriction_{\prec n}))\}$. Thus, still by Corollary 7.3.2, at least one of the sets Y_n belongs to $\mathbb{P}_{\mathsf{E}_0\mathsf{E}_1}$. Fix such an index n. Then $x(n) \neq f^n_{q_k}(x \restriction_{\prec n})$ for all k and all $x \in Y_n$. It easily follows that Y_n forces $\mathbf{x}(n) \neq f^n_{q_k}(\mathbf{x} \restriction_{\prec n})$ for all k. However $Y_n \subseteq Y \subseteq X$, a contradiction with the choice of X. □

11.8. Above E_1

Our main goal here is to prove the following important theorem of KECHRIS and LOUVEAU [**KL97**]. It shows that E_1 is not Borel reducible to orbit equivalence relations of Polish group actions.

THEOREM 11.8.1. *Suppose that $\mathbb{G}$ is a Polish group and $\mathbb{X}$ is a Borel $\mathbb{G}$-space. Then E_1 is* not *Borel reducible to $\mathsf{E}^{\mathbb{X}}_{\mathbb{G}}$.*

Our proof is based on HJORTH's proof in [**Hjo00b**, Chapter 8], but the following lemma [**KL97**] provides us with an essential simplification. Recall that $\sqsubseteq_{\mathrm{C}}$ means the existence of a continuous embedding, that is, a continuous 1-to-1 reduction.

LEMMA 11.8.2. *Suppose that $\mathsf{E}_1 \leq_{\mathrm{B}} \mathsf{F}$, where F is a Σ^1_1 equivalence relation on a Polish space $\mathbb{Y}$. Then $\mathsf{E}_1 \sqsubseteq_{\mathrm{C}} \mathsf{F}$.*

PROOF (Lemma). Let, as above, $\preccurlyeq$ be the inverted order on $\mathbb{N}$, *i.e.*, $m \preccurlyeq n$ iff $n \le m$. Let $\mathfrak{P}$ be the collection of all sets $P \subseteq (2^{\mathbb{N}})^{\mathbb{N}}$ such that there is a continuous 1-to-1 map $\eta : (2^{\mathbb{N}})^{\mathbb{N}} \xrightarrow{\text{onto}} P$ satisfying

$$x \restriction_{\preccurlyeq n} = y \restriction_{\preccurlyeq n} \iff \eta(x) \restriction_{\preccurlyeq n} = \eta(y) \restriction_{\preccurlyeq n}$$

for all n and $x, y \in (2^{\mathbb{N}})^{\mathbb{N}}$, where $x \restriction_{\preccurlyeq n} = \{x(i)\}_{i \preccurlyeq n}$ for every $x \in (2^{\mathbb{N}})^{\mathbb{N}}$. Clearly, any such a map is a continuous embedding of E_1 into itself.

This set $\mathfrak{P}$ can be used as a forcing notion to extend the universe by a sequence of reals x_i so that each x_n is SACKS-generic over $\{x_i\}_{i \prec n}$. This is an example of iterated SACKS extensions with an ill-founded "skeleton", defined in [**Kan99**]. (See [**Zap04, KS05**] on more recent developments on ill-iterated forcing.) Here, the "skeleton" is $\mathbb{N}$ with the inverted order $\preccurlyeq$. It was shown in [**Kan99**] that Borel maps admit the following *canonization scheme* on sets in $\mathfrak{P}$: if $\mathbb{Y}$ is Polish, $P' \in \mathfrak{P}$, and $\vartheta : P' \to \mathbb{Y}$ is a Borel map then there is a set $P \in \mathfrak{P}$, $P \subseteq P'$, on which ϑ is continuous, and either a constant or, for some n, $1-1$ on $P \restriction_{\preccurlyeq n}$, that is,

$$(*) \qquad \text{for all } x, y \in P : \quad x \restriction_{\preccurlyeq n} = y \restriction_{\preccurlyeq n} \iff \vartheta(x) = \vartheta(y)\,.$$

We apply this to a Borel map $\vartheta : (2^{\mathbb{N}})^{\mathbb{N}} \to \mathbb{Y}$ which reduces E_1 to F. Starting with $P' = (2^{\mathbb{N}})^{\mathbb{N}}$, find a set $P \in \mathfrak{P}$ as indicated. Since ϑ cannot be a constant on P (indeed, every $P \in \mathfrak{P}$ contains many pairwise E_1-inequivalent elements), we have $*$ for some n. In other words, there is a 1-to-1 continuous map $f : P \restriction_{\preccurlyeq n} \to \mathbb{Y}$ (where $P \restriction_{\preccurlyeq n} = \{x \restriction_{\preccurlyeq n} : x \in P\}$) such that $\vartheta(x) = f(x \restriction_{\preccurlyeq n})$ for all $x \in P$.

Now, suppose that $x \in (2^{\mathbb{N}})^{\mathbb{N}}$. Define $\zeta(x) = z \in (2^{\mathbb{N}})^{\mathbb{N}}$ so that $z(i) = \mathbb{N} \times \{0\}$ for $i < n$ and $z(n+i) = x(i)$ for all i. Finally, put $\vartheta'(x) = f(\eta(\zeta(x)) \restriction_{\preccurlyeq n})$ for all $x \in (2^{\mathbb{N}})^{\mathbb{N}}$, where $\eta : (2^{\mathbb{N}})^{\mathbb{N}} \xrightarrow{\text{onto}} P$ witnesses $P \in \mathfrak{P}$. This map ϑ' is a continuous embedding of E_1 in F. □ (Lemma)

PROOF (Theorem 11.8.1). Toward the contrary, let $\vartheta : (2^{\mathbb{N}})^{\mathbb{N}} \to \mathbb{X}$ be a Borel reduction of E_1 to E. We can assume, by Lemma 11.8.2, that ϑ is continuous and 1-to-1. We are going to define a sequence of points $x_n \in (2^{\mathbb{N}})^{\mathbb{N}}$, elements $g_n \in \mathbb{G}$, and natural numbers ℓ_n satisfying

(i) $x_0 \in (2^{\mathbb{N}})^{\mathbb{N}}$ is the total 0, that is, $x_0(i)(k) = 0$ for all i, k;

(ii) $x_n \restriction_{\preccurlyeq n} = x_{n+1} \restriction_{\preccurlyeq n}$, that is, $x_n(i) = x_{n+1}(i)$ whenever $i \ge n$;

(iii) $\ell_0 < \ell_1 < \ell_2 < \cdots$;

(iv) if $i < n$, then $x_n(i) \restriction \ell_n = x_{n+1}(i) \restriction \ell_n$;

(v) if $i < n$, then $x_n(i)(\ell_n) \ne x_{n+1}(i)(\ell_n)$;

(vi) $\vartheta(x_n) = g_n \cdot \vartheta(x_0)$;

(vii) $d_{\mathbb{G}}(g_n, g_{n+1}) < 2^{-n}$, where $d_{\mathbb{G}}$ is the Polish distance on $\mathbb{G}$.

If this is done, then obviously $x = \mathtt{lim}_{n\to\infty} x_n \in (2^{\mathbb{N}})^{\mathbb{N}}$ exists, and $x \mathrel{\mathsf{E}_1} x_0$ is not true, basically $x(i) \ne x_0(i)$ for all i by (iv), (v). Moreover $g = \mathtt{lim}_{n\to\infty} g_n \in \mathbb{G}$ exists by (vii). In addition $\vartheta(x) = g \cdot \vartheta(x_0)$ by (vi) and because both ϑ and the group action are continuous. Therefore, $\vartheta(x) \mathrel{\mathsf{F}} \vartheta(x_0)$, a contradiction.

Thus it remains to carry out the construction. Suppose that x_n, g_n, ℓ_{n-1} are defined. (For $n = 0$, x_0 is defined by (i), and we can take $g_0 = 1_{\mathbb{G}}$.) First of all pick an open nbhd W of $1_{\mathbb{G}}$ such that $d_{\mathbb{G}}(g_n, gg_n) < 2^{-n}$ for all $g \in W$. By the continuity, there is a number $\ell > \ell_{n-1}$ (or just $\ell > 0$ in the case $n = 0$) such that

$$\left(x_n \restriction_{\preccurlyeq n} = y \restriction_{\preccurlyeq n} \wedge \forall\, i < n\, (x_n(i) \restriction \ell = y(i) \restriction \ell)\right) \Longrightarrow \vartheta(y) \in W \cdot \vartheta(x_n)$$

for all $y \in (2^{\mathbb{N}})^{\mathbb{N}}$. Put $\ell_n = \ell$. Obviously, there exists $x_{n+1} \in (2^{\mathbb{N}})^{\mathbb{N}}$ satisfying (ii), (iv), (v). Then $\vartheta(x_{n+1}) \in W \cdot \vartheta(x_n)$ by the choice of ℓ. In other words, there exists $g \in W$ such that $\vartheta(x_{n+1}) = g \cdot \vartheta(x_n) = gg_n \cdot \vartheta(x_0)$. Thus $g_{n+1} = gg_n$ satisfies (vi) (for $n+1$). Finally, (vii) holds by the choice of W. □

Let us mention several notable corollaries of Theorem 11.8.1.

COROLLARY 11.8.3. *If $\mathscr{I}_1$ is a Σ^1_1 ideal on $\mathbb{N}$, then $\mathscr{I}_1 \le_{\mathrm{RB}} \mathscr{I}$ iff $\mathsf{E}_1 \le_{\mathrm{B}} \mathsf{E}_{\mathscr{I}}$.*

PROOF. The non-trivial direction is $\Longleftarrow$. Suppose that $\mathscr{I}_1 \le_{\mathrm{RB}} \mathscr{I}$ fails. Then $\mathscr{I}$ is polishable by Theorem 3.5.1; therefore, $\mathsf{E}_{\mathscr{I}}$ is induced by a Polish action of the Δ-group of $\mathscr{I}$ on $\mathscr{P}(\mathbb{N})$. It remains to apply Theorem 11.8.1. □

Thus condition $\mathsf{E}_1 \le_{\mathrm{B}} \mathsf{E}_{\mathscr{I}}$ can be added to the equivalence of the five conditions of Theorem 3.5.1.

COROLLARY 11.8.4. *Suppose that $\mathscr{J}$ is a Σ^1_1 P-ideal. Then every ideal $\mathscr{I}$ satisfying $\mathscr{I} \le_{\mathrm{B}} \mathscr{J}$ is a Σ^1_1 P-ideal, too.* □

We are now able to give another proof of a result already obtained by different method (see Corollary 11.2.2).

COROLLARY 11.8.5. $\mathsf{E}_\infty \not\le_{\mathrm{B}} \mathsf{E}_1$.

PROOF. If $\mathsf{E}_\infty \le_{\mathrm{B}} \mathsf{E}_1$, then by Theorem 11.3.1 "either" $\mathsf{E}_\infty \le_{\mathrm{B}} \mathsf{E}_0$ "or" $\mathsf{E}_\infty \sim_{\mathrm{B}} \mathsf{E}_1$. The "either" case contradicts Theorem 7.5.1. The "or" case contradicts Theorem 11.8.1 since E_∞ is induced by a Polish action of F_2, a countable hence Polish group. □

CHAPTER 12

Actions of the infinite symmetric group

This chapter is connected with the next one (on turbulence). We concentrate on a principal result in this area, due to HJORTH, that turbulent equivalence relations are not reducible to those induced by actions of S_∞, the infinite symmetric group. The focal point here will be equivalence relations induced by S_∞, especially isomorphism relations of countable structures of various types. In particular, we shall prove the following:

I. The LOPEZ-ESCOBAR theorem: every invariant Borel set of countable models is the truth domain of a certain formula of the infinitary language $\mathscr{L}_{\omega_1\omega}$.

II. Every orbit equivalence relation of a Polish action of a closed subgroup of S_∞ is classifiable by countable structures; that is, it is Borel reducible to the isomorphism of a certain kind of countable structure.

III. Every equivalence relation, classifiable by countable structures, is Borel reducible to the isomorphism of countable ordered graphs.

IV. Every *Borel* equivalence relation, classifiable by countable structures, is Borel reducible to one of equivalence relations T_ξ (see Section 4.2).

SCOTT's analysis, involved in proof of IV, will appear only in a rather restricted and self-contained version.

12.1. Infinite symmetric group S_∞ and isomorphisms

Let S_∞ be the group of all permutations (*i.e.*, 1-to-1 maps $\mathbb{N} \xrightarrow{\text{onto}} \mathbb{N}$) of $\mathbb{N}$, with the superposition as the group operation. Clearly, S_∞ is a $\mathbf{G}_\delta$ subset of $\mathbb{N}^\mathbb{N}$, hence, a Polish group.

EXERCISE 12.1.1. Prove that a compatible complete metric on S_∞ can be defined by $D(x,y) = d(x,y) + d(x^{-1}, y^{-1})$, where d is the ordinary complete metric of $\mathbb{N}^\mathbb{N}$, *i.e.*, $d(x,y) = 2^{-m-1}$, where m is the least such that $x(m) \neq y(m)$. □

Yet S_∞ admits no compatible left-invariant complete metric [**BK96**, 1.5].

For instance, isomorphism relations of various kinds of countable structures are orbit equivalence relations induced by S_∞. Indeed, suppose that $\mathscr{L} = \{R_i\}_{i\in I}$ is a countable relational language, *i.e.*, $0 < \mathtt{card}\, I \le \aleph_0$ and each R_i is an m_i-ary relational symbol. We put[1]

$$\mathtt{Mod}_\mathscr{L} = \prod_{i\in I} \mathscr{P}(\mathbb{N}^{m_i}),$$

[1] $X_\mathscr{L}$ is often used to denote $\mathtt{Mod}_\mathscr{L}$.

the space of (coded) $\mathscr{L}$-*structures* on $\mathbb{N}$. The *logic action* $j_{\mathscr{L}}$ of S_∞ on $\mathtt{Mod}_{\mathscr{L}}$ is defined as follows: if $x = \{x_i\}_{i\in I} \in \mathtt{Mod}_{\mathscr{L}}$ and $g \in S_\infty$, then

$$y = j_{\mathscr{L}}(g,x) = g\cdot x = \{y_i\}_{i\in I} \in \mathtt{Mod}_{\mathscr{L}},$$

where

$$\langle k_1,\ldots,k_{m_i}\rangle \in x_i \iff \langle g(k_1),\ldots,g(k_{m_i})\rangle \in y_i$$

for all $i \in I$ and $\langle k_1,\ldots,k_{m_i}\rangle \in \mathbb{N}^{m_i}$.

EXERCISE 12.1.2. Prove that $\langle \mathtt{Mod}_{\mathscr{L}}\,; j_{\mathscr{L}}\rangle$ is a Polish S_∞-space and $j_{\mathscr{L}}$-orbits in $\mathtt{Mod}_{\mathscr{L}}$ are exactly the isomorphism classes of $\mathscr{L}$-structures on $\mathbb{N}$. □

This is a reason to denote the associated equivalence relation $\mathsf{E}_{j_{\mathscr{L}}}^{\mathtt{Mod}_{\mathscr{L}}}$ as $\cong_{\mathscr{L}}$.

If G is a subgroup of S_∞, then $j_{\mathscr{L}}$ restricted to G is still an action of G on $\mathtt{Mod}_{\mathscr{L}}$ whose orbit equivalence relation will be denoted by $\cong_{\mathscr{L}}^G$, that is, for x, $y \in \mathtt{Mod}_{\mathscr{L}}$: $x \cong_{\mathscr{L}}^G y$ iff $\exists g \in G\,(g\cdot x = y)$.

12.2. Borel invariant sets

A set $M \subseteq \mathtt{Mod}_{\mathscr{L}}$ is *invariant* if $[M]_{\cong_{\mathscr{L}}} = M$. There is a convenient characterization of *Borel* invariant sets, in terms of $\mathscr{L}_{\omega_1\omega}$, an infinitary extension of $\mathscr{L} = \{R_i\}_{i\in I}$ by countable conjunctions and disjunctions. To be more exact, the language $\mathscr{L}_{\omega_1\omega}$ is defined as follows:

a) every $R_i(v_0,\ldots,v_{m_i-1})$ is an atomic formula of $\mathscr{L}_{\omega_1\omega}$ (all v_i being variables over $\mathbb{N}$ and m_i is the arity of R_i), and propositional connectives and quantifiers $\exists,\forall$ can be applied as usual;

b) if φ_i, $i \in \mathbb{N}$, are formulas of $\mathscr{L}_{\omega_1\omega}$ whose free variables are among a finite list $v_0,\ldots,v_n$, then $\bigvee_i \varphi_i$ and $\bigwedge_i \varphi_i$ are formulas of $\mathscr{L}_{\omega_1\omega}$.

If $x \in \mathtt{Mod}_{\mathscr{L}}$, $\varphi(v_1,\ldots,v_n)$ is a formula of $\mathscr{L}_{\omega_1\omega}$, and $i_1,\ldots,i_n \in \mathbb{N}$, then the relation of *satisfiability*

$$x \models \varphi(i_1,\ldots,i_n)$$

means that $\varphi(i_1,\ldots,i_n)$ is satisfied on x, in the usual sense that involves transfinite induction on the "depth" of φ; see [**Kec95**, 16.C].

THEOREM 12.2.1 (LOPEZ-ESCOBAR, see, e.g., 16.8 in [**Kec95**]). *A set $M \subseteq \mathtt{Mod}_{\mathscr{L}}$ is invariant and Borel iff $M = \{x \in \mathtt{Mod}_{\mathscr{L}} : x \models \varphi\}$ for a closed formula φ of $\mathscr{L}_{\omega_1\omega}$.*

PROOF. To prove the non-trivial direction, let $M \subseteq \mathtt{Mod}_{\mathscr{L}}$ be invariant and Borel. Put $B_s = \{g \in S_\infty : s \subset g\}$ whenever $s \in \mathbb{N}^{<\omega}$ is injective (i.e., $s_i \neq s_j$ for $i \neq j$). This is a clopen subset of S_∞ (in the Polish topology of S_∞ inherited from $\mathbb{N}^{\mathbb{N}}$). If $A \subseteq S_\infty$, then let $s \Vdash A(\mathring{g})$ mean that the set $B_s \cap A$ is comeager in B_s, i.e., $g \in A$ holds for a.a. $g \in S_\infty$ with $s \subset g$. The proof consists of two parts:

(1) $M = \{x \in \mathtt{Mod}_{\mathscr{L}} : \Lambda \Vdash \mathring{g}\cdot x \in M\}$, where $g\cdot x = j_{\mathscr{L}}(g,x)$ (see above);

(2) For every Borel set $M \subseteq \mathtt{Mod}_{\mathscr{L}}$ and every $n \in \mathbb{N}$ there is a formula $\varphi_M^n(v_0,\ldots,v_{n-1})$ of $\mathscr{L}_{\omega_1\omega}$ such that for every $x \in \mathtt{Mod}_{\mathscr{L}}$ and every injective $s \in \mathbb{N}^n$ we have $x \models \varphi_M^n(s_0,\ldots,s_{n-1})$ iff $s \Vdash \mathring{g}^{-1}\cdot x \in M$.

Claim (1) is clear: since M is invariant, we have $g\cdot x \in M$ for all $x \in M$ and $g \in S_\infty$. On the other hand, if $g\cdot x \in M$ for at least one $g \in S_\infty$, then $x \in M$.

To prove (2) we argue by induction on the Borel complexity of M. Suppose, for the sake of simplicity, that $\mathscr{L}$ contains a single binary predicate, say, $R(\cdot,\cdot)$. Then $\mathtt{Mod}_{\mathscr{L}} = \mathscr{P}(\mathbb{N}^2)$. If $M = \{x \subseteq \mathbb{N}^2 : \langle k, l\rangle \notin x\}$ for some $k, l \in \mathbb{N}$, then take

$$\forall u_0 \dots \forall u_m \left(\bigwedge_{i<j\le m}(u_i \ne u_j) \wedge \bigwedge_{i<n}(u_i = v_i) \implies \neg R(u_k, u_l)\right),$$

where $m = \max\{l, k, n\}$, as $\varphi^n_M(v_0, \dots, v_{n-1})$. Further, take the formula

$$\bigwedge_{k\ge n} \forall u_0 \dots \forall u_{k-1} \bigvee_{m\ge k} \exists w_0 \dots \exists w_{m-1} \Big(\bigwedge_{i<j<k}(u_i \ne u_j) \wedge \bigwedge_{i<n}(u_i = v_i)$$
$$\implies \bigwedge_{i<j<m}(w_i \ne w_j) \wedge \bigwedge_{i<k}(w_i = v_i) \wedge \varphi^m_M(w_0, \dots, w_{m-1})\Big)$$

as $\varphi^n_{\neg M}(v_0, \dots, v_{n-1})$. Finally, if $M = \bigcap_j M_j$, then we take $\bigwedge_j \varphi^n_{M_j}(v_0, \dots, v_{n-1})$ as $\varphi^n_M(v_0, \dots, v_{n-1})$. □ (Theorem 12.2.1)

12.3. Equivalence relations classifiable by countable structures

The classifiability by countable structures means that we can associate, in a Borel way, a countable $\mathscr{L}$-structure, say, $\vartheta(x)$ with any point $x \in \mathbb{X} = \mathsf{dom}\,\mathsf{E}$ so that $x \mathrel{\mathsf{E}} y$ iff $\vartheta(x)$ and $\vartheta(y)$ are isomorphic. More exactly,

DEFINITION 12.3.1 (HJORTH [**Hjo00b**, 2.38]). An equivalence relation E is *classifiable by countable structures* if there is a countable relational language $\mathscr{L}$ such that $\mathsf{E} \le_{\mathrm{B}} \cong_{\mathscr{L}}$. □

REMARK 12.3.2. For instance, the equivalence relations T_ξ, in particular, T_2, belong to this class by Proposition 12.5.1 below. It follows that E_3 and all countable Borel equivalence relations (see Figure 1 on page 68) are classifiable by countable structures by the results in Section 6.1.

However, it will be proved in the next chapter that E_2 and the density-0 equivalence relation Z_0 are not classifiable by countable structures. □

Every equivalence relation E classifiable by countable structures is Σ^1_1, of course, and many of them are Borel. On the other hand, we have the following theorem of BECKER and KECHRIS [**BK96**]:

THEOREM 12.3.3. *Every orbit equivalence relation of a Polish action of a closed subgroup of S_∞ is classifiable by countable structures.*

Thus all orbit equivalence relations of Polish actions of S_∞ and its closed subgroups are Borel reducible to a very special kind of action of S_∞.

PROOF (sketch). First show that every orbit equivalence relation of a Polish action of S_∞ itself is classifiable by countable structures. HJORTH's simplified argument [**Hjo00b**, 6.19] is as follows. Let $\mathbb{X}$ be a Polish S_∞-space with basis $\{U_l\}_{l\in\mathbb{N}}$, and let $\mathscr{L}$ be the language with relations R_{lk} where each R_{lk} has arity k. If $x \in \mathbb{X}$, then define $\vartheta(x) \in \mathtt{Mod}_{\mathscr{L}}$ by stipulation that $\vartheta(x) \models R_{lk}(s_0, \dots, s_{k-1})$ iff 1) $s_i \ne s_j$ whenever $i < j < k$, and 2) $\forall g \in B_s\, (g^{-1}\cdot x \in U_l)$, where $B_s = \{g \in S_\infty : s \subset g\}$ and $s = \langle s_0, \dots, s_{k-1}\rangle \in \mathbb{N}^k$. Then ϑ reduces $\mathsf{E}^{\mathbb{X}}_{S_\infty}$ to $\cong_{\mathscr{L}}$.

To accomplish the proof of the theorem, it remains to apply the following result (an immediate corollary of Theorem 2.3.5b in [**BK96**]):

PROPOSITION 12.3.4. *If $\mathbb{G}$ is a closed subgroup of a Polish group $\mathbb{H}$ and $\mathbb{X}$ is a Polish $\mathbb{G}$-space, then there is a Polish $\mathbb{H}$-space $\mathbb{Y}$ such that $\mathsf{E}^{\mathbb{X}}_{\mathbb{G}} \le_{\mathrm{B}} \mathsf{E}^{\mathbb{Y}}_{\mathbb{H}}$.*

PROOF. HJORTH [**Hjo00b**, 7.18] outlines a proof as follows. Let $Y = \mathbb{X} \times \mathbb{H}$; define $\langle x, h\rangle \approx \langle x', h'\rangle$ if $x' = g\cdot x$ and $h' = gh$ for some $g \in \mathbb{G}$, and consider the quotient space $\mathbb{Y} = Y/\approx$ with the topology induced by the Polish topology of Y via the surjection $\langle x, h\rangle \mapsto [\langle x, h\rangle]_\approx$, on which $\mathbb{H}$ acts by $h'\cdot[\langle x, h\rangle]_\approx = [\langle x, hh'^{-1}\rangle]_\approx$. Obviously, $\mathsf{E}^{\mathbb{X}}_{\mathbb{G}} \le_{\mathrm{B}} \mathsf{E}^{\mathbb{Y}}_{\mathbb{H}}$ via the map $x \mapsto [\langle x, 1\rangle]_\approx$, hence, it remains to prove that $\mathbb{Y}$ is a Polish $\mathbb{H}$-space, which is not really elementary; we refer the reader to [**Hjo00b**, 7.18] or [**BK96**, 2.3.5b]. □ (Proposition)

To bypass Proposition 12.3.4 in the proof of Theorem 12.3.3, we can use a characterization of all closed subgroups of S_∞. Let $\mathscr{L}$ be a language as above, and $x \in \mathtt{Mod}_{\mathscr{L}}$. Define $\mathtt{Aut}_x = \{g \in S_\infty : g\cdot x = x\}$: the group of all automorphisms of x.

PROPOSITION 12.3.5 (see [**BK96**, 1.5]). *$G \subseteq S_\infty$ is a closed subgroup of S_∞ iff there is an $\mathscr{L}$-structure $x \in \mathtt{Mod}_{\mathscr{L}}$ of a countable language $\mathscr{L}$, such that $G = \mathtt{Aut}_x$.*

PROOF. For the non-trivial direction, let G be a closed subgroup of S_∞. For any $n \ge 1$, let I_n be the set of all G-orbits in $\mathbb{N}^n$, *i.e.*, equivalence classes of the equivalence relation $s \sim t$ iff $\exists g \in G\,(t = g \circ s)$, thus, I_n is an at most countable subset of $\mathscr{P}(\mathbb{N}^n)$. Let $I = \bigcup_n I_n$, and, for any $i \in I_n$, let R_i be an n-ary relational symbol, and $\mathscr{L} = \{R_i\}_{i\in I}$. Let $x \in \mathtt{Mod}_{\mathscr{L}}$ be defined as follows: if $i \in I_n$, then $x \models R_i(k_0, \dots, k_{n-1})$ iff $\langle k_0, \dots, k_{n-1}\rangle \in i$. Then $G = \mathtt{Aut}_x$, actually, if G is not a necessarily closed subgroup, then $\mathtt{Aut}_x = \overline{G}$. □ (Proposition)

Now come back to Theorem 12.3.3. The same argument as in the beginning of the proof shows that every orbit equivalence relation of a Polish action of G, a closed subgroup of S_∞, is $\le_{\mathrm{B}} \cong^G_{\mathscr{L}}$ for an appropriate countable language $\mathscr{L}$. Yet, by Proposition 12.3.5, $G = \mathtt{Aut}_{y_0}$ where $y_0 \in \mathtt{Mod}_{\mathscr{L}'}$ and $\mathscr{L}'$ is a countable language disjoint from $\mathscr{L}$. The map $x \longmapsto \langle x, y_0\rangle$ witnesses that $\cong^G_{\mathscr{L}} \le_{\mathrm{B}} \cong_{\mathscr{L}\cup\mathscr{L}'}$.
□ (Theorem 12.3.3)

12.4. Reduction to countable graphs

It could be expected that the more complicated a language $\mathscr{L}$ is, accordingly, the more complicated isomorphism equivalence relation $\cong_{\mathscr{L}}$ it produces. However this is not the case. Let $\mathscr{G}$ be the language of (oriented binary) graphs, *i.e.*, $\mathscr{G}$ contains a single binary predicate, say $R(\cdot,\cdot)$.

THEOREM 12.4.1. *If $\mathscr{L}$ is a countable relational language, then $\cong_{\mathscr{L}} \le_{\mathrm{B}} \cong_{\mathscr{G}}$. Therefore, an equivalence relation E is classifiable by countable structures iff $\mathsf{E} \le_{\mathrm{B}} \cong_{\mathscr{G}}$. In other words, a single binary relation can code structures of any countable language.*

BECKER and KECHRIS [**BK96**, 6.1.4] outline a proof based on coding in terms of lattices, unlike the following argument, yet it may in fact involve the same idea.

PROOF. Let $\mathrm{HF}(\mathbb{N})$ be the set of all hereditarily finite sets over the set $\mathbb{N}$ considered as the set of atoms, and ε be the associated "membership". (Thus none of $n \in \mathbb{N}$ has ε-elements, $\{0,1\}$ is different from 2, *etc.*) Let $\simeq_{\mathrm{HF}(\mathbb{N})}$ be the $\mathrm{HF}(\mathbb{N})$ version of $\cong_{\mathscr{G}}$, *i.e.*, if $P, Q \subseteq \mathrm{HF}(\mathbb{N})^2$, then $P \simeq_{\mathrm{HF}(\mathbb{N})} Q$ means that there is a bijection b of $\mathrm{HF}(\mathbb{N})$ such that $Q = b\cdot P = \{\langle b(s), b(t)\rangle : \langle s, t\rangle \in P\}$. Obviously, $(\cong_{\mathscr{G}}) \sim_{\mathrm{B}} (\simeq_{\mathrm{HF}(\mathbb{N})})$, thus, we have to prove that $\cong_{\mathscr{L}} \le_{\mathrm{B}} \simeq_{\mathrm{HF}(\mathbb{N})}$ for every $\mathscr{L}$.

An action of S_∞ on $\mathrm{HF}(\mathbb{N})$ is defined as follows. If $g \in S_\infty$, then $g \circ n = g(n)$ for every $n \in \mathbb{N}$, and, by ε-induction, $g \circ \{a_1, \ldots, a_n\} = \{g \circ a_1, \ldots, g \circ a_n\}$ for all $a_1, \ldots, a_n \in \mathrm{HF}(\mathbb{N})$. Clearly, the map $a \mapsto g \circ a$ ($a \in \mathrm{HF}(\mathbb{N})$) is an ε-isomorphism of $\mathrm{HF}(\mathbb{N})$ for any fixed $g \in S_\infty$.

LEMMA 12.4.2. *Suppose that $X, Y \subseteq \mathrm{HF}(\mathbb{N})$ are ε-transitive subsets of $\mathrm{HF}(\mathbb{N})$, the sets $\mathbb{N} \smallsetminus X$ and $\mathbb{N} \smallsetminus Y$ are infinite, and $\varepsilon \restriction X \simeq_{\mathrm{HF}(\mathbb{N})} \varepsilon \restriction Y$. Then there is $f \in S_\infty$ such that $Y = f \circ X = \{f \circ s : s \in X\}$.*

PROOF. It follows from the assumption $\varepsilon \restriction X \cong_{\mathrm{HF}(\mathbb{N})} \varepsilon \restriction Y$ that there is an ε-isomorphism $\pi : X \xrightarrow{\text{onto}} Y$. Easily, $\pi \restriction (X \cap \mathbb{N})$ is a bijection of $X_0 = X \cap \mathbb{N}$ onto $Y_0 = Y \cap \mathbb{N}$; hence, there is $f \in S_\infty$ such that $f \restriction X_0 = \pi \restriction X_0$, and then we have $f \circ s = \pi(s)$ for every $s \in X$. □ (Lemma)

Coming back to the proof of Theorem 12.4.1, we first show that $\cong_{\mathscr{G}(m)} \le_{\mathrm{B}} \simeq_{\mathrm{HF}(\mathbb{N})}$ for every $m \ge 3$, where $\mathscr{G}(m)$ is the language with a single m-ary predicate. Note that $\langle i_1, \ldots, i_m \rangle \in \mathrm{HF}(\mathbb{N})$ whenever $i_1, \ldots, i_m \in \mathbb{N}$.

Put $\Theta(x) = \{\vartheta(s) : s \in x\}$ for every element $x \in \mathtt{Mod}_{\mathscr{G}(m)} = \mathscr{P}(\mathbb{N}^m)$, where $\vartheta(s) = \mathrm{TC}_\varepsilon(\{\langle 2i_1, \ldots, 2i_m \rangle\})$ for each $s = \langle i_1, \ldots, i_m \rangle \in \mathbb{N}^m$, and finally, for $X \subseteq \mathrm{HF}(\mathbb{N})$, $\mathrm{TC}_\varepsilon(X)$ is the least ε-transitive set $T \subseteq \mathrm{HF}(\mathbb{N})$ with $X \subseteq T$. It easily follows from Lemma 12.4.2 that $x \cong_{\mathscr{G}(m)} y$ iff $\varepsilon \restriction \Theta(x) \simeq_{\mathrm{HF}(\mathbb{N})} \varepsilon \restriction \Theta(y)$. This ends the proof of $\cong_{\mathscr{G}(m)} \le_{\mathrm{B}} \simeq_{\mathrm{HF}(\mathbb{N})}$.

It remains to show that $\cong_{\mathscr{L}'} \le_{\mathrm{B}} \simeq_{\mathrm{HF}(\mathbb{N})}$, where $\mathscr{L}'$ is the language with infinitely many binary predicates. In this case $\mathtt{Mod}_{\mathscr{L}'} = \mathscr{P}(\mathbb{N}^2)^{\mathbb{N}}$, so that we can assume that every $x \in \mathtt{Mod}_{\mathscr{L}'}$ has the form $x = \{x_n\}_{n \ge 1}$, with $x_n \subseteq (\mathbb{N} \smallsetminus \{0\})^2$ for all n. Let $\Theta(x) = \{s_n(k,l) : n \ge 1 \wedge \langle k, l \rangle \in x_n\}$ for every such x, where

$$s_n(k,l) = \mathrm{TC}_\varepsilon(\{\{\ldots\{\langle k,l \rangle\}\ldots\}, 0\}), \text{ with } n+2 \text{ pairs of brackets } \{\,,\}.$$

Then Θ is a continuous reduction of $\cong_{\mathscr{L}'}$ to $\simeq_{\mathrm{HF}(\mathbb{N})}$. □ (Theorem)

12.5. Reduction of Borel classifiability to T_ξ

Equivalence relations T_ξ introduced in Section 4.2 offer a perfect calibration tool for those Borel equivalence relations which admit classification by countable structures. First of all,

PROPOSITION 12.5.1. *Every equivalence relation T_ξ is classifiable by countable structures.*

The relations T_α are known in different versions, which reflect the same idea of coding sets of α-th cumulative level over $\mathbb{N}$, as, e.g., in [**HKL98**, § 1], where results similar to Proposition 12.5.1 are obtained in much more precise form.

PROOF. T_0, the equality on $\mathbb{N}$, is the orbit equivalence relation of the action of S_∞ by $g \cdot x = x$ for all g, x. The operation (o2) of Section 4.2 (countable disjoint union) easily preserves the property of being Borel reducible to an orbit equivalence relation of a continuous action of S_∞.

Now consider operation (o5) of countable power. Suppose that an equivalence relation E on a Polish space $\mathbb{X}$ is Borel reducible to F, the orbit relation of a continuous action of S_∞ on some Polish $\mathbb{Y}$. Let D be the set of all points $x = \{x_k\}_{k \in \mathbb{N}} \in \mathbb{X}^{\mathbb{N}}$ such that either $x_k \not\mathrel{\mathsf{E}} x_l$ whenever $k \ne l$, or there is m such that $x_k \mathrel{\mathsf{E}} x_l$ iff m divides $|k - l|$. Then $\mathsf{E}^+ \le_{\mathrm{B}} (\mathsf{E}^+ \restriction D)$ (via a Borel map $\vartheta : \mathbb{X}^{\mathbb{N}} \to D$

such that $x \mathsf{E}^+ \vartheta(x)$ for all x). On the other hand, obviously $(\mathsf{E}^+ \upharpoonright D) \le_{\mathrm{B}} \mathsf{F}'$, where, for $y, y' \in \mathbb{Y}^{\mathbb{N}}$, $y \mathsf{F}' y'$ means that there is $f \in S_\infty$ such that $y_k \mathsf{F} y'_{f(k)}$ for all k. Finally, F' is the orbit equivalence relation of a continuous action of $S_\infty \times {S_\infty}^{\mathbb{N}}$, which can be realized as a closed subgroup of S_∞, so it remains to apply Theorem 12.3.3. □

The next theorem shows that the relations T_ξ are cofinal in the $\le_{\mathrm{B}}$-structure of Borel equivalence relations induced by Polish actions of S_∞. The theorem does not extend to arbitrary (that is, $\mathbf{\Sigma}^1_1$) orbit equivalence relations of Polish actions of S_∞ since every equivalence relation on a Borel domain Borel reducible to a Borel equivalence relation is Borel itself. However Lemma 13.3.3 below will show that a wider class of reduction maps leads to a certain reducibility of arbitrary Polish actions of S_∞ to T_ξ.

THEOREM 12.5.2. *If E is a Borel equivalence relation classifiable by countable structures, then $\mathsf{E} \le_{\mathrm{B}} \mathsf{T}_\xi$ for some $\xi < \omega_1$.*

PROOF. The proof (a version of the proof can be found in [**Fri00**]) is based on Scott's analysis. Define, by induction on $\alpha < \omega_1$, a family of Borel equivalence relations $\equiv^\alpha$ on $\mathbb{N}^{<\omega} \times \mathscr{P}(\mathbb{N}^2)$:

- $A \equiv^\alpha_{st} B$ means $\langle s, A\rangle \equiv^\alpha \langle t, B\rangle$;

thus, all relations of the form $\equiv^\alpha_{st}$ $(s, t \in \mathbb{N}^{<\omega})$ are binary relations on $\mathscr{P}(\mathbb{N}^2)$, and among them all relations of the form $\equiv^\alpha_{ss}$ are equivalence relations. We define them by transfinite induction on α.

- $A \equiv^0_{st} B$ iff $A(s_i, s_j) \iff B(t_i, t_j)$ for all $i, j < \mathtt{lh}\, s = \mathtt{lh}\, t$;
- $A \equiv^{\alpha+1}_{st} B$ iff $\forall k\, \exists l\, (A \equiv^\alpha_{s^\frown k,\, t^\frown l} B)$ and $\forall l\, \exists k\, (A \equiv^\alpha_{s^\frown k,\, t^\frown l} B)$;
- if $\lambda < \omega_1$ is limit, then $A \equiv^\lambda_{st} B$ iff $A \equiv^\alpha_{st} B$ for all $\alpha < \lambda$.

Easily, $\equiv^\beta \subseteq \equiv^\alpha$ whenever $\alpha < \beta$.

Recall that, for $A, B \subseteq \mathbb{N}^2$, $A \cong_{\mathscr{G}} B$ means that there is $f \in S_\infty$ with $A(k, l) \iff B(f(k), f(l))$ for all k, l. Then we have $\cong_{\mathscr{G}} \subseteq \bigcap_{\alpha<\omega_1} \equiv^\alpha_{\Lambda\Lambda}$ by induction on α (in fact $=$ rather than $\subseteq$; see below), where Λ is the empty sequence. Call a set $P \subseteq \mathscr{P}(\mathbb{N}^2) \times \mathscr{P}(\mathbb{N}^2)$ *unbounded* if $P \cap \equiv^\alpha_{\Lambda\Lambda} \ne \varnothing$ for all $\alpha < \omega_1$.

LEMMA 12.5.3. *Every unbounded $\mathbf{\Sigma}^1_1$ set P contains a pair $\langle A, B\rangle \in P$ such that $A \cong_{\mathscr{G}} B$.*

It follows that $A \cong_{\mathscr{G}} B$ iff $A \equiv^\alpha_{\Lambda\Lambda} B$ for all $\alpha < \omega_1$. (For take $P = \{\langle A, B\rangle\}$.)

PROOF. Since P is $\mathbf{\Sigma}^1_1$, there is a continuous map $F : \mathbb{N}^{\mathbb{N}} \xrightarrow{\text{onto}} P$. For $u \in \mathbb{N}^{<\omega}$, let $P_u = \{F(a) : u \subset a \in \mathbb{N}^{\mathbb{N}}\}$. There is a number n_0 such that $P_{\langle n_0\rangle}$ is still unbounded. Let $k_0 = 0$. By a simple cofinality argument, there is l_0 such that $P_{\langle n_0\rangle}$ is still unbounded *over* $\langle k_0\rangle, \langle l_0\rangle$ in the sense that there is no ordinal $\alpha < \omega_1$ such that $P_{\langle i_0\rangle} \cap \equiv^\alpha_{\langle k_0\rangle\langle l_0\rangle} = \varnothing$. Following this idea, we can define infinite sequences of numbers n_m, k_m, l_m such that both $\{k_m\}_{m\in\mathbb{N}}$ and $\{l_m\}_{m\in\mathbb{N}}$ are permutations of $\mathbb{N}$ and, for each m, the set $P_{\langle n_0,\dots,n_m\rangle}$ is still unbounded over $\langle k_0, \dots, k_m\rangle, \langle l_0, \dots, l_m\rangle$ in the same sense. Note that $a = \{n_m\}_{m\in\mathbb{N}} \in \mathbb{N}$ and $F(a) = \langle A, B\rangle \in P$. (Both A, B are subsets of $\mathbb{N}^2$.)

Prove that the map $f(k_m) = l_m$ witnesses $A \cong_{\mathscr{G}} B$, i.e., $A(k_j, k_i)$ iff $B(l_j, l_i)$ for all j, i. Take $m > \mathtt{max}\{j, i\}$ big enough for the following. If $\langle A', B' \rangle \in P_{\langle i_0,\dots,i_m \rangle}$, then $A(k_j, k_i)$ iff $A'(k_j, k_i)$, and similarly $B(l_j, l_i)$ iff $B'(l_j, l_i)$. By the construction, there is a pair $\langle A', B' \rangle \in P_{\langle i_0,\dots,i_m \rangle}$ with $A' \equiv^0_{\langle k_0,\dots,k_m \rangle \langle l_0,\dots,l_m \rangle} B'$, in particular, $A'(k_j, k_i)$ iff $B'(l_j, l_i)$, as required. □ (Lemma)

COROLLARY 12.5.4 (see, e.g., FRIEDMAN [**Fri00**]). *If* E *is a Borel equivalence relation and* $\mathsf{E} \le_{\mathrm{B}} \cong_{\mathscr{G}}$, *then* $\mathsf{E} \le_{\mathrm{B}} \equiv^{\alpha}_{\mathrm{AA}}$ *for some* $\alpha < \omega_1$.

PROOF. Let ϑ be a Borel reduction of E to $\cong_{\mathscr{G}}$. Then $\{\langle \vartheta(x), \vartheta(y) \rangle : x \not\mathsf{E}\, y\}$ is a Σ^1_1 subset of $\mathscr{P}(\mathbb{N}^2) \times \mathscr{P}(\mathbb{N}^2)$ which does not intersect $\cong_{\mathscr{G}}$, hence, it is bounded by Lemma 12.5.3. Take an ordinal $\alpha < \omega_1$ which witnesses the boundedness. □

Now, if E is a Borel equivalence relation classifiable by countable structures, then $\mathsf{E} \le_{\mathrm{B}} \cong_{\mathscr{G}}$ by Theorem 12.4.1. Hence, it remains to establish the following:

PROPOSITION 12.5.5. *Every equivalence relation* $\equiv^{\alpha}$ *is Borel reducible to an equivalence relation of the form* T_ξ.

PROOF. We have $\equiv^0 \le_{\mathrm{B}} \mathsf{T}_0$ since $\equiv^0$ has countably many equivalence classes, all of which are clopen sets. To carry out the step $\alpha \mapsto \alpha + 1$, note that the map $\langle s, A \rangle \mapsto \{\langle s^\frown k, A \rangle\}_{k \in \mathbb{N}}$ is a Borel reduction of $\equiv^{\alpha+1}$ to $(\equiv^{\alpha})^{\infty}$. To carry out the limit step, let $\lambda = \{\alpha_n : n \in \mathbb{N}\}$ be a limit ordinal, and $\mathsf{R} = \bigvee_{n \in \mathbb{N}} \equiv^{\alpha_n}$, i.e., R is an equivalence relation on $\mathbb{N} \times \mathbb{N}^{<\omega} \times \mathscr{P}(\mathbb{N}^2)$ defined so that $\langle m, s, A \rangle \;\mathsf{R}\; \langle n, t, B \rangle$ iff $m = n$ and $A \equiv^{\alpha_m}_{st} B$. However, the map $\langle s, A \rangle \mapsto \{\langle m, s, A \rangle\}_{m \in \mathbb{N}}$ is a Borel reduction of $\equiv^{\lambda}$ to R^{∞}. □ (Proposition)

□ (Theorem 12.5.2)

CHAPTER 13

Turbulent group actions

This family of Polish actions is characterized by a condition of "somewhere-density" of orbits, and even of local orbits. It contains such principal equivalence relations as E_2 and $\mathbf{c}_0$. The main results of this chapter show that equivalence relations in this family are not classifiable by countable structures, and in fact do not belong to a family of Borel equivalence relations much bigger than the countably classifiable family. The following will be established. (We continue the list from the beginning of Chapter 12.)

V. Every generically (gen.) turbulent equivalence relation, Baire measurable reducible to a Polish action of S_∞, is Borel reducible to one of the equivalence relations of the form T_ξ on a comeager set. (Lemma 13.3.3; compare with Theorem 12.5.2 above.)

VI. Every gen. turbulent equivalence relation E is gen. T_ξ-ergodic for each ordinal $\xi < \omega_1$. (Lemma 13.3.4.) Therefore, E is not Borel reducible to T_ξ.

VII. Gen. turbulent equivalence relations are not classifiable by countable structures. (Theorem 13.1.2, a corollary of VI and V.)

VIII. A generalization of VII: gen. turbulent equivalence relations are not Borel reducible to equivalence relations that can be obtained from the equality $\Delta_{\mathbb{N}}$ with the help of the operations defined in Section 4.2. (Theorem 13.5.3.)

13.1. Local orbits and turbulence

Suppose that a group $\mathbb{G}$ acts on a space $\mathbb{X}$. If $G \subseteq \mathbb{G}$ and $X \subseteq \mathbb{X}$, then let

$$\mathsf{R}_G^X = \{\langle x, y\rangle \in X^2 : \exists g \in G\,(x = g \cdot y)\}$$

and let $\sim_G^X$ denote the equivalence-hull of R_G^X, that is, the $\subseteq$-least equivalence relation on X such that $x \mathrel{\mathsf{R}_G^X} y \Longrightarrow x \sim_G^X y$. In particular $\sim_{\mathbb{G}}^{\mathbb{X}} = \mathsf{E}_{\mathbb{G}}^{\mathbb{X}}$, but generally we have $\sim_G^X \subsetneqq \mathsf{E}_{\mathbb{G}}^{\mathbb{X}} \restriction X$. Finally, define, for $x \in X$, the *local orbit*

$$\mathscr{O}(x, X, G) = [x]_{\sim_G^X} = \{y \in X : x \sim_G^X y\}$$

of x. In particular, $[x]_{\mathbb{G}} = [x]_{\mathsf{E}_{\mathbb{G}}^{\mathbb{X}}} = \mathscr{O}(x, \mathbb{X}, \mathbb{G})$ is the full $\mathbb{G}$-orbit of a point $x \in \mathbb{X}$.

DEFINITION 13.1.1 (This particular version is taken from KECHRIS [**Kec02**]). Suppose that $\mathbb{X}$ is a Polish space and $\mathbb{G}$ is a Polish group acting on $\mathbb{X}$ continuously.

(t1) A point $x \in \mathbb{X}$ is *turbulent* if for every non-empty open set $X \subseteq \mathbb{X}$ containing x and every nbhd $G \subseteq \mathbb{G}$ (not necessarily a subgroup) of $1_{\mathbb{G}}$, the local orbit $\mathscr{O}(x, X, G)$ is somewhere dense (that is, not a nowhere dense set) in $\mathbb{X}$.

(t2) An orbit $[x]_{\mathbb{G}}$ is *turbulent* if x is such. (Then all $y \in [x]_{\mathbb{G}}$ are turbulent since this notion is invariant *w.r.t.* homeomorphisms.)

(t3) The action (of $\mathbb{G}$ on $\mathbb{X}$) is *generically,*[1] or *gen., turbulent* and $\mathbb{X}$ is a *gen. turbulent* Polish $\mathbb{G}$-space if the union of all dense (topologically), turbulent, and meager orbits $[x]_{\mathbb{G}}$ is comeager. □

Thus, turbulence means that orbits, and even local orbits, of the action considered behave rather chaotically in some exact sense. According to the following theorem of HJORTH [**Hjo00b**], this property is incompatible with the classifiability by countable structures.

THEOREM 13.1.2. *Suppose that* $\mathbb{G}$ *is a Polish group and* $\mathbb{X}$ *is a gen. turbulent Polish* $\mathbb{G}$*-space. Then* $\mathsf{E}_{\mathbb{G}}^{\mathbb{X}}$ *is* not *Baire measurable reducible*[2] *to a Polish action of* S_∞, *hence, it is* not *classifiable by countable structures.*

The proof given below is based on general ideas in [**Hjo00b, Kec02, Fri00**]. Yet it is designed so that only quite common tools of descriptive set theory are involved. It will also be shown (Theorem 13.5.3) that "turbulent" equivalence relations are not reducible actually to a much bigger family of relations than orbit equivalence relations of Polish actions of S_∞.

It is worth noting that turbulent equivalence relations and those classifiable by countable structures are not only disjoint, but also in some sense complementary families. This is asserted by the *fifth* dichotomy theorem of HJORTH. This complicated result (see, e.g., [**HK97**]) will not be considered in this book.

13.2. Shift actions of summable ideals are turbulent

Quite a lot of examples of turbulent actions are known (see, *e.g.*, [**Hjo00b**]). The following example will be used in the proof of some irreducibility results at the end of this chapter. Recall that every summable ideal

$$\mathscr{S}_{\{r_n\}} = \{x \subseteq \mathbb{N} : \sum_{n \in x} r_n < +\infty\}$$

(where $r_n \geq 0$ for all n) generates the equivalence relation $\mathsf{E}_{\{r_n\}} = \mathsf{E}_{\mathscr{S}_{\{r_n\}}}$ on $\mathscr{P}(\mathbb{N})$, defined so that $x \mathrel{\mathsf{E}_{\{r_n\}}} y$ iff $x \mathbin{\triangle} y \in \mathscr{S}_{\{r_n\}}$.

PROPOSITION 13.2.1. *If* $r_n > 0$, $\{r_n\} \to 0$, *and* $\sum_n r_n = +\infty$, *then the* $\triangle$*-action of* $\mathscr{S}_{\{r_n\}}$ *on* $\mathscr{P}(\mathbb{N})$ *is Polish and gen. turbulent.*

The condition $\{r_n\} \to 0$ here implies that $\mathscr{S}_{\{r_n\}}$ contains some infinite sets. The condition $\sum_n r_n = +\infty$ means that $\mathscr{S}_{\{r_n\}}$ does not contain co-infinite sets.

PROOF. That $\langle \mathscr{S}_{\{r_n\}} ; \triangle \rangle$ is a Polish group with the distance $d_{\{r_n\}}(a, b) = \varphi_{\{r_n\}}(a \mathbin{\triangle} b)$, where $\varphi_{\{r_n\}}(x) = \sum_{n \in x} r_n$, and its $\triangle$-action on $\mathscr{P}(\mathbb{N})$ is Polish as well; see Lemma 3.6.1 and Example 4.4.1. To prove the turbulence, consider any $x \in \mathscr{P}(\mathbb{N})$. That $[x]_{\mathscr{S}_{\{r_n\}}} = \mathscr{S}_{\{r_n\}} \mathbin{\triangle} x$ is dense and meager is an easy exercise. Thus,

[1] In this research direction, "generically", or, in our abbreviation "gen.", (property) intends to mean that (property) holds on a comeager domain.

[2] Reducible via a Baire measurable function. This is weaker than the Borel reducibility.

it suffices to check that x is turbulent. Consider an open nbhd $X = \{y \in \mathscr{P}(\mathbb{N}) : y \cap [0,k) = u\}$ of x, where $k \in \mathbb{N}$ and $u = x \cap [0,k)$, and a $d_{\{r_n\}}$-nbhd $G = \{g \in \mathscr{S}_{\{r_n\}} : \varphi_{r_n}(g) < \varepsilon\}$ of $\varnothing$ (the neutral element), where $\varepsilon > 0$. Prove that the local orbit $\mathscr{O}(x, X, G)$ is somewhere dense in X.

Let $l \geq k$ be large enough for $r_n < \varepsilon$ to hold for all $n \geq l$. Prove that the orbit $\mathscr{O}(x, X, G)$ is dense in $Y = \{y \in \mathscr{P}(\mathbb{N}) : y \cap [0,l) = v\}$, where $v = x \cap [0,l)$. Consider an open set $Z = \{z \in Y : z \cap [l,j) = w\}$, where $j \geq l$, $w \subseteq [l,j)$. Let z be the only point of Z satisfying $z \cap [j,+\infty) = x \cap [j,+\infty)$. Thus, $x \,\triangle\, z = \{l_1, \ldots, l_m\} \subseteq [l,j)$. Note that every element of the form $g_i = \{l_i\}$ belongs to G by the choice of l since $l_i \geq l$. Moreover, $x_i = g_i \,\triangle\, g_{i-1} \,\triangle \cdots \triangle\, g_1 \,\triangle\, x = \{l_1, \ldots, l_i\} \,\triangle\, x$ belongs to X for each $i = 1, \ldots, m$. On the other hand $x_m = z$. It follows that $z \in \mathscr{O}(x, X, G)$, as required. □

A suitable modification of this argument can be used to prove the turbulence of the $\triangle$-action of some other ideals, including the density ideal $\mathscr{Z}_0$, but as far as some irreducibility results are concerned, the turbulence of summable ideals will suffice!

13.3. Ergodicity

The non-reducibility in Theorem 13.1.2 will be established in a special stronger form. Let E, F be equivalence relations on Polish spaces $\mathbb{X}, \mathbb{Y}$, respectively. A map $\vartheta : \mathbb{X} \to \mathbb{Y}$ is

- $(\mathsf{E} \to \mathsf{F})$-invariant if $x \mathrel{\mathsf{E}} y \Longrightarrow \vartheta(x) \mathrel{\mathsf{F}} \vartheta(y)$ for all $x, y \in \mathbb{X}$;
- *gen.* $(\mathsf{E} \to \mathsf{F})$-invariant if the implication $x \mathrel{\mathsf{E}} y \Longrightarrow \vartheta(x) \mathrel{\mathsf{F}} \vartheta(y)$ holds for all x, y in a comeager subset of $\mathbb{X}$;[3]
- *gen. reduction of* E *to* F if the equivalence $x \mathrel{\mathsf{E}} y \Longleftrightarrow \vartheta(x) \mathrel{\mathsf{F}} \vartheta(y)$ holds for all x, y in a comeager subset of $\mathbb{X}$;
- *gen.* F-constant if $\vartheta(x) \mathrel{\mathsf{F}} \vartheta(y)$ for all x, y in a comeager subset of $\mathbb{X}$.

Finally, following HJORTH and KECHRIS, say that E is *gen.* F-*ergodic* if every Borel gen. $(\mathsf{E} \to \mathsf{F})$-invariant map is gen. F-constant.

The ergodicity preserves $\leq_{\mathrm{B}}$ in the sense of the next lemma.

LEMMA 13.3.1. *If* $\mathsf{E}, \mathsf{F}, \mathsf{F}'$ *are Borel equivalence relations,* E *is gen.* F-*ergodic, and* $\mathsf{F}' \leq_{\mathrm{B}} \mathsf{F}$, *then* E *is gen.* F'-*ergodic as well.*

PROOF. Let ϑ be a Borel reduction of F' to F. Given a Borel gen. $(\mathsf{E} \to \mathsf{F}')$-invariant map f, the map $f'(x) = \vartheta(f(x))$ is obviously gen. $(\mathsf{E} \to \mathsf{F})$-invariant, hence it is a gen. F-constant — then it is easily a gen. F'-constant, too. □

The following lemma shows that ergodicity implies irreducibility.

LEMMA 13.3.2. *If an equivalence relation* E *is gen.* F-*ergodic and does not have comeager equivalence classes, then* E *does not admit a Borel gen. reduction to* F. *In addition,* E *does not admit a Baire measurable reduction to* F.

PROOF. Suppose toward the contrary that a Borel map $\vartheta : \mathbb{X} \to \mathbb{Y}$ (where $\mathbb{X}$, $\mathbb{Y}$ are the domains of E, F, respectively) is a gen. reduction of E to F, that is, ϑ is a true reduction on a comeager set $C \subseteq \mathbb{X}$. Then ϑ is a gen. F-constant by the

[3] Recall that "gen." means "generic" or "generically".

ergodicity, that is, there exists a comeager set $C' \subseteq \mathbb{X}$ such that $\vartheta(x) \mathsf{F} \vartheta(x')$ for all $x, x' \in C'$. The set $D = C \cap C'$ is comeager as well, hence there exist $x, x' \in D$ such that $x \not\mathsf{E} x'$. Then $\vartheta(x) \not\mathsf{F} \vartheta(x')$ holds since ϑ is a reduction on C. On the other hand, we know that $\vartheta(x) \mathsf{F} \vartheta(x')$, a contradiction.

The additional result follows because it is known that every Baire measurable map is continuous on a comeager set. □

The proof of Theorem 13.1.2 consists of the next two lemmas.[4]

LEMMA 13.3.3. *If $\mathbb{G}$ is a Polish group, $\mathbb{X}$ is a gen. turbulent Polish $\mathbb{G}$-space, and $\mathsf{E}_{\mathbb{G}}^{\mathbb{X}}$ is Baire measurable reducible to a Polish action of S_∞, then $\mathsf{E}_{\mathbb{G}}^{\mathbb{X}}$ admits a Borel gen. reduction to an equivalence relation of the form T_ξ.*

Saying it differently, every equivalence relation, Baire measurable reducible to a Polish action of S_∞, is Borel reducible to one of T_ξ on a comeager set. Note that every equivalence relation, Borel reducible (in proper sense) to one of T_ξ, is Borel itself. Yet this cannot be applied to $\mathsf{E}_{\mathbb{G}}^{\mathbb{X}}$ in the lemma, since only a generic (on a comeager set) reduction is claimed.

LEMMA 13.3.4. *Every equivalence relation induced by a gen. turbulent Polish action of a Polish group is gen. T_ξ-ergodic for all ξ.*

PROOF OF THEOREM 13.1.2 FROM LEMMAS 13.3.3 AND 13.3.4. If $\mathsf{E}_{\mathbb{G}}^{\mathbb{X}}$ is Baire measurable reducible to a Polish action of S_∞, then $\mathsf{E}_{\mathbb{G}}^{\mathbb{X}}$ also is Borel gen. reducible to one of T_ξ by Lemma 13.3.3. On the other hand, $\mathsf{E}_{\mathbb{G}}^{\mathbb{X}}$ is gen. T_ξ-ergodic by Lemma 13.3.4. Thus, $\mathsf{E}_{\mathbb{G}}^{\mathbb{X}}$ has a comeager equivalence class by Lemma 13.3.2. But this contradicts the assumption of gen. turbulence.

□ (Theorem 13.1.2 from Lemmas 13.3.3 and 13.3.4)

The proof of the lemmas follows below in this chapter.

13.4. "Generic" reduction to T_ξ

In this Section we *prove Lemma* 13.3.3.

Suppose that $\mathbb{G}$ is a Polish group and $\mathbb{X}$ is a gen. turbulent Polish $\mathbb{G}$-space. In particular, the set W_0 of all points $x \in \mathbb{X}$ that belong to dense turbulent orbits $[x]_G$ is comeager in $\mathbb{X}$. It follows that there exists a dense $\mathbf{G}_\delta$ set $W \subseteq W_0$.

Assume further that the orbit equivalence relation $\mathsf{E} = \mathsf{E}_{\mathbb{G}}^{\mathbb{X}}$ is Baire measurable reducible to a Polish action of S_∞. As the latter is Borel reducible to the isomorphism $\cong_{\mathscr{G}}$ of binary relations on $\mathbb{N}$ according to Theorems 12.3.3 and 12.4.1, E itself admits a Baire measurable reduction $\rho : \mathbb{X} \to \mathscr{P}(\mathbb{N}^2)$ to $\cong_{\mathscr{G}}$. The remainder of the argument borrows elements of notation from the proof of Theorem 12.5.2.

There is a dense $\mathbf{G}_\delta$ set $D_0 \subseteq \mathbb{X}$ such that the restricted map $\vartheta = \rho \upharpoonright D_0$ is continuous on D_0. By definition, we have

$$x \mathsf{E} y \Longrightarrow \vartheta(x) \cong_{\mathscr{G}} \vartheta(y) \quad \text{and} \quad x \not\mathsf{E} y \Longrightarrow \vartheta(x) \not\cong_{\mathscr{G}} \vartheta(y)$$

for all $x, y \in D_0$. We are mostly interested in the second implication, and the aim is to find a dense $\mathbf{G}_\delta$ set $D \subseteq D_0$ such that, for some $\alpha < \omega_1$:

[4] There are slightly different ways to the same goal. HJORTH [**Hjo00b**, 3.18] proves outright and with different technique, that gen. turbulent equivalence relations are gen. ergodic *w.r.t.* any Polish action of S_∞. KECHRIS [**Kec02**] proves that 1) gen. T_2-ergodic equivalence relations are gen. ergodic *w.r.t.* any Polish action of S_∞, and 2) turbulent ones are gen. T_2-ergodic.

13.4.1. *The implication* $x \not\mathsf{E}\, y \Longrightarrow \vartheta(x) \not\equiv^{\alpha}_{\Lambda\Lambda} \vartheta(y)$ *holds for all* $x, y \in D$.

Recall that $A \cong_{\mathscr{G}} B$ iff $\forall\, \alpha < \omega_1\, A \equiv^{\alpha}_{\Lambda\Lambda} B$. (See the remark after Lemma 12.5.3.) It follows that $x \mathsf{E}\, y \Longrightarrow \vartheta(x) \equiv^{\alpha}_{\Lambda\Lambda} \vartheta(y)$ holds for all $x, y \in D_0$. Thus 13.4.1 implies that $\mathsf{E} \restriction D$ is Borel reducible to $\equiv^{\alpha}_{\Lambda\Lambda}$. Now, to accomplish the proof of Lemma 13.3.3, apply Proposition 12.5.5.

To find an ordinal α and a dense $\mathbf{G}_\delta$ set D satisfying 13.4.1, we make use of Cohen forcing. See Appendix A.1, A.2, A.4 on relevant definitions and results.

DEFINITION 13.4.2. Under the *w.l.o.g.* assumption that the Polish spase $\mathbb{X}$ and the Polish group $\mathbb{G}$ belong to the set $\mathrm{H}_{\mathfrak{c}^+}$ (see Definition A.1.2), there exists a countable transitive model $\mathfrak{M}$ of $\mathbf{ZFC}^-$ containing codes (in the sense of Appendix A.2) of $\mathbb{X}$, $\mathbb{G}$, the action (as a continuous map $\mathbb{G} \times \mathbb{X} \to \mathbb{X}$), of the $\mathbf{G}_\delta$ sets D_0 and W and of the continuous restricted map $\vartheta = \rho \restriction D_0$.

Such a model $\mathfrak{M}$ is fixed below in this section. □

The notion of a point of $\mathbb{X}$ or an element of $\mathbb{G}$ *Cohen-generic over* $\mathfrak{M}$ makes sense (see Appendix A.4). In addition, by Corollary A.4.4, the set D of all Cohen-generic, over $\mathfrak{M}$, points of $\mathbb{X}$ is a dense $\mathbf{G}_\delta$ subset of $\mathbb{X}$ satisfying $D \subseteq D_0$. We are going to prove that D satisfies 13.4.1 for an appropriate ordinal $\alpha < \omega_1$.

Suppose that $x, y \in D$, and $\langle x, y\rangle$ is a Cohen-generic pair over $\mathfrak{M}$. If $x \mathsf{E}^{\mathbb{X}}_{\mathbb{G}}\, y$ is false, then $\vartheta(x) \not\cong_{\mathscr{G}} \vartheta(y)$. Moreover, this fact holds in the extended model $\mathfrak{M}[x, y]$ by the Mostowski absoluteness. This allows us to find, arguing in $\mathfrak{M}[x, y]$ (which is still a model of $\mathbf{ZFC}^-$), an ordinal $\alpha \in \mathtt{Ord}^{\mathfrak{M}} = \mathtt{Ord}^{\mathfrak{M}[x,y]}$ such that $\vartheta(x) \not\equiv^{\alpha}_{\Lambda\Lambda} \vartheta(y)$. Moreover, since the Cohen forcing satisfies CCC, there is an ordinal $\alpha \in \mathfrak{M}$ such that $\vartheta(x) \not\equiv^{\alpha}_{\Lambda\Lambda} \vartheta(y)$ holds for *every* Cohen-generic, over $\mathfrak{M}$, pair $\langle x, y\rangle \in D^2$ such that $x \mathsf{E}^{\mathbb{X}}_{\mathbb{G}}\, y$ is false. It remains to show that this also holds provided $x, y \in D$ (are generic separately, but) do not necessarily form a pair Cohen-generic over $\mathfrak{M}$.

Now we prove

LEMMA 13.4.3. *If* $\mathfrak{N}$ *is a countable transitive model of* $\mathbf{ZFC}^-$ *with* $\mathfrak{M} \subseteq \mathfrak{N}$, *a point* $x \in \mathbb{X} \cap \mathfrak{N}$ *is Cohen-generic over* $\mathfrak{M}$, *and an element* $g \in \mathbb{G}$ *is Cohen-generic over* $\mathfrak{N}$, *then* $x' = g \cdot x$ *is Cohen-generic over* $\mathfrak{N}$.

PROOF. It follows from the genericity that x belongs to the set W introduced in the beginning of Section 13.4. Thus the $\mathbb{G}$-orbit $\{g' \cdot x : g' \in G\}$ is turbulent, in particular it is dense in $\mathbb{X}$.

Now consider any dense open set $X \subseteq \mathbb{X}$ coded in $\mathfrak{N}$. The set $H = \{g' \in \mathbb{G} : g' \cdot x \in X\}$ is also open and coded in $\mathfrak{N}$. Moreover, H is dense in $\mathbb{G}$. (Indeed otherwise there is an open non-empty set $G \subseteq \mathbb{G}$ such that the partial orbit $G \cdot x = \{g \cdot x : g \in G\}$ is nowhere dense. This leads to a contradiction with the turbulence of x.) We conclude that $g \in H$, and further $g \cdot x \in X$, as required. □

To make use of the lemma, let $\mathfrak{N}$ be a countable transitive model of $\mathbf{ZFC}^-$ containing a given pair of points $x, y \in D$ and all sets in $\mathfrak{M}$. Note that $\mathfrak{N}$ may contain more ordinals than $\mathfrak{M}$ does since the pair $\langle x, y\rangle$ is not assumed to be generic over $\mathfrak{M}$.

Fix an element $g \in \mathbb{G}$ Cohen-generic over $\mathfrak{N}$. Then $x' = g \cdot x$ is Cohen-generic over $\mathfrak{N}$ by the lemma, hence over $\mathfrak{M}[y]$. Yet y is generic over $\mathfrak{M}$, thus the pair $\langle x', y\rangle$ is Cohen-generic over $\mathfrak{M}$ by the product forcing theorem (Theorem A.3.2). This implies $\vartheta(x') \not\equiv^{\alpha}_{\Lambda\Lambda} \vartheta(y)$ by the choice of α. On the other hand, we have $x' \mathsf{E}^{\mathbb{X}}_{\mathbb{G}}\, x$ and hence $\vartheta(x) \equiv^{\alpha}_{\Lambda\Lambda} \vartheta(x')$. Thus we finally obtain $\vartheta(x') \not\equiv^{\alpha}_{\Lambda\Lambda} \vartheta(y)$, as required.

□ (Lemma 13.3.3)

13.5. Ergodicity of turbulent actions *w.r.t.* T_ξ

The proof of Lemma 13.3.4 involves a somewhat stronger property than gen. ergodicity in Section 13.3.

DEFINITION 13.5.1. Suppose that F is an equivalence relation on a Polish space $\mathbb{X}$. An action of $\mathbb{G}$ on $\mathbb{X}$ and the induced equivalence relation $\mathsf{E}_{\mathbb{G}}^{\mathbb{X}}$ are *locally generically* (*loc. gen.*, for brevity) F-ergodic if the equivalence relation $\sim_G^X$ is gen. F-ergodic whenever $X \subseteq \mathbb{X}$ is a non-empty open set, $G \subseteq \mathbb{G}$ is a non-empty open set containing $1_{\mathbb{G}}$, and the local orbit $\mathscr{O}(x, X, G)$ is dense in X for comeager (in X) many $x \in X$. □

This obviously implies gen. F-ergodicity of $\mathsf{E}_{\mathbb{G}}^{\mathbb{X}}$ provided the action is gen. turbulent. Therefore, Lemma 13.3.4 is a corollary of the following Theorem 13.5.3.

DEFINITION 13.5.2. We let $\mathscr{F}_0$ denote the least family of equivalence relations that contain $\Delta_{\mathbb{N}}$ (the equality on $\mathbb{N}$) and are closed under operations (o1)–(o5) of Section 4.2. □

THEOREM 13.5.3. *Let $\mathbb{X}$ be a gen. turbulent Polish $\mathbb{G}$-space. If an equivalence relation F belongs to $\mathscr{F}_0$, then $\mathsf{E}_{\mathbb{G}}^{\mathbb{X}}$ is loc. gen. F-ergodic, and hence $\mathsf{E}_{\mathbb{G}}^{\mathbb{X}}$ is not Borel reducible to F.*

Due to the operation of the Fubini product, the family $\mathscr{F}_0$ contains a lot of equivalence relations very different from T_ξ, among them some Borel equivalence relations not classifiable by countable structures, e.g., all equivalence relations of the form $\mathsf{E}_{\mathscr{I}}$, where $\mathscr{I}$ is one of FRÉCHET ideals, indecomposable ideals, or WEISS ideals of Section 3.8. (In fact it is not so easy to show that ideals of the two last families produce relations in $\mathscr{F}_0$.) In particular, it follows that *no gen. turbulent equivalence relation is Borel reducible to a* FRÉCHET, *or indecomposable, or* WEISS *ideal.*

PROOF (Theorem 13.5.3). We begin with two rather elementary technical results of a topological nature.

LEMMA 13.5.4. *Suppose that $\mathbb{G}$ is a Polish group and $\mathbb{X}$ is a gen. turbulent Polish $\mathbb{G}$-space. Let $\varnothing \neq X \subseteq \mathbb{X}$ be an open set, $G \subseteq \mathbb{G}$ be a nbhd of $1_{\mathbb{G}}$, and $\mathscr{O}(x, X, G)$ be dense in X for X-comeager many $x \in X$. Let $U, U' \subseteq X$ be non-empty open and $D \subseteq X$ be comeager in X. Then there exist points $x \in D \cap U$ and $x' \in D \cap U'$ with $x \sim_G^X x'$.*

PROOF. Under our assumptions there exist points $x_0 \in U$ and $x_0' \in U'$ with $x_0 \sim_G^X x_0'$, that is, there exist elements $g_1, \dots, g_n \in G \cup G^{-1}$ such that $x_0' = g_n g_{n-1} \cdots g_1 \cdot x_0$ and in addition $g_k \cdots g_1 \cdot x_0 \in X$ for all $k \le n$. Since the action is continuous, there is a nbhd $U_0 \subseteq U$ of x_0 such that $g_k \cdots g_1 \cdot x \in X$ for all k and $g_n g_{n-1} \cdots g_1 \cdot x \in U'$ for all $x \in U_0$. Since D is comeager, easily there is $x \in U_0 \cap D$ such that $x' = g_n g_{n-1} \cdots g_1 \cdot x \in U' \cap D$. □ (Lemma)

LEMMA 13.5.5. *Suppose that $\mathbb{G}$ is a Polish group, and $\mathbb{X}$ is a gen. turbulent Polish $\mathbb{G}$-space. Then for every open non-empty $U \subseteq \mathbb{X}$ and $G \subseteq \mathbb{G}$ with $1_{\mathbb{G}} \in G$, there is an open non-empty $U' \subseteq U$ such that the local orbit $\mathscr{O}(x, U', G)$ is dense in U' for U'-comeager many $x \in U'$.*

PROOF. Let $\operatorname{INT} \overline{X}$ be the interior of the closure of X. If $x \in U$ and $\mathscr{O}(x, U, G)$ is somewhere dense (in U), then the set $U_x = U \cap \operatorname{INT} \overline{\mathscr{O}(x, U, G)} \subseteq U$ is open and

$\sim_G^U$-invariant (an observation made, e.g., in [**Kec02**, proof of 8.4]). Moreover, $\mathscr{O}(x,U,G) \subseteq U_x$, hence, $\mathscr{O}(x,U,G) = \mathscr{O}(x,U_x,G)$. It follows from the invariance that the sets U_x are pairwise disjoint, and it follows from the turbulence that the union of them is dense in U. Take any non-empty U_x as U'. □ (Lemma)

The proof of Theorem 13.5.3 goes on by induction on the number of applications of the basic operations, in several following sections until the end of Section 13.8.

Right now, we begin with the initial step: prove that, under the assumptions of the theorem, $\mathsf{E}_{\mathbb{G}}^{\mathbb{X}}$ is loc. gen. $\Delta_{\mathbb{N}}$-ergodic. Suppose that $X \subseteq \mathbb{X}$ and $G \subseteq \mathbb{G}$ are non-empty open sets, $1_{\mathbb{G}} \in G$, and $\mathscr{O}(x,X,G)$ is dense in X for X-comeager many $x \in X$, and prove that $\sim_G^X$ is generically $\Delta_{\mathbb{N}}$-ergodic.

Consider a Borel gen. $(\sim_G^X \to \Delta_{\mathbb{N}})$-invariant map $\vartheta : X \to \mathbb{N}$. Suppose toward the contrary that ϑ is not gen. $\Delta_{\mathbb{N}}$-constant. Then there exist two open non-empty sets $U_1, U_2 \subseteq X$, two numbers $\ell_1 \neq \ell_2$, and a comeager set $D \subseteq X$ such that $\vartheta(x) = \ell_1$ for all $x \in D \cap U_1$, $\vartheta(x) = \ell_2$ for all $x \in D \cap U_2$, and $\vartheta \restriction D$ is "strictly" $(\sim_G^X \to \Delta_{\mathbb{N}})$-invariant. Lemma 13.5.4 yields a pair of points $x_1 \in U_1 \cap D$ and $x_2 \in U_2 \cap D$ with $x_1 \sim_G^X x_2$, a contradiction.

13.6. Inductive step of countable power

To carry out this step in the proof of Theorem 13.5.3, suppose that

13.6.1. $\mathbb{X}$ is a gen. turbulent Polish $\mathbb{G}$-space, F is a Borel equivalence relation on a Polish space $\mathbb{Y}$, and the action of $\mathbb{G}$ on $\mathbb{X}$ is loc. gen. F-ergodic.

Prove that the action is loc. gen. F^+-ergodic. Fix a non-empty open set $X_0 \subseteq \mathbb{X}$ and a nbhd G_0 of $1_{\mathbb{G}}$ in $\mathbb{G}$, such that $\mathscr{O}(x,X_0,G_0)$ is dense in X_0 for all x in a comeager $\mathbf{G}_\delta$-set $D_0 \subseteq X_0$. Consider a Borel function $\vartheta : X_0 \to \mathbb{Y}^{\mathbb{N}}$, continuous and $(\sim_{G_0}^{X_0} \to \mathsf{F}^+)$-invariant on D_0, so that

$$x \sim_{G_0}^{X_0} x' \implies \forall k\, \exists l\, (\vartheta_k(x) \mathrel{\mathsf{F}} \vartheta_l(x')) \quad : \qquad \text{for all } x, x' \in D_0\,,$$

where $\vartheta_k(x) = \vartheta(x)(k)$, $\vartheta_k : X_0 \to \mathbb{Y}$. Prove that ϑ is gen. F^+-constant.

Let us fix a countable transitive model $\mathfrak{M}$ of $\mathbf{ZFC}^-$ as in Definition 13.4.2; that is, $\mathfrak{M}$ contains all relevant objects, or their codes (in the sense of Appendix A.2), in particular, codes of the spaces $\mathbb{X}, \mathbb{G}, \mathbb{Y}$, of the set D_0, and of the Borel map ϑ. Then every point $x \in X_0$ Cohen-generic over $\mathfrak{M}$ belongs to D_0. It follows that ϑ is $(\sim_{G_0}^{X_0} \to \mathsf{F}^+)$-invariant on Cohen-generic points of X_0, and local orbits $\mathscr{O}(x,X_0,G_0)$ of Cohen-generic points $x \in X_0$ are dense in X_0.

Coming back to the step of countable power, fix $k \in \mathbb{N}$. Consider any open non-empty set $U_0 \subseteq X_0$.

LEMMA 13.6.2. *There exist a number l and open non-empty sets $U \subseteq U_0$ and $H \subseteq G_0$ such that both $g \cdot x \in X_0$ and $\vartheta_k(x) \mathrel{\mathsf{F}} \vartheta_l(g \cdot x)$ hold for all pairs $\langle g, x\rangle$ in $H \times U$ Cohen-generic over $\mathfrak{M}$.*

PROOF. Consider any point $x_0 \in U_0$ Cohen-generic over $\mathfrak{M}$. Note that $1_{\mathbb{G}} \cdot x_0 = x_0 \in X_0$, hence there exist a nbhd $U_1 \subseteq U_0$ of x_0 and a nbhd $G_1 \subseteq G_0$ of $1_{\mathbb{G}}$ such that $G_1 \cdot U_1 \subseteq U_0$, i.e., $g \cdot x \in X_0$ for all $g \in G_1$ and $x \in U_1$.

Suppose that a pair $\langle g, x\rangle \in G_1 \times U_1$ is Cohen-generic over $\mathfrak{M}$. Then $g \cdot x \in U_0$. In addition, x is Cohen-generic over $\mathfrak{M}$ while g is Cohen-generic over $\mathfrak{M}[x]$ by the forcing product theorem (Theorem A.3.2). It follows that $g \cdot x$ is Cohen-generic over $\mathfrak{M}[x]$, and hence over $\mathfrak{M}$, by Lemma 13.4.3.

Furthermore, we have $x \sim^{X_0}_{G_0} g \cdot x$. By the invariance of ϑ on generic points, this implies $\vartheta(x) \mathsf{F}^+ \vartheta(g \cdot x)$. It follows that there is an index l such that $\vartheta_k(x) \mathsf{F} \vartheta_l(g \cdot x)$. Thus there exist Cohen conditions, *i.e.*, non-empty open sets $U \subseteq U_1$ and $H \subseteq G_1$ such that $x \in U$, $g \in H$, and every pair $\langle g', x' \rangle \in H \times U$ Cohen-generic over $\mathfrak{M}$ satisfies $g' \cdot x' \in X_0$ and $\vartheta_k(x') \mathsf{F} \vartheta_l(g' \cdot x')$. □ (Lemma)

Fix l, U, H as provided by the lemma. Since H is open, there is $h_0 \in H \cap \mathfrak{M}$ and a symmetric nbhd $G \subseteq G_1$ of $1_{\mathbb{G}}$ such that $g_0 \, G \subseteq H$.

LEMMA 13.6.3 (The key point of the turbulence). *If $x, x' \in U$ are Cohen-generic over $\mathfrak{M}$ and $x \sim^U_G x'$, then we have $\vartheta_k(x) \mathsf{F} \vartheta_k(x')$.*

PROOF. We argue by induction on $n(x, x') =$ the least number n such that there exist $g_1, \ldots, g_n \in G$ (recall: $G = G^{-1}$) satisfying

$$(1) \qquad x' = g_n \, g_{n-1} \cdots g_1 \cdot x, \qquad \text{and} \qquad g_k \cdots g_1 \cdot x \in U' \ \text{ for all } k \le n.$$

Suppose that $n(x, x') = 1$, thus, $x = g \cdot x'$ for some $g \in G$. Let $\mathfrak{N}$ be any countable transitive model of $\mathbf{ZFC}^-$ containing x, x', g, and all sets in $\mathfrak{M}$. Consider any element $h \in H$ Cohen-generic over $\mathfrak{N}$ and close enough to h_0 for $h' = hg^{-1}$ to belong to H. (Note that $h_0 g^{-1} \in H$ by the choice of G.) Then h is generic over $\mathfrak{M}[x]$, too, and hence $\langle h, x \rangle \in H \times U$ is Cohen-generic over $\mathfrak{M}$ by the product forcing theorem. It follows, by the choice of H, that $h \cdot x \in X_0$ and $\vartheta_k(x) \mathsf{F} \vartheta_l(h \cdot x)$.

Moreover, $h' = hg^{-1}$ also is $\mathbf{C}_{\mathbb{G}}$-generic over $\mathfrak{N}$ (because $g \in \mathfrak{N}$), so that $\vartheta_k(x') \mathsf{F} \vartheta_l(h' \cdot x')$ by the same argument. Finally, $g' \cdot x' = gh^{-1} \cdot (h \cdot x) = g \cdot x$, and hence $\vartheta_k(x') \mathsf{F} \vartheta_k(x)$, as required.

As for the inductive step, prove that (1) holds for some $n \ge 2$ assuming that it holds for $n - 1$. Consider an element $g_1' \in G$ close enough to g_1 for $g_2' = g_2 \, g_1 \, {g_1'}^{-1}$ to belong to G and for $x^* = g_1' \cdot x$ to belong to U, and Cohen-generic over a fixed transitive countable model $\mathfrak{N}$ of $\mathbf{ZFC}^-$ containing x, x', g_1, g_2. Then, as above, g_2' is Cohen-generic over $\mathfrak{N}$ while x^* is Cohen-generic over $\mathfrak{M}$, and obviously $n(x^*, x') \le n - 1$ because $g_2' \cdot x^* = g_2 \cdot g_1 \cdot x$. It remains to use the induction hypothesis. □ (Lemma)

To summarize, we have shown that for every k and every open $\varnothing \ne U_0 \subseteq X_0$ there exist an open non-empty set $U \subseteq U_0$ and an open $G \subseteq G_0$ with $1_{\mathbb{G}} \in G$, such that the map ϑ_k is $(\sim^U_G \to \mathsf{F})$-invariant on U. We can also assume that the orbit $\mathscr{O}(x, U, G)$ is dense in U for U-comeager many $x \in U$ by Lemma 13.5.5. Then, by the loc. gen. F-ergodicity of the action considered, ϑ_k is gen. F-constant on U. That is, there exist a comeager $\mathbf{G}_\delta$ set $D \subseteq U$ and a point $y \in \mathbb{Y}$ such that $\vartheta_k(x) \mathsf{F} y$ for all $x \in D$.

We conclude that there exist an X_0-comeager set $D \subseteq X_0$ and a countable set $Y = \{y_j : j \in \mathbb{N}\} \subseteq \mathbb{Y}$ such that, for every k and for every $x \in D$ there is j with $\vartheta_k(x) \mathsf{F} y_j$. Put $\eta(x) = \bigcup_{k \in \mathbb{N}} \{j : \vartheta_k(x) \mathsf{F} y_j\}$. Then, for every pair $x, x' \in D$, we have $\vartheta(x) \mathsf{F}^+ \vartheta(x')$ iff $\eta(x) = \eta(x')$, so that, by the invariance of ϑ,

$$(2) \qquad x \sim^{U_0}_{G_0} x' \implies \eta(x) = \eta(x') \quad : \quad \text{for all } x, x' \in D.$$

It remains to show that η is a constant on a comeager subset of D.

Suppose, on the contrary, that there exist two non-empty open sets $U_1, U_2 \subseteq U_0$, a number $j \in \mathbb{N}$, and a comeager set $D' \subseteq D$ such that $j \in \eta(x_1)$ and $j \notin \eta(x_2)$ for all $x_1 \in D' \cap U_1$ and $x_2 \in D' \cap U_2$. Now Lemma 13.5.4 yields a contradiction to (2) above as in the end of Section 13.6.

□ (Inductive step of countable power in Theorem 13.5.3)

13.7. Inductive step of the Fubini product

To carry out this step in the proof of Theorem 13.5.3, suppose that

13.7.1. $\mathbb{X}$ is a gen. turbulent Polish $\mathbb{G}$-space, and for every k, F_k is a Borel equivalence relation on a Polish space $\mathbb{Y}_k$, the action of $\mathbb{G}$ on $\mathbb{X}$ is loc. gen. F_k-ergodic, and $\mathsf{F} = \prod_k \mathsf{F}_k \,/\, \mathtt{Fin}$ is, accordingly, a Borel equivalence relation on $\mathbb{Y} = \prod_k \mathbb{Y}_k$.

Prove that the action is loc. gen. F-ergodic.

Fix a non-empty open set $X_0 \subseteq \mathbb{X}$, a nbhd G_0 of $1_{\mathbb{G}}$ in $\mathbb{G}$, and a comeager $\mathbf{G}_\delta$ set $D_0 \subseteq X_0$ such that all local orbits $\mathscr{O}(x, X_0, G_0)$ with $x \in D_0$ are dense in X_0. Consider a Borel function $\vartheta : X_0 \to \mathbb{Y}$, $(\sim_{G_0}^{X_0} \to \mathsf{F})$-invariant on D_0, *i.e.*,

$$x \sim_{G_0}^{X_0} y \implies \exists\, k_0\, \forall\, k \geq k_0\, \left(\vartheta_k(x)\; \mathsf{F}_k\; \vartheta_k(y)\right) \quad : \qquad \text{for all } x, y \in D_0\,,$$

where $\vartheta_k(x) = \vartheta(x)(k)$, and prove that ϑ is gen. F-constant.

Choose a countable transitive model $\mathfrak{M}$ of $\mathbf{ZFC}^-$ as in Section 13.6.

Consider an open non-empty set $U_0 \subseteq X_0$. Similar to Lemma 13.6.2, there exist non-empty open sets $U \subseteq U_0$ and $H \subseteq G_0$, and a number k_0, such that both $g \cdot x \in X_0$ and $\vartheta_k(x)\; \mathsf{F}_k\; \vartheta_k(g \cdot x)$ hold for all indices $k \geq k_0$ and for all pairs $\langle g, x\rangle \in H \times U$ Cohen-generic over $\mathfrak{M}$.

As H is open, there exist an element $h_0 \in H \cap \mathfrak{M}$ and a symmetric nbhd $G \subseteq G_0$ of $1_{\mathbb{G}}$ such that $h_0\, G \subseteq H$.

LEMMA 13.7.2. *If $k \geq k_0$, points $x, y \in U$ are Cohen-generic over $\mathfrak{M}$, and $x \sim_G^U y$, then $\vartheta_k(x)\; \mathsf{F}_k\; \vartheta_k(y)$.* (Similar to Lemma 13.6.3.) □

Thus, for every open non-empty set $U_0 \subseteq X_0$ there exist a number k_0, an open non-empty $U \subseteq U_0$, and a nbhd $G \subseteq G_0$ of $1_{\mathbb{G}}$, such that $\vartheta_k(x)$ is gen. $(\sim_G^U \to \mathsf{F}_k)$-invariant on U for every $k \geq k_0$. We can assume that U-comeager many orbits $\mathscr{O}(x, U, G)$ are dense in U, by Lemma 13.5.5. Now, by F_k-ergodicity, every ϑ_k with $k \geq k_0$ is gen. F_k-constant on such a set U. Hence, ϑ itself is gen. F-constant on U because $\mathsf{F} = \prod_k \mathsf{F}_k \,/\, \mathtt{Fin}$. It remains to show that these constants are F-equivalent to each other.

Suppose, on the contrary, that there exist two non-empty open sets $U_1, U_2 \subseteq U_0$ and a pair of $y_1 \not\mathsf{F}\, y_2$ in $\mathbb{Y}$ such that $\vartheta(x_1)\, \mathsf{F}\, y_1$ and $\vartheta(x_2)\, \mathsf{F}\, y_2$ for comeager many $x_1 \in U_1$ and $x_2 \in U_2$. A contradiction follows as in the end of Section 13.6.

□ (Inductive step of Fubini product in Theorem 13.5.3)

13.8. Other inductive steps

Here we accomplish the proof of Theorem 13.5.3, by carrying out induction steps, related to operations (o1), (o2), (o3) of Section 4.2.

Countable union. Suppose that $\mathsf{F}_1, \mathsf{F}_2, \mathsf{F}_3, \dots$ are Borel equivalence relations on a Polish space $\mathbb{Y}$, $\mathsf{F} = \bigcup_k \mathsf{F}_k$ is still an equivalence relation, and the Polish and gen. turbulent action of $\mathbb{G}$ on $\mathbb{X}$ is loc. gen. F_k-ergodic for every k. Prove that the action remains loc. gen. F-ergodic.

Fix a non-empty open set $X_0 \subseteq \mathbb{X}$, a comeager $\mathbf{G}_\delta$ set $D_0 \subseteq X_0$, and a nbhd G_0 of $1_{\mathbb{G}}$ in $\mathbb{G}$ such that all local orbits $\mathscr{O}(x, X_0, G_0)$ with $x \in D_0$ are dense in X_0. Consider a Borel function $\vartheta : X_0 \to \mathbb{Y}$, $(\sim_{G_0}^{X_0} \to \mathsf{F})$-invariant on D_0. It follows from the invariance that for every open non-empty $U \subseteq U_0$ there exist a number

k and open non-empty sets $U \subseteq U_0$ and $H \subseteq G_0$ such that both $g \cdot x \in X_0$ and $\vartheta(x) \mathsf{F}_k \vartheta(g \cdot x)$ hold for every pair $\langle x, g\rangle \in U \times H$ Cohen-generic over a fixed countable transitive model $\mathfrak{M}$ of $\mathbf{ZFC}^-$ chosen as above. Further, there exist $h_0 \in H \cap \mathfrak{M}$ and a symmetric nbhd $G \subseteq G_0$ of $1_{\mathbb{G}}$ such that $h_0 G \subseteq H$.

Similar to Lemmas 13.6.3 and 13.7.2, $\vartheta(x) \mathsf{F}_k \vartheta(x')$ holds for every pair of elements $x, x' \in U$ Cohen-generic over $\mathfrak{M}$ and satisfying $x \sim_G^U x'$. It follows, by ergodicity, that ϑ is F_k-constant, hence, F-constant, on a comeager subset of U. It remains to show that these F-constants are F-equivalent to each other, which is demonstrated exactly as in the end of Section 13.6.

Disjoint union. Let F_k be Borel equivalence relations on Polish spaces $\mathbb{Y}_k$, $k = 0, 1, 2, \dots$. By definition, $\bigvee_k \mathsf{F}_k = \bigcup_k \mathsf{F}'_k$, where each F'_k is a Borel equivalence relation defined on the space $\mathbb{Y} = \bigcup_k \{k\} \times \mathbb{Y}_k$ as follows: $\langle l, y\rangle \mathsf{F}'_k \langle l', y'\rangle$ iff either $l = l'$ and $y = y'$ or $l = l' = k$ and $y \mathsf{F}_k y'$.

Countable product. Let F_k be equivalence relations on Polish spaces $\mathbb{Y}_k$. Then $\mathsf{F} = \prod_k \mathsf{F}_k$ is an equivalence relation on the space $\mathbb{Y} = \prod_k \mathbb{Y}_k$. For any map $\vartheta : \mathbb{X} \to \mathbb{Y}$, to be gen. $(\mathsf{E} \to \mathsf{F})$-invariant (where E is an equivalence relation on $\mathbb{X}$), it is necessary and sufficient that every coordinate map $\vartheta_k(x) = \vartheta(x)(k)$ is gen. $(\mathsf{E} \to \mathsf{F}_k)$-invariant. This allows us to easily accomplish this induction step.

□ (Theorem 13.5.3, Lemma 13.3.4, Theorem 13.1.2)

13.9. Applications to the shift action of ideals

We are going to apply the results above in this chapter in order to prove that equivalence relations generated by many Borel ideals (in particular almost all polishable ideals) are not Borel reducible to Borel actions of the permutation group S_∞, and hence not classifiable by countable structures. The difficult problem of verification of the turbulence can fortunately be circumvented by reference to Theorem 13.5.3 and Proposition 13.2.1 (the turbulence of summable ideals).

Say that a Borel ideal $\mathscr{Z} \subseteq \mathscr{P}(\mathbb{N})$ is *special* if there is a sequence of reals $r_n > 0$ with $\{r_n\} \to 0$, such that $\mathscr{S}_{\{r_n\}} \subseteq \mathscr{Z}$. *Non-trivial* in the next theorem means containing no cofinite sets. In the context of summable ideals, non-triviality means simply that $\sum_n r_n = +\infty$.

THEOREM 13.9.1. *Suppose that $\mathscr{Z}$ is a non-trivial Borel special ideal and F belongs to the family $\mathscr{F}_0$ of Definition 13.5.2. Then $\mathsf{E}_{\mathscr{Z}}$ is generically F-ergodic, hence, is not Borel reducible to F.*

PROOF. The "hence" statement follows because, by non-triviality, all $\mathsf{E}_{\mathscr{Z}}$-equivalence classes are meager subsets of $\mathscr{P}(\mathbb{N})$.

As $\mathscr{Z}$ is special, let $\{r_k\} \to 0$ be a sequence of positive reals such that $\sum_n r_n = +\infty$ and $\mathscr{S}_{\{r_n\}} \subseteq \mathscr{Z}$. Note that $x \mathsf{E}_{\{r_n\}} y$ implies $x \mathsf{E}_{\mathscr{Z}} y$, and hence every gen. $(\mathsf{E}_{\mathscr{Z}} \to \mathsf{F})$-invariant map is gen. $(\mathsf{E}_{\{r_n\}} \to \mathsf{F})$-invariant as well (on the same comeager set). Thus it suffices to prove that $\mathsf{E}_{\{r_n\}} = \mathsf{E}_{\mathscr{S}_{\{r_n\}}}$ is gen. F-ergodic.

Recall that the shift action of the Δ-group of the ideal $\mathscr{S}_{\{r_n\}}$ on $\mathscr{P}(\mathbb{N})$ is Polish and gen. turbulent by Proposition 13.2.1. Thus $\mathsf{E}_{\mathscr{S}_{\{r_n\}}}$ is gen. F-ergodic by Theorem 13.5.3, as required. □

The next corollary returns us to the discussion at the end of Section 5.5.

COROLLARY 13.9.2. *The equivalence relations* c_0 *and* E_2 *are not Borel reducible to any equivalence relation* F *in the family* $\mathscr{F}_0$, *in particular, they are not Borel reducible to* T_2 *and therefore to* E_∞ *and any other countable Borel equivalence relation by Lemma* 6.1.3.

PROOF. According to Lemmas 6.2.3 and 6.2.4, it suffices to prove that the ideals $\mathscr{Z}_0$ (density 0) and $\mathscr{S}_{\{1/n\}}$ are special. (Their non-triviality is obvious.) The ideal $\mathscr{S}_{\{1/n\}}$ is special by definition. As for $\mathscr{Z}_0$, it suffices to prove that $\mathscr{S}_{\{1/n\}} \subseteq \mathscr{Z}_0$. Consider a set $x \subseteq \mathbb{N}$, $x \notin \mathscr{Z}_0$. There is a real $\varepsilon > 0$ such that $\frac{\mathtt{card}\,(x\cap[0,n))}{n} > 2\varepsilon$ for infinitely many numbers n. One easily defines an increasing sequence $n_0 < n_1 < n_2 < \cdots$ such that $n_{i+1} \geq 2n_i$ and $\frac{\mathtt{card}\,(x\cap[n_i,n_{i+1}))}{n_{i+1}-n_i} > \varepsilon$ for all i. Then $\sum_{n\in x}\frac{1}{n} \geq \varepsilon\sum_i \frac{n_{i+1}-n_i}{n_{i+1}} = +\infty$, hence $x \notin \mathscr{S}_{\{1/n\}}$. □

The next theorem shows that, with three exceptions, there exist no polishable ideals Borel reducible to equivalence relations in $\mathscr{F}_0$. (Note that $\mathscr{F}_0$ contains various equivalence relations of the form $\mathsf{E}_{\mathscr{I}}$, generated by non-polishable ideals $\mathscr{I}$, for instance, by the FRÉCHET ideals.) KECHRIS [**Kec98**] proved a similar theorem, in which the assumption of reducibility to a relation in $\mathscr{F}_0$ is replaced by the assumption of reducibility to a Borel action of S_∞. Recall that $\mathscr{I} \cong \mathscr{J}$ means isomorphism via bijection between the underlying sets of the ideals.

THEOREM 13.9.3. *If* $\mathscr{I}$ *is a non-trivial Borel polishable ideal on* $\mathbb{N}$, F *an equivalence relation in* $\mathscr{F}_0$, *and* $\mathsf{E}_{\mathscr{I}} \leq_{\mathrm{B}} \mathsf{F}$, *then* $\mathscr{I}$ *is isomorphic to one of the following three ideals:* $\mathscr{I}_3$, $\mathtt{Fin}$, $\mathtt{Fin} \oplus \mathscr{P}(\mathbb{N})$.

Note that in each of the three cases $\mathsf{E}_{\mathscr{I}} \leq_{\mathrm{B}} \mathsf{E}_3$ holds.

PROOF. It follows from Theorem 3.5.1 and Corollary 11.8.3 that $\mathscr{I} = \mathrm{Exh}_\varphi$ for an LSC submeasure φ on $\mathbb{N}$. We can assume that $\varphi(x) \leq 1$ for all $x \in \mathscr{P}(\mathbb{N})$. (Otherwise, put $\varphi'(x) = \mathtt{min}\{1, \varphi(x)\}$.) Consider the sets $U_n = \{k : \varphi(\{k\}) \leq \frac{1}{n}\}$ and $U_\infty = \{k : \varphi(\{k\}) = 0\}$. Clearly, $U_{n+1} \subseteq U_n$ and $\varphi(U_\infty) = 0$; therefore, $\varphi(x) = \varphi(x \smallsetminus U_\infty)$ for all $x \in \mathscr{P}(\mathbb{N})$.

We claim that $\mathtt{lim}_{n\to\infty}\varphi(U_n) = 0$.

Suppose toward the contrary that there exists $\varepsilon > 0$ such that $\varphi(U_n) > \varepsilon$ for all n. By definition, for every m there is $n \geq m$ satisfying $U_n \subseteq [m,\infty) \cup U_\infty$; hence, $\varphi(U_n \smallsetminus m) > \varepsilon$ as well. Moreover, there exists $n' \geq n$ satisfying $\varphi(U_n \cap [m, n')) > \varepsilon$. This leads to a sequence $n_1 < n_2 < n_3 < \cdots$ of numbers and a sequence of finite sets $w_j \subseteq U_{n_j} \smallsetminus U_{n_{j+1}}$ such that $\varphi(w_j) > \varepsilon$. The sets w_j are pairwise disjoint, hence every "tail" $W \cap [n,\infty)$ of their union $W = \bigcup_j w_j$ includes at least one of w_j as a subset. It follows that $W \notin \mathscr{I} = \mathrm{Exh}_\varphi$. The ideal $\mathscr{J} = \mathscr{I} \cap \mathscr{P}(W)$ on W is then non-trivial. We also have $\{\varphi(\{k\})\}_{k\in W} \to 0$ and $\sum_k \varphi(\{k\}) = +\infty$ since for every n all but finite sets w_l satisfy $w_l \subseteq W$. Finally, the equivalence $x \,\Delta\, y \in \mathscr{I} \iff x \,\Delta\, y \in \mathscr{J}$ holds for all $x, y \subseteq W$. It follows that $\mathsf{E}_{\mathscr{J}} \leq_{\mathrm{B}} \mathsf{E}_{\mathscr{I}}$ by means of the identity map.

Since φ is an LSC submeasure, we have $\varphi(y) \leq \sum_{k\in y}\varphi(\{k\})$ for all $y \subseteq \mathbb{N}$. It follows that every set $x \subseteq W$ satisfying $\sum_{k\in x}\varphi(\{k\}) < +\infty$ belongs to $\mathscr{I}$, hence to $\mathscr{J}$ as well. Thus $\mathscr{J}$ is isomorphic to a special ideal via a bijection of W onto $\mathbb{N}$. We conclude that $\mathsf{E}_{\mathscr{J}}$, and hence $\mathsf{E}_{\mathscr{I}}$, are Borel irreducible to relations in the family $\mathscr{F}_0$ by Theorem 13.9.1, a contradiction.

Thus $\varphi(U_n) \to 0$. It follows that for every set $x \in \mathscr{P}(\mathbb{N})$ to belong to $\mathscr{I}$, it is necessary and sufficient that $x \cap (U_n \smallsetminus U_{n+1})$ is finite for every n. This observation allows us to accomplish the proof: if the difference $U_n \smallsetminus U_{n+1}$ is infinite for infinitely many indices n, then $\mathscr{I} \cong \mathscr{I}_3$; if there exist only finitely many infinite differences $U_n \smallsetminus U_{n+1}$ and their union is co-finite in $\mathbb{N}$, then $\mathscr{I} \cong \mathsf{Fin}$; and finally $\mathscr{I} \cong \mathsf{Fin} \oplus \mathscr{P}(\mathbb{N})$ iff there exist only finitely many (but > 0) infinite differences $U_n \smallsetminus U_{n+1}$ but their union is co-infinite in $\mathbb{N}$. □

COROLLARY 13.9.4. *There is no Borel ideal* $\mathscr{I}$ *such that* $\mathsf{E}_{\mathscr{I}} \sim_{\mathrm{B}} \mathsf{T}_2$.

PROOF. Suppose toward the contrary that $\mathscr{I}$ is such an ideal. Then $\mathscr{I}$ is polishable. (Indeed otherwise $\mathsf{E}_1 \le_{\mathrm{B}} \mathscr{I}$ by Corollary 11.8.3, and hence $\mathsf{E}_1 \le_{\mathrm{B}} \mathsf{T}_2$. But this contradicts Theorem 11.8.1 since T_2 is easily Borel reducible to a Polish action.) Thus $\mathsf{E}_{\mathscr{I}} \le_{\mathrm{B}} \mathsf{E}_3$ by Theorem 13.9.3. On the other hand, recall that the ideal $\mathscr{I}_3 = 0 \times \mathsf{Fin}$ is a P-ideal (Exercise 3.3.4), hence it is polishable by Theorem 3.5.1. Thus $\mathsf{T}_2 \not\le_{\mathrm{B}} \mathsf{E}_3$ by Theorem 17.1.3, which is applicable in this case because the Δ-group of $\mathscr{I}_3$ (basically, of any ideal) is abelian. Therefore $\mathsf{T}_2 \not\le_{\mathrm{B}} \mathsf{E}_{\mathscr{I}}$, as required. □

The next application of Theorem 13.9.1 is related to the structure of ideals Borel reducible to E_3. The result is similar to Theorem 11.1.1. We begin with the following irreducibility lemma.

LEMMA 13.9.5. $\mathsf{E}_0 <_{\mathrm{B}} \mathsf{E}_3$. *Equivalence relations* E_3 *and* E_1 *are* $\le_{\mathrm{B}}$*-incomparable. Equivalence relations* E_2 *and* E_1 *are* $\le_{\mathrm{B}}$*-incomparable as well.*

PROOF. It is quite obvious that $\mathsf{E}_0 \le_{\mathrm{B}} \mathsf{E}_3$ and $\mathsf{E}_0 \le_{\mathrm{B}} \mathsf{E}_1$. Thus $\mathsf{E}_0 <_{\mathrm{B}} \mathsf{E}_3$ strictly since we have $\mathsf{E}_3 \not\le_{\mathrm{B}} \mathsf{E}_1$ by Corollary 11.1.4. To prove $\mathsf{E}_1 \not\le_{\mathrm{B}} \mathsf{E}_3$, recall that the ideal $\mathscr{I}_3$ is polishable (see above). Now $\mathsf{E}_1 \not\le_{\mathrm{B}} \mathsf{E}_3$ follows from Corollary 11.8.3.

The proof of the second claim is similar. □

The following result of KECHRIS **[Kec98]** should be compared with Theorem 11.1.1.

COROLLARY 13.9.6. *If* $\mathscr{I}$ *is a non-trivial Borel ideal on* $\mathbb{N}$ *and* $\mathsf{E}_{\mathscr{I}} \le_{\mathrm{B}} \mathsf{E}_3$, *then* $\mathscr{I}$ *is isomorphic to one of the following three ideals*: $\mathscr{I}_3$, Fin, $\mathsf{Fin} \oplus \mathscr{P}(\mathbb{N})$.

PROOF. We have $\mathsf{E}_1 \not\le_{\mathrm{B}} \mathsf{E}_{\mathscr{I}}$ by Lemma 13.9.5. Therefore $\mathscr{I}$ is a polishable ideal by Corollary 11.8.3. It remains to apply Theorem 13.9.3. □

CHAPTER 14

The ideal $\mathscr{I}_3$ and the equivalence relation E_3

This chapter is devoted to the ideal $\mathscr{I}_3$ and the corresponding equivalence relation E_3. Recall that $\mathscr{I}_3$ (also denoted by $0\times\mathtt{Fin}$) consists of all sets $x \subseteq \mathscr{P}(\mathbb{N}\times\mathbb{N})$ such that all cross-sections $(x)_n = \{k : \langle n,k\rangle \in x\}$ are finite. Accordingly, the relation $\mathsf{E}_3 = \mathsf{E}_{\mathscr{I}_3}$ is defined on $\mathscr{P}(\mathbb{N}\times\mathbb{N})$ by $x \mathrel{\mathsf{E}_3} y$ iff $x \mathbin{\Delta} y \in \mathscr{I}_3$. But we'll rather consider E_3 to be an equivalence relation on $(2^{\mathbb{N}})^{\mathbb{N}}$ defined so that $x \mathrel{\mathsf{E}_3} y$ iff $x(n) \mathrel{\mathsf{E}_0} y(n)$ for all n : here x,y belong to $(2^{\mathbb{N}})^{\mathbb{N}}$. Formally,

$$x \mathrel{\mathsf{E}_3} y \iff \forall n\, \exists k_0\, \forall k \ge k_0\, (x(n,k) = y(n,k)).$$

The main goal of this chapter will be the proof of Theorem 5.7.6 of HJORTH and KECHRIS, the 6th dichotomy theorem. Recall that it asserts that every Borel equivalence relation E such that $\mathsf{E} \le_{\mathrm{B}} \mathsf{E}_3$ satisfies "either" $\mathsf{E} \le_{\mathrm{B}} \mathsf{E}_0$ "or" $\mathsf{E} \sim_{\mathrm{B}} \mathsf{E}_3$. Thus, similar to E_1, E_3 is an immediate successor of E_0 in the structure of Borel reducibility. Let us mention an immediate corollary.

COROLLARY 14.0.1 (of HJORTH and KECHRIS, Theorem 5.7.6). $\mathsf{E}_\infty \not\le_{\mathrm{B}} \mathsf{E}_3$.

PROOF. Suppose toward the contrary that $\mathsf{E}_\infty \le_{\mathrm{B}} \mathsf{E}_3$. Then by Theorem 5.7.6 "either" $\mathsf{E}_\infty \le_{\mathrm{B}} \mathsf{E}_0$ "or" $\mathsf{E}_\infty \sim_{\mathrm{B}} \mathsf{E}_3$. The "either" case contradicts Theorem 7.5.1. To derive a contradiction from the "or" case assumption, note that $\mathsf{E}_\infty \le_{\mathrm{B}} \ell^\infty$ by Theorem 6.6.1, but on the other hand $\mathsf{E}_3 \not\le_{\mathrm{B}} \ell^\infty$ by Lemma 6.1.1. □

The proof of Theorem 5.7.6 employs the Gandy–Harrington forcing in a manner rather similar to the proof of Theorem 5.7.3 (3rd dichotomy theorem). The proof given here is designed on the basis of the proofs of Theorems 7.2 and 7.3 in [**HK01**]. The first of them contains a dichotomy that somewhat generalizes the result of Theorem 5.7.6. Recall that E_∞ is a $\le_{\mathrm{B}}$-largest countable Borel equivalence relation, realized in the form of a certain equivalence relation on the Polish space 2^{F_2}, where F_2 is the free group with two generators. Let $(\mathsf{E}_\infty)^{\aleph_0}$ denote the equivalence relation on $(2^{F_2})^{\mathbb{N}}$, defined so that $x \mathrel{(\mathsf{E}_\infty)^{\aleph_0}} y$ iff $x(n) \mathrel{\mathsf{E}_\infty} y(n)$ for all n. Thus $(\mathsf{E}_\infty)^{\aleph_0}$ is related to E_∞ just as E_3 is to E_0. Theorem 7.2 in [**HK01**] asserts that every Borel equivalence relation E such that $\mathsf{E} \le_{\mathrm{B}} (\mathsf{E}_\infty)^{\aleph_0}$ satisfies either $\mathsf{E} \le_{\mathrm{B}} \mathsf{E}_\infty$ or $\mathsf{E}_3 \le_{\mathrm{B}} \mathsf{E}$. Theorem 7.3 in [**HK01**] contains a result that allows us to derive Theorem 5.7.6 from Theorem 7.2 (also in [**HK01**]). In our proof, the effect of Theorem 7.3 in [**HK01**] is reduced to Theorem 14.1.1.

14.1. Continual assembling of equivalence relations

The next theorem will be used in the proof of Theorem 5.7.6. The result is somewhat similar to Theorem 7.3.1 in that it evaluates the type of equivalence relation E on the basis of the types of certain fragments of E. But in this case the number of fragments can be uncountable!

THEOREM 14.1.1. *Suppose that* $\mathbb{X}, \mathbb{Y}$ *are Polish spaces,* $P \subseteq \mathbb{X} \times \mathbb{Y}$ *is a Borel set,* E *is a Borel equivalence relation on* P, $\mathbb{G}$ *is a countable group acting on* $\mathbb{X}$ *in a Borel way, and* $\langle x, y\rangle \,\mathsf{E}\, \langle x', y'\rangle$ *implies* $x \,\mathsf{E}_{\mathbb{G}}^{\mathbb{X}}\, x'$. *Finally, suppose that* $\mathsf{E} \restriction (P)_x$ *is smooth for each* $x \in \mathbb{X}$, *where* $(P)_x = \{\langle x', y\rangle \in P : x' = x\}$. *Then* E *is Borel-reducible to a Borel action of* $\mathbb{G}$.

PROOF. We can assume that $\mathbb{X} = \mathbb{Y} = 2^{\mathbb{N}}$ and both P and E are $\varDelta^1_1$. We can also assume that the underlying set of $\mathbb{G}$ (a countable group) is $\mathbb{N}$, and both the group operation and the action of $\mathbb{G}$ are $\varDelta^1_1$. Then clearly $x \,\mathsf{E}_{\mathbb{G}}^{\mathbb{X}}\, x' \Longrightarrow \varDelta^1_1(x) = \varDelta^1_1(x')$.

Define $P^*(x) = \bigcup_{a \in \mathbb{G}} (P)_{a \cdot x}$ for $x \in \mathbb{X}$.

LEMMA 14.1.2. *Suppose that pairs* $\langle x, y\rangle$ *and* $\langle x', y'\rangle$ *belong to* P *and* $x \,\mathsf{E}_{\mathbb{G}}^{\mathbb{X}}\, x'$. *Then* $\langle x, y\rangle \,\mathsf{E}\, \langle x', y'\rangle$ *iff the equivalence* $\langle x, y\rangle \in U \Longleftrightarrow \langle x', y'\rangle \in U$ *holds for every* $\mathsf{E} \restriction P^*(x)$*-invariant set* $U \subseteq P^*(x)$ *of class* $\varDelta^1_1(x)$.

PROOF. Note that the restricted equivalence relation $\mathsf{E} \restriction P^*(x)$ is still smooth by Corollary 7.3.2 since $\mathbb{G}$ is countable. In addition $\mathsf{E} \restriction P^*(x)$ is $\varDelta^1_1(x)$. This observation yields the result. Indeed otherwise the equivalence relation, defined on $P^*(x)$ by intersections with $\mathsf{E} \restriction P^*(x)$-invariant sets of class $\varDelta^1_1(x)$, is strictly coarser than $\mathsf{E} \restriction P^*(x)$. It follows by the $\varDelta^1_1(x)$-version of Theorem 10.4.5 that $\mathsf{E}_0 \le_{\mathrm{B}} \mathsf{E} \restriction P^*(x)$, a contradiction with the smoothness. □ (Lemma)

In the continuation of the proof of Theorem 14.1.1 we make use of a standard enumeration of $\varDelta^1_1$ sets. By Theorem 2.8.2 there exist $\varPi^1_1$ sets $\mathtt{Cod}(\varDelta^1_1) \subseteq \mathbb{X} \times \mathbb{N}$ and $W \subseteq \mathbb{X} \times \mathbb{N} \times \mathbb{X} \times \mathbb{Y}$ and a $\varSigma^1_1$ set $W' \subseteq \mathbb{X} \times \mathbb{N} \times \mathbb{X} \times \mathbb{Y}$ such that the sets

$$(W)_{xe} = \{\langle x', y'\rangle : \langle x, e, x', y'\rangle \in W\} \quad \text{and} \quad (W')_{xe} = \{\langle x', y'\rangle : \langle x, e, x', y'\rangle \in W'\}$$

coincide whenever $\langle x, e\rangle \in \mathtt{Cod}(\varDelta^1_1)$, and for every $x \in \mathbb{X}$ a set $R \subseteq \mathbb{X} \times \mathbb{Y}$ is $\varDelta^1_1(x)$ iff there is e such that $\langle x, e\rangle \in \mathtt{Cod}(\varDelta^1_1)$ and $X = (W)_{xe} = (W')_{xe}$.

Let J be the set of all pairs $\langle x, e\rangle \in \mathtt{Cod}(\varDelta^1_1)$ such that $(W)_{xe} \subseteq P^*(x)$ and the set $(W)_{xe}$ is $\mathsf{E} \restriction P^*(x)$-invariant. Easily J is $\varPi^1_1$.

COROLLARY 14.1.3. *Suppose that pairs* $\langle x, y\rangle, \langle x', y'\rangle$ *are as in Lemma* 14.1.2. *Then* $\langle x, y\rangle \,\mathsf{E}\, \langle x', y'\rangle$ *iff the equivalence* $\langle x, y\rangle \in (W)_{xe} \Longleftrightarrow \langle x', y'\rangle \in (W)_{xe}$ *holds for every* e *with* $\langle x, e\rangle \in J$. □

Let us change "iff" here to $\Longleftarrow$. Such a reduced claim can be formally represented in the form $(P \times P) \cap \mathsf{E}_{\mathbb{G}}^{\mathbb{X}} \subseteq U \cup \mathsf{E}$, where $U = \bigcup_{e \in \mathbb{N}} U_e$ and

$$U_e = \{\langle\langle x, y\rangle, \langle x', y'\rangle\rangle : \langle x, e\rangle \in J \wedge \neg\,(\langle x, y\rangle \in (W)_{xe} \Longleftrightarrow \langle x', y'\rangle \in (W)_{xe})\}.$$

As $J \subseteq \mathtt{Cod}(\varDelta^1_1)$, we can rewrite the negation of $\Longleftrightarrow$ in the last formula in the following $\varPi^1_1$ form:

$$\big(\langle x, y\rangle \in (W)_{xe} \wedge \langle x', y'\rangle \notin (W')_{xe}\big) \wedge \big(\langle x, y\rangle \notin (W')_{xe} \wedge \langle x', y'\rangle \in (W)_{xe}\big).$$

Thus the inclusion $(P \times P) \cap \mathsf{E}_{\mathbb{G}}^{\mathbb{X}} \subseteq U \cup \mathsf{E}$ as a property of a $\varPi^1_1$ set J is $\varPi^1_1$ in the codes. It follows by Theorem 2.7.1 (Reflection) that there is a $\varDelta^1_1$ set $J' \subseteq J$ such that still $(P \times P) \cap \mathsf{E}_{\mathbb{G}}^{\mathbb{X}} \subseteq U' \cap \mathsf{E}$ holds, where $U' = \bigcup_e U'_e$ is defined in terms of J' similar to the definition of U in terms of J.

Let us fix such a $\varDelta^1_1$ set J'.

COROLLARY 14.1.4. *Suppose that pairs* $\langle x, y\rangle, \langle x', y'\rangle$ *are as in Lemma* 14.1.2. *Then* $\langle x, y\rangle \,\mathsf{E}\, \langle x', y'\rangle$ *iff the equivalence* $\langle x, y\rangle \in (W)_{xe} \Longleftrightarrow \langle x', y'\rangle \in (W)_{xe}$ *holds for every* e *with* $\langle x, e\rangle \in J'$. □

To continue the proof of Theorem 14.1.1, define, for any $\langle x,y\rangle \in P$,

$$D_{xy} = \{\langle a,e\rangle : a \in \mathbb{G} \wedge \langle a\cdot x, e\rangle \in J' \wedge \langle x,y\rangle \in (W)_{a\cdot x, e}\}.$$

Clearly $\langle x,y\rangle \mapsto D_{x,y}$ is a Δ^1_1 map $P \to \mathscr{P}(\mathbb{G}\times\mathbb{N})$.

If $D \subseteq \mathbb{G}\times\mathbb{N}$ and $b\in\mathbb{G}$, then put $b\circ D = \{\langle ab^{-1}, e\rangle : \langle a,e\rangle \in D\}$.

LEMMA 14.1.5. *Suppose that $\langle x,y\rangle$ and $\langle x',y'\rangle$ belong to P, $b \in \mathbb{G}$, and $x' = b\cdot x$. Then $\langle x,y\rangle \mathsf{E} \langle x',y'\rangle$ iff $b\circ D_{xy} = D_{x'y'}$.*

PROOF. Assume that $b\circ D_{xy} = D_{x'y'}$. According to Corollary 14.1.4, to establish $\langle x,y\rangle \mathsf{E} \langle x',y'\rangle$, it suffices to prove that $\langle x,y\rangle \in (W)_{xe} \Longleftrightarrow \langle x',y'\rangle \in (W)_{xe}$ holds whenever $\langle x,e\rangle \in J'$. We have

$$\begin{aligned}\langle x,y\rangle \in (W)_{xe} \quad\Longleftrightarrow\quad \langle \Lambda, e\rangle \in D_{xy} \quad&\Longleftrightarrow\quad \langle b^{-1}, e\rangle \in D_{x'y'}\\ &\Longleftrightarrow\quad \langle x',y'\rangle \in (W)_{b^{-1}\cdot x', e} = (W)_{xe},\end{aligned}$$

as required. To prove the converse, suppose that $\langle x,y\rangle \mathsf{E} \langle x',y'\rangle$. If $\langle a,e\rangle \in D_{xy}$, then $\langle a\cdot x, e\rangle$ belongs to J' and $\langle x,y\rangle \in (W)_{a\cdot x, e}$. Hence, $\langle x',y'\rangle \in (W)_{a\cdot x, e}$ also, because the set $(W)_{a\cdot x, e}$ is invariant and $\langle x,y\rangle \mathsf{E} \langle x',y'\rangle$. Yet $a\cdot x = ab^{-1}\cdot x'$, therefore, by definition, $\langle ab^{-1}, e\rangle \in D_{x'y'}$. The same argument can be carried out in the opposite direction, so that $\langle a,e\rangle \in D_{xy}$ iff $\langle ab^{-1}, e\rangle \in D_{x'y'}$, that means $b\circ D_{xy} = D_{x'y'}$. □ (Lemma)

To end the proof of the theorem, consider $\mathbb{S} = \mathbb{X}\times\mathscr{P}(\mathbb{G}\times\mathbb{N})$, a Polish space. Define a Borel action $b\cdot\langle x, D\rangle = \langle b\cdot x, b\circ D\rangle$ of $\mathbb{G}$ on $\mathbb{S}$. We assert that $\vartheta(x,y) = \langle x, D_{xy}\rangle$ is a Borel reduction of $\mathsf{E}\restriction P$ to the action $\mathsf{E}^{\mathbb{S}}_{\mathbb{G}}$. Indeed, let $\langle x,y\rangle$ and $\langle x',y'\rangle$ belong to P. Suppose that $\langle x,y\rangle \mathsf{E} \langle x',y'\rangle$. Then $x \mathsf{E}^{\mathbb{X}}_{\mathbb{G}} x'$, so that $x' = b\cdot x$ for some $b\in\mathbb{G}$. Moreover, $b\circ D_{xy} = D_{x'y'}$ by Lemma 14.1.5, hence, $\vartheta(x',y') = b\cdot\vartheta(x,y)$. Conversely, let $\vartheta(x',y') = b\cdot\vartheta(x,y)$, so that $x' = b\cdot x$ and $D_{x'y'} = b\circ D_{xy}$. Then $\langle x,y\rangle \mathsf{E} \langle x',y'\rangle$ by Lemma 14.1.5, as required. □

Theorem 14.1.1 is a particular case of the following conjecture which in general remains open. The conjecture can hardly be true in all cases, but surprisingly enough we do not know of any counterexample.

CONJECTURE 14.1.6 (ZAPLETAL). Suppose that X, Y, and $P \subseteq X\times Y$ are Borel sets, F and E are Borel equivalence relations on X and P, respectively, such that $\langle x,y\rangle \mathsf{E} \langle x',y'\rangle$ implies $x \mathsf{F} x'$, and for every E-class $[x_0]_{\mathsf{E}} \subseteq X$ the saturated cross-section $P^*(x_0) = \{\langle x,y\rangle \in P : x \mathsf{E} x_0\}$ satisfies $\mathsf{F}\restriction P^*(x_0) \le_{\mathrm{B}} \mathsf{G}$, where G is a one more Borel equivalence relation. Then $\mathsf{E}\restriction P \le_{\mathrm{B}} (\mathsf{F}\times\mathsf{G})$. □

Theorem 14.1.1 is related to the case when G is smooth and F is a countable Borel equivalence relation in Conjecture 14.1.6. It will be applied below in the particular case when $\mathbb{G}$ is the Δ-group on $\mathscr{P}_{\mathtt{fin}}(\mathbb{N})$ that induces hyperfinite equivalence relations, that is, the case $\mathsf{F} = \mathsf{G} = \mathsf{E}_0$ and E =smooth in Conjecture 14.1.6.

14.2. The two cases

Similar to the proof of Theorem 5.7.3 (see Section 11.3), Corollary 5.2.2 reduces Theorem 5.7.6 to the following particular form.

THEOREM 14.2.1. *Let $D \subseteq (2^{\mathbb{N}})^{\mathbb{N}}$ be a $\mathbf{\Sigma}^1_1$ set. Then* either $\mathsf{E}_3\restriction D \le_{\mathrm{B}} \mathsf{E}_0$ or $\mathsf{E}_3 \sqsubseteq_{\mathrm{B}} \mathsf{E}_3\restriction D$ — *and* then $\mathsf{E}_3\restriction D \sim_{\mathrm{B}} \mathsf{E}_3$.

PROOF. As usual, *w.l.o.g.* we can assume that the set D in the theorem is a lightface Σ^1_1 set. The proof of the theorem begins with a few definitions necessary to set up the partition onto cases here.

For $x, y \in (2^{\mathbb{N}})^{\mathbb{N}}$ and $n \in \mathbb{N}$, define $x \equiv_n y$ iff $x \mathsf{E}_3 y$ and $x \restriction_{\le n} = y \restriction_{\le n}$. (The latter requirement means $x(k) = y(k)$ for all $k \le n$.) Let $\mathscr{A}^{\mathtt{fin}}_{nk}$, resp., $\mathscr{A}^1_{nk}$, denote the collection of all non-empty Σ^1_1 sets $A \subseteq (2^{\mathbb{N}})^{\mathbb{N}}$ satisfying the first, resp., second, of the following two conditions:

$$\forall x \in A \,(\text{the set } \{y(k) : y \in A \wedge x \equiv_n y\} \text{ is finite});$$
$$\forall x \in A \,(\text{the set } \{y(k) : y \in A \wedge x \equiv_n y\} \text{ is a singleton}).$$

LEMMA 14.2.2. *If $A \in \mathscr{A}^1_{nk}$, then there is a Δ^1_1 set $B \in \mathscr{A}^1_{nk}$ with $A \subseteq B$.*

PROOF. The Π^1_1 set

$$P = \{y \in (2^{\mathbb{N}})^{\mathbb{N}} : \forall x \in A \; (x \equiv_n y \Longrightarrow x(k) = y(k))\}$$

satisfies $A \subseteq P$, hence by Theorem 2.3.2 (Separation) there is a Δ^1_1 set U such that $A \subseteq U \subseteq P$. The set

$$X = \{x \in U : \forall y \in U \; (x \equiv_n y \Longrightarrow x(k) = y(k))\}$$

is still Π^1_1 and $A \subseteq X$ because $A \subseteq U \subseteq P$. Therefore, any Δ^1_1 set B such that $A \subseteq B \subseteq X$ is as required. □

COROLLARY 14.2.3. *The sets*[1] *$A^1_{nk} = \bigcup \mathscr{A}^1_{nk}$ belong to Π^1_1 uniformly on n, k. Therefore, the set $\mathbf{S} = \bigcup_n \bigcap_{k>n} A^1_{nk}$ also belongs to Π^1_1.*

The set $\mathbf{S}$ is the structural domain here.

PROOF. The result follows from Lemma 14.2.2 by standard computations based on the coding of Δ^1_1 sets (see Section 2.8) and Theorem 2.8.1. □

LEMMA 14.2.4. *$\mathscr{A}^1_{nk} \subseteq \mathscr{A}^{\mathtt{fin}}_{nk}$, and conversely, every $A \in \mathscr{A}^{\mathtt{fin}}_{nk}$ is covered by a union of sets in $\mathscr{A}^1_{nk}$. Therefore, each set A^1_{nk} is equal to $A^{\mathtt{fin}}_{nk} = \bigcup \mathscr{A}^{\mathtt{fin}}_{nk}$.*

PROOF. Given A in $\mathscr{A}^{\mathtt{fin}}_{nk}$, consider the Π^1_1 set B of all $y \in (2^{\mathbb{N}})^{\mathbb{N}}$ such that

$$\exists s \in 2^{<\omega} \; (s \subset y(k) \;\wedge\; \forall x \in A \; (x \equiv_n y \wedge s \subset x(k) \Longrightarrow x(k) = y(k))).$$

Then obviously $A \subseteq B$, and hence there is a Δ^1_1 set U such that $A \subseteq U \subseteq B$. As $U \subseteq B$, Theorem 2.4.5 (Kreisel Selection) yields a Δ^1_1 map $\sigma : U \to 2^{<\omega}$ such that

$$\forall y \in U \; (\sigma(y) \subset y(k) \;\wedge\; \forall x \in A \; (x \equiv_n y \wedge \sigma(y) \subset x(k) \Longrightarrow x(k) = y(k))).$$

Then $A = \bigcup_s A_s$, where each $A_s = \{x \in A : \sigma(x) = s \subset x(k)\}$ is a set in $\mathscr{A}^1_{nk}$ or the empty set. □

Now let us come back to Theorem 14.2.1. We have two cases.

Case 1. $D \subseteq \mathbf{S}$. We will show below that in this case $\mathsf{E}_3 \restriction D \le_{\mathrm{B}} \mathsf{E}_0$.

Case 2. $D \setminus \mathbf{S} \ne \varnothing$. We will prove that then $\mathsf{E}_3 \sqsubseteq_{\mathrm{B}} \mathsf{E}_3 \restriction D$.

Let us finish this section with a few remarks connecting the partition to cases with the material in Sections 10.8 and 10.9. Recall that, for $x, y \in (2^{\mathbb{N}})^{\mathbb{N}}$, $x \equiv_n y$ means that $x \restriction_{\le n} = y \restriction_{\le n}$ and $x \mathsf{E}_3 y$, that is, $x(j) \mathsf{E}_0 y(j)$ for all j. Let $x \equiv^k_n y$ mean that still $x \restriction_{\le n} = y \restriction_{\le n}$ and $x(j) \mathsf{E}_0 y(j)$ for all $j \ne k$. In these terms, sets $A \in \mathscr{A}^1_{nk}$

[1] A^1_{nk} is the union of all sets in $\mathscr{A}^1_{nk}$.

are characterized by the requirement that for every $x \in A$ the set $A'_k(x) = \{y(k) : y \in A \wedge x \equiv_n^k y\}$ is an E_0-transversal. Thus the ground idea behind the definitions here can be formulated as a kind of unusual and not easily formalizable countable product of the ideal $\mathscr{I}_{\mathsf{E}_0}$ defined in Section 10.8.

14.3. Case 1

The proof of the next theorem shows that the Case 1 assumption makes all E_3-classes inside the domain R look in a sense similar to E_0-classes. This will allow us to employ Theorem 14.1.1 to obtain the result required.

THEOREM 14.3.1. *If* $D \subseteq (2^{\mathbb{N}})^{\mathbb{N}}$ *is an arbitrary* Σ^1_1 *set and* $D \subseteq \mathbf{S}$, *then* $\mathsf{E}_3 \restriction D \leq_{\mathrm{B}} \mathsf{E}_0$, *the "either" case of Theorem* 14.2.1.

PROOF. By Separation there is a Δ^1_1 set D' such that $D \subseteq D'$ and still $D' \subseteq \mathbf{S}$. Thus it can be assumed that D is Δ^1_1. Then by Kreisel Selection (Theorem 2.4.5) there exists a Δ^1_1 map $\nu : D \to \mathbb{N}$ such that

$$\forall k > \nu(x)\, \exists B \in \mathscr{A}^1_{\nu(x)k}\ (x \in B \in \Delta^1_1)$$

for all $x \in D$. Put $D_n = \{x \in D : \nu(x) \leq n\}$, these are increasing Δ^1_1 subsets of D, and $D = \bigcup_n D_n$. According to Corollary 7.3.2, it suffices to prove that $\mathsf{E}_3 \restriction D_n \leq_{\mathrm{B}} \mathsf{E}_0$ for every n. Thus let us fix n. Then by definition

$$(*) \qquad \forall x \in D_n\, \forall k > n\, \exists B \in \mathscr{A}^1_{nk}\ (x \in B \in \Delta^1_1).$$

Recall that $\mathbf{C}$ is the least class of sets containing all open sets and closed under the A-operation and the complement. A map f is called $\mathbf{C}$-*measurable* iff all f-preimages of open sets belong to $\mathbf{C}$.

LEMMA 14.3.2. *For every n there is a* $\mathbf{C}$-*measurable map* $f : D_n \to (2^{\mathbb{N}})^{\mathbb{N}}$ *such that* $f(x) = f(y) \equiv_n x$ *whenever* $x, y \in D_n$ *satisfy* $x \equiv_n y$.

PROOF. Let $\mathsf{Cod}(\Delta^1_1) \subseteq \mathbb{N}$ be the Π^1_1 set of codes for Δ^1_1 subsets of $(2^{\mathbb{N}})^{\mathbb{N}}$, and let $(W)_e \subseteq (2^{\mathbb{N}})^{\mathbb{N}}$ be the Δ^1_1 set coded by $e \in \mathsf{Cod}(\Delta^1_1)$. (We refer to Theorem 2.8.1.) Then, by $(*)$ above,

$$\forall x \in D_n\, \forall k > n\, \exists e \in \mathsf{Cod}(\Delta^1_1)\ (x \in (W)_e \in \mathscr{A}^1_{nk}).$$

Now a straightforward application of the Kreisel Selection yields a Δ^1_1 map $\varepsilon : D_n \times \mathbb{N} \to \mathsf{Cod}(\Delta^1_1)$ such that $x \in (W)_{\varepsilon(x,k)} \in \mathscr{A}^1_{nk}$ holds whenever $x \in D_n$ and $k > n$. Let $\tilde{\varepsilon}(x,k)$ be the least of all numbers $\varepsilon(x',k)$ with $x' \in D_n \cap [x]_{\equiv_n}$. Then $\tilde{\varepsilon}$ is $\equiv_n$-invariant in the first argument. In addition, $(W)_{\tilde{\varepsilon}(x,k)} \in \mathscr{A}^1_{nk}$ and the set $Z_{xk} = D_n \cap [x]_{\equiv_n} \cap (W)_{\tilde{\varepsilon}(x,k)}$ is non-empty whenever $x \in D_n$ and $k > n$.

Let $x \in D_n$. For every $k > n$, the set $Y_{xk} = \{y(k) : y \in Z_{xk}\} \subseteq 2^{\mathbb{N}}$ is a singleton by the definition of $\mathscr{A}^1_{nk}$. Let $f_k(x)$ be its only element. Define $f(x) \in (2^{\mathbb{N}})^{\mathbb{N}}$ so that $f(x)(k) = x(k)$ for $k \leq n$ and $f(x)(k) = f_k(x)$ for $k > n$.

That $f(x) = f(y)$ whenever $x \equiv_n y$ follows from the invariance of ε. To see that $f(x) \equiv_n x$, note that by definition $f_k(x) \mathsf{E}_0 x_k$ for $k > n$: indeed, $f_k(x) = y(k)$ for some $y \in [x]_{\equiv_n}$, but $x \equiv_n y$ implies $x(k)\ \mathsf{E}_0\ y(k)$ for all k. Finally, the $\mathbf{C}$-measurability needs a routine check. □ (Lemma)

For any $u \in (2^{\mathbb{N}})^n$ define $D_n(u) = \{x \in D_n : x \restriction_{<n} = u\}$.

LEMMA 14.3.3. *If* $u \in (2^{\mathbb{N}})^n$, *then* $\mathsf{E}_3 \restriction D_n(u)$ *is smooth.*

PROOF. As E_3 and $\equiv_n$ coincide on $D_n(u)$, the relation $\mathsf{E}_3 \restriction D_n(u)$ is smooth by means of a **C**-measurable, hence, a Baire measurable map. Suppose toward the contrary, that it is not really smooth, i.e., smooth by means of a Borel map. Then, by the 2nd dichotomy theorem (Theorem 10.4.1), we have $\mathsf{E}_0 \leq_{\mathrm{B}} \mathsf{E}_3 \restriction D_n(u)$, hence, E_0 turns out to be smooth by means of a Baire measurable map, which is easily impossible; see the proof of Corollary 10.7.2. □ (Lemma)

To complete the proof of Theorem 14.3.1, let $\mathbb{G}$ denote the group $\mathscr{P}_{\mathtt{fin}}(\mathbb{N})^n$, that is, the product of n copies of $\langle \mathscr{P}_{\mathtt{fin}}(\mathbb{N})\,;\triangle\rangle$. Let $\mathbb{G}$ act on $\mathbb{X} = (2^{\mathbb{N}})^n$ componentwise and as defined in Example 4.4.2 on each of the n coordinates. Then, for every pair of $u, v \in \mathbb{X}$, $u \,\mathsf{E}_{\mathbb{G}}^{\mathbb{X}}\, v$ is equivalent to $\forall\, k < n\,(u(k)\,\mathsf{E}_0\, v(k))$. Apply Theorem 14.1.1 with $\mathbb{G}$ and $\mathbb{X}$ as indicated, and $P = D_n$, $\mathsf{E} = \mathsf{E}_3 \restriction D_n$. Lemma 14.3.3 witnesses the principal condition. Thus $\mathsf{E}_3 \restriction D_n$ is Borel reducible to an equivalence relations induced by a Borel action of $\mathbb{G}$. Yet the group $\mathbb{G}$ is the increasing union of a countable sequence of its finite subgroups. It follows that all equivalence relations induced by a Borel action of $\mathbb{G}$ are hyperfinite, therefore Borel reducible to E_0.

□ (Theorem 14.3.1 and Case 1 in Theorem 14.2.1)

14.4. Case 2

In this case the Σ^1_1 set $D \cap \mathbf{H}$, where $\mathbf{H} = 2^{\mathbb{N}} \smallsetminus \mathbb{S}$ is the chaotic domain, is non-empty. A rather typical example is

$$D = \{x \in (2^{\mathbb{N}})^{\mathbb{N}} : \forall\, n, k, l\, (x(\prec n, k\succ) = x(\prec n, l\succ))\},$$

where $n, k \mapsto \prec n, k\succ = 2^n(2k+1)-1$ is a pairing function on $\mathbb{N}$. Thus members of D are those infinite sequences of elements of $2^{\mathbb{N}}$ in which every term is duplicated in infinitely many copies. It can be verified that the intersection $\mathbb{S} \cap D$ consists of all sequences $x \in D$ that contain a finite number of terms $x(0), \ldots, x(n)$ such that all other terms are Δ^1_1 in $x(0), \ldots, x(n)$. Obviously the difference $D \smallsetminus \mathbb{S}$ is non-empty.

We are going to prove

THEOREM 14.4.1. *If $D \subseteq (2^{\mathbb{N}})^{\mathbb{N}}$ is an arbitrary Σ^1_1 set and $D \not\subseteq \mathbb{S}$, then $\mathsf{E}_3 \sqsubseteq_{\mathrm{B}} \mathsf{E}_3 \restriction D$, the "or" case of Theorem 14.2.1.*

Let us take some space for technical notation involved in the proof of the theorem. Put

$$L(n) = \mathtt{max}\{r : \exists\, q\, (\prec r, q\succ \leq n)\} = \{r : 2^r - 1 \leq n\}$$

for any n. Then for instance $L(0) = 0$ and $L(1) = L(2) = 1$. If $r \leq L(n)$, then define $(n)_r = \{q : \prec r, q\succ \leq n\}$; this is a natural number ≥ 1 (assuming $r \leq L(n)$). For instance $(0)_0 = 1$ (since $\prec 0, 0\succ = 0$), $(1)_0 = 2$, and $(1)_1 = 1$. Obviously, $n = \sum_{r=0}^{L(n)-1} (n)_r$.

Suppose that $n \in \mathbb{N}$ and $s \in 2^n$ (a dyadic sequence of length n). For any $r < L(n)$ define $(s)_r \in 2^{(n)_r}$ so that $(s)_r(q) = s(\prec r, q\succ)$ for all $q < (n)_r$. Thus the original sequence $s \in 2^{<\omega}$ of length $\mathtt{lh}\, s = n$ is split into an $L(n)$-sequence of dyadic sequences of lengths $\mathtt{lh}\,(s)_r = (n)_r$. Formally this secondary sequence $\{(s)_r\}_{r<L(n)}$ belongs to the product set $\prod_{r=0}^{L(n)-1} 2^{(n)_r}$.

We consider $2^{\mathbb{N}}$ to be a group with componentwise operation; that is, if $a, b \in 2^{\mathbb{N}}$, then $a \cdot b \in 2^{\mathbb{N}}$ and $(a \cdot b)(k) = a(k) +_2 b(k)$, $\forall\, k$, where $+_2$ is the addition modulo 2. The neutral element is the constant-0 sequence $\mathbf{0} = \mathbb{N} \times \{0\}$ (that is, $\mathbf{0}(k) = 0$, $\forall k$), clearly $\mathbf{0} \cdot a = a$ for all $a \in 2^{\mathbb{N}}$.

Accordingly, consider $(2^{\mathbb{N}})^{\mathbb{N}}$ as the product of $\mathbb{N}$-many copies of $2^{\mathbb{N}}$, a group with the componentwise operation still denoted by $\cdot$, so that $(f \cdot g)(n)(k) = f(n)(k) +_2 g(n)(k)$ for all n, k. The neutral element is the constant-$\mathbf{0}$ sequence $\mathbf{0}^{\mathbb{N}} \in (2^{\mathbb{N}})^{\mathbb{N}}$. Define $\operatorname{supp} g = \{n \in \mathbb{N} : g(n) \neq \mathbf{0}\}$, *the domain of non-triviality* of an element $g \in (2^{\mathbb{N}})^{\mathbb{N}}$.

The group $(2^{\mathbb{N}})^{\mathbb{N}}$ contains the subgroup

$$\mathbf{F} = \{g \in (2^{\mathbb{N}})^{\mathbb{N}} : \forall n \, \exists k_0 \, \forall k \geq k_0 \, (f(n)(k) = 0)\},$$

essentially the ideal $\mathscr{I}_3$, acting on $(2^{\mathbb{N}})^{\mathbb{N}}$ by the group operation $\cdot$, and three more series of subgroups:

$$\begin{aligned} \mathbf{F}_{>n} &= \{g \in \mathbf{F} : \operatorname{supp} g \subseteq (n, \infty)\} &&= \{g \in \mathbf{F} : \forall k \leq n \, (g(k) = \mathbf{0})\}, \\ \mathbf{F}_{\geq n} &= \{g \in \mathbf{F} : \operatorname{supp} g \subseteq [n, \infty)\} &&= \{g \in \mathbf{F} : \forall k < n \, (g(k) = \mathbf{0})\}, \\ \mathbf{F}_{\leq n} &= \{g \in \mathbf{F} : \operatorname{supp} g \subseteq [0, n]\} &&= \{g \in \mathbf{F} : \forall k > n \, (g(k) = \mathbf{0})\}, \end{aligned}$$

for all n. Obviously $x \mathsf{E}_3 y$ iff $y \in \mathbf{F} \cdot x$, and $x \equiv_n y$ iff $y \in \mathbf{F}_{>n} \cdot x$,

Finally, if $X \subseteq (2^{\mathbb{N}})^{\mathbb{N}}$, then put $g \cdot X = \{g \cdot x : x \in X\}$.

PROOF (Theorem 14.4.1, the proof ends in Section 14.8). Put $H = D \smallsetminus \mathbf{S}$. Then $H = \bigcap_n \bigcup_{k>n} H_{nk}$, where

14.4.2. $$\begin{aligned} H_{nk} = D \smallsetminus A^1_{nk} &= \{x \in D : \forall A \, (x \in A \Longrightarrow A \notin \mathscr{A}^{\mathtt{fin}}_{nk})\} \\ &= \{x \in D : \forall A \in \Delta^1_1 \, (x \in A \Longrightarrow A \notin \mathscr{A}^1_{nk})\} \end{aligned}$$

by Lemmas 14.2.2 and 14.2.4, and H_{nk} is Σ^1_1 by Corollary 14.2.3. Note also that for every Σ^1_1 set $A \subseteq (2^{\mathbb{N}})^{\mathbb{N}}$ and every n, k the following holds:

14.4.3. $A \notin \mathscr{A}^{\mathtt{fin}}_{nk} \Longrightarrow \forall p \in \mathbb{N} \, \exists y, z \in A \, \exists j \geq p \, (y \equiv_n z \wedge y(k)(j) \neq z(k)(j))$.

14.5. Splitting system

To prove Theorem 14.4.1, we are going to define a rather complicated splitting system of non-empty Σ^1_1 sets $X_s \subseteq (2^{\mathbb{N}})^{\mathbb{N}}$, $s \in 2^{<\omega}$, the increasing sequence of numbers $k_0 < k_1 < k_2 < \cdots \in \mathbb{N}$, a collection of natural numbers p_{mj}, $m, j \in \mathbb{N}$, and elements $g_{st} \in \mathbf{F}$, where $s, t \in 2^{<\omega}$, $\mathtt{lh}\, s = \mathtt{lh}\, t$, satisfy the following list of requirements **1°**–**9°**:

1°. $X_\Lambda \subseteq H = D \smallsetminus \mathbf{S}$, $X_{s^\frown i} \subseteq X_s$, $\mathtt{diam}\, X_s \leq 2^{-\mathtt{lh}\, s}$.

2°. The same as 7° in Section 11.5, so that, as a consequence, $\bigcap_n X_{a \restriction n}$ is a singleton for every $a \in 2^{\mathbb{N}}$.

3°. If $s \in 2^{n+1}$, then $X_s \subseteq \bigcap_{r \leq L(n)} H_{rk_r}$.

4°. If $s, t \in 2^{n+1}$, then $\operatorname{supp} g_{st} \subseteq [0, k_{L(n)}]$, that is, $g_{st} \in \mathbf{F}_{\leq k_{L(n)}}$.

5°. $k_0 < k_1 < k_2 < \cdots$, and $p_{m0} < p_{m1} < p_{m2} < \cdots$ for all m.

6°. $g_{su} = g_{tu} \cdot g_{st}$ for all $s, t, u \in 2^{n+1}$. It easily follows that $g_{ss} = \mathbf{0}^{\mathbb{N}}$, $\forall s$.

To formulate three more requirements, define, for arbitrary sets $X, Y \subseteq (2^{\mathbb{N}})^{\mathbb{N}}$, $X \equiv_m Y$ iff $[X]_{\equiv_m} = [Y]_{\equiv_m}$; that is, for every $x \in X$ there exists $y \in Y$ satisfying $x \equiv_m y$ and *vice versa* for every $y \in Y$ there exists $x \in X$ satisfying $x \equiv_m y$. This is equivalent to $\mathbf{F}_{>m} \cdot X = \mathbf{F}_{>m} \cdot Y$.

7°. $g_{st} \cdot X_s \equiv_{k_{L(n)}} X_t$ holds for all $s, t \in 2^{n+1}$.

8°. If $s, t \in 2^{n+1}$, $\ell \le L(n)$, $n' \le n$, and $s', t' \in 2^{n'}$ satisfy $s' \subseteq s$, $t' \subseteq t$, and the equality $(s)_r(q) = (t)_r(q)$ holds whenever $r \le \ell$ and $q \in \mathbb{N}$ satisfy $n' \le \prec r, q \succ \le n$, then $g_{st}(i) = g_{s't'}(i)$ for all $i \le \ell$.

9°. If $s, t \in 2^{n+1}$, $s(n) = 0 \ne 1 = t(n)$, and $n = \prec m, j \succ$, then $x(k_m)(p_{mj}) = 0$ for all $x \in X_s$ but $y(k_m)(p_{mj}) = 1$ for all $y \in X_t$.

14.6. The embedding

Suppose that a system of sets X_s, elements g_{st}, and numbers k_m and p_{mj} satisfying **1°–9°** has been defined. Let us show that this leads to *the proof of Theorem* 14.4.1.

As usual it follows from **1°** and **2°** that for every $a \in 2^{\mathbb{N}}$ the intersection $\bigcap_n X_{a \restriction n}$ is a singleton in $(2^{\mathbb{N}})^{\mathbb{N}}$. Let us denote by $\vartheta(a) = \{\vartheta_n(a)\}_{n \in \mathbb{N}}$ its only element. Thus $a \mapsto \vartheta(a)$ is a map $2^{\mathbb{N}} \to H$ while each ϑ_n is a map $2^{\mathbb{N}} \to 2^{\mathbb{N}}$. In addition both ϑ and all ϑ_n are continuous (in the Polish product topology). We claim that the *set* $X = \{\vartheta(a) : a \in 2^{\mathbb{N}}\} = \bigcap_n \bigcup_{s \in 2^n} X_s$ *satisfies* $\mathsf{E}_3 \le_{\mathrm{B}} \mathsf{E}_3 \restriction X$. It follows that $\mathsf{E}_3 \le_{\mathrm{B}} \mathsf{E}_3 \restriction D$, and this proves Theorem 14.4.1.

To prove the claim, define a point $\varphi(a) = \{\varphi_n(a)\}_{n \in \mathbb{N}} \in (2^{\mathbb{N}})^{\mathbb{N}}$ for every $a \in 2^{\mathbb{N}}$ such that $\varphi_n(a)(k) = a(\prec n, k \succ)$ for all n, k. The map φ is a homeomorphism of $2^{\mathbb{N}}$ onto $(2^{\mathbb{N}})^{\mathbb{N}}$, while each $a \mapsto \varphi_n(a)$ is a continuous map $2^{\mathbb{N}} \to 2^{\mathbb{N}}$. Thus the following lemma implies the claim.

LEMMA 14.6.1. *The equivalence* $\varphi(a) \mathrel{\mathsf{E}_3} \varphi(b) \iff \vartheta(a) \mathrel{\mathsf{E}_3} \vartheta(b)$ *holds for all* $a, b \in 2^{\mathbb{N}}$. *Therefore the map* $f(x) = \vartheta(\varphi^{-1}(x)) : (2^{\mathbb{N}})^{\mathbb{N}} \to (2^{\mathbb{N}})^{\mathbb{N}}$ *is a continuous embedding (that is, an injective reduction) of* E_3 *in* $\mathsf{E}_3 \restriction X$.

PROOF. Assume that $\varphi(a) \mathrel{\mathsf{E}_3} \varphi(b)$, take an arbitrary $\ell \in \mathbb{N}$, and prove that $\vartheta_\ell(a) \mathrel{\mathsf{E}_0} \vartheta_\ell(b)$. In our assumptions there exists a number n' such that $\ell \le L(n')$ and for every $r \le \ell$ and q, if $\prec r, q \succ \ge n'$, then $a(\prec r, q \succ) = b(\prec r, q \succ)$. Put $s' = a \restriction n'$ and $t' = b \restriction n'$. Then $g' = g_{s't'} \in \mathbf{F}$. Our goal is to prove that $\vartheta_\ell(b) = (g')_\ell \cdot \vartheta_\ell(a)$; this obviously implies $\vartheta_\ell(a) \mathrel{\mathsf{E}_0} \vartheta_\ell(b)$.

It suffices to show that $g_{s't'} \cdot X_s \equiv_\ell X_t$ holds for all $n > n'$, where $s = a \restriction n$ and $t = b \restriction n$. We observe that $g_{st} \cdot X_s \equiv_\ell X_t$ by **7°** because $\ell \le L(n') \le k_{L(n')} \le k_{L(n)}$. On the other hand, $g_{st}(i) = g_{s't'}(i)$ for all $i \le \ell$ by **8°** and the choice of n'. It follows that $g_{s't'} \cdot X_s = g_{st} \cdot X_s \equiv_\ell X_t$, as required.

To prove the converse, suppose that $\varphi(a) \mathrel{\mathsf{E}_3} \varphi(b)$ fails, and hence there is at least one index m such that $\varphi_m(a) \mathrel{\mathsf{E}_0} \varphi_m(b)$ fails as well, meaning that $a(\prec m, j \succ) \ne b(\prec m, j \succ)$ holds for infinitely many numbers $j \in \mathbb{N}$. Then by **9°** we obtain $\vartheta_{k_m}(a)(p_{mj}) = 0 \ne 1 = \vartheta_{k_m}(b)(p_{mj})$ for all j, and hence $\vartheta_{k_m}(a) \mathrel{\mathsf{E}_0} \vartheta_{k_m}(b)$ fails since the numbers p_{mj}, $j \in \mathbb{N}$, strictly increase by **5°**. □

□ (Theorem 14.4.1 modulo the construction **1°–9°**)

This accomplishes Case 2 in the proof of Theorem 14.2.1.

□ (Theorems 14.2.1 and 5.7.6 modulo the construction **1°–9°**)

REMARK 14.6.2. The first part of the proof of the lemma demonstrates that for every ℓ, if $\varphi_r(a) \mathrel{\mathsf{E}_0} \varphi_r(b)$ for all $r \le \ell$, then $\vartheta_r(a) \mathrel{\mathsf{E}_0} \vartheta_r(b)$ for all $r \le \ell$ as well. Quite a similar argument shows that, for any $a, b \in 2^{\mathbb{N}}$, if $\varphi_r(a) = \varphi_r(b)$ for all $r \le \ell$, then $\vartheta_r(a) = \vartheta_r(b)$ for all $r \le \ell$. The principal ingredient here is 8° for $n' = 0$ and $s' = t' = \Lambda$. Then $g_{s't'} = \mathbf{0}^{\mathbb{N}}$. □

14.7. The construction of a splitting system: warmup

Now, to prove Theorems 14.4.1, 14.2.1, and 5.7.6, it remains to carry out the construction of a system of sets X_s and g_{st} and numbers k_m and p_{mj} satisfying conditions **1°–9°** of Section 14.5. The construction goes on by induction on n, so that at each step n we define the sets X_s, $s \in 2^n$ and elements g_{st}, $s, t \in 2^n$. Here we present only the transition from 0 to 1 as a warmup.

Put $X_\Lambda = H$ and[2] by default $g_{\Lambda\Lambda} = \mathbf{0}^{\mathbb{N}}$ for the sequence Λ of length 0.

At the next stage, we have to define Σ^1_1 sets $X_{\langle 0\rangle}, X_{\langle 1\rangle} \subseteq X_\Lambda$, an element $g_{\langle 0\rangle\langle 1\rangle} = g_{\langle 1\rangle\langle 0\rangle} \in \mathbf{F}$, and numbers k_0 and p_{00} such that a relevant fragment of **1°–9°** is satisfied. Note that $L(0) = 0$.

Stage 1. We shrink X_Λ to make sure that conditions **1°** and **2°** of Section 14.5 are satisfied; the resulting Σ^1_1 set is still denoted by X_Λ.

Stage 2. Consider any $x \in X_\Lambda$. Then $x \in \bigcap_{k>0} H_{0k}$ (see the beginning of the proof of Theorem 14.4.1). Fix a number $k = k_0 > 0$ such that $x \in H_{0k_0}$. The set $X'_\Lambda = X_\Lambda \cap H_{0k_0}$ is still of class Σ^1_1, and it does not belong to the family $\mathscr{A}^{\mathtt{fin}}_{0k_0}$ by 14.4.2. Thus by 14.4.3 there exist points $y_0, z_0 \in X'_\Lambda$ satisfying $y_0 \equiv_{L(0)} z_0$ and numbers $k_0 > 0 = L(0)$ and p_{00} such that $y_0(k_0)(p_{00}) = 0 \ne 1 = z_0(k_0)(p_{00})$. The Σ^1_1 sets

$$\begin{aligned} Y &= \{y \in X'_\Lambda : y \equiv_{L(0)} y_0 \wedge y(k_0)(p_{00}) = 0\}, \quad \text{and} \\ Z &= \{z \in X'_\Lambda : z \equiv_{L(0)} z_0 \wedge z(k_0)(p_{00}) = 1\} \end{aligned}$$

still contain elements y_0, z_0, respectively; therefore, so do the Σ^1_1 sets

$$\begin{aligned} Y' &= \{y' \in Y : \exists z \in Z\,(y' \equiv_{L(0)} z)\}, \quad \text{and} \\ Z' &= \{z' \in Z : \exists y \in Y\,(y \equiv_{L(0)} z')\}. \end{aligned}$$

Finally, define $g_{\langle 0\rangle\langle 1\rangle} = g_{\langle 1\rangle\langle 0\rangle} \in \mathbf{F}$ so that $g_{\langle 0\rangle\langle 1\rangle}(k_0)(p_{00}) = 1$ and $g_{\langle 0\rangle\langle 1\rangle}(m)(j) = 0$ for every other pair of m, j. Then easily $g_{\langle 0\rangle\langle 1\rangle} \cdot y_0 \equiv_{k_0} z_0$, hence $g_{\langle 0\rangle\langle 1\rangle} \cdot Y' \equiv_{k_0} Z'$. Thus we get a pair of sets $X_{\langle 0\rangle} = Y'$ and $X_{\langle 1\rangle} = Z'$ compatible with **7°**. This ends the construction for $n = 1$.

14.8. The construction of a splitting system: the step

Now suppose that $n = \prec m, j \succ \ge 1$, and the construction has been accomplished up to the level n; that is, there exist sets $X_s \subseteq H$ and elements $g_{st} \in \mathbf{F}$, where $s, t \in 2^{n'}$, $n' \le n$, and numbers $k_0, \dots, k_{L(n-1)}$ and $p_{m'j'}$, where $\prec m', j' \succ < n$, such that conditions **1°–9°** of Section 14.5 are satisfied in this domain. The goal is to define X_s and g_{st}, where $s, t \in 2^{n+1}$, and numbers k_n and p_{mj}, such that conditions **1°–9°** are satisfied in the extended domain.

The numbers n, m, j are fixed in the course of the arguments in this section.

LEMMA 14.8.1. *Suppose that collections of Σ^1_1 sets $P_s \subseteq (2^{\mathbb{N}})^{\mathbb{N}}$, $s \in 2^n$ and elements $g_{st} \in \mathbf{F}$, $s, t \in 2^n$, satisfy both* **6°** *and* **7°** *of Section 14.5 for a fixed k; that is, $g_{su} = g_{tu} \cdot g_{st}$ and $g_{st} \cdot P_s \equiv_k P_t$ for all $s, t, u \in 2^{n+1}$.*

[2] But just every non-empty Σ^1_1 set $H' \subseteq H$ can be considered to be X_Λ.

If $\sigma \in 2^n$ *and* $P \subseteq P_\sigma$ *is a non-empty* Σ^1_1 *set, then the sets*

$$P'_s = \{x \in P_s : \exists y \in P\,(g_{\sigma s} \cdot y \equiv_k x)\}, \quad s \in 2^n,$$

are non-empty Σ^1_1 *sets still satisfying* 7°*, i.e.* $g_{st} \cdot P'_s \equiv_k P'_t$ *for all* $s, t \in 2^{n+1}$.

Proof. Fix $s, t \in 2^n$. To show that $g_{st} \cdot X'_s \equiv_k X'_t$, consider any $x \in X'_s$, so that $g_{st} \cdot x \in g_{st} \cdot X'_s$. By definition there exists $y \in X$ such that $g_{\sigma s} \cdot y \equiv_k x$. It follows from **4°** that $g_{st} \in \mathbf{F}_{\le k}$, therefore $g_{st} \cdot g_{\sigma s} \cdot y \equiv_k g_{st} \cdot x$, that is, $g_{\sigma t} \cdot y \equiv_k g_{st} \cdot x$ by **6°**. However by definition $g_{\sigma t} \cdot y \in X'_t$, as required.

The converse is similar. □ (Lemma)

It follows from **4°** (in the domain 2^n) that there is a number $\mu \in \mathbb{N}$ such that $g_{st}(r)(q) = 1$ holds only in the case when both $r \le k_{L(n-1)}$ and $q \le \mu$. We proceed with several stages of successive reduction and splitting of the Σ^1_1 sets X_s, $s \in 2^n$. These further stages depend on whether the number $n = \prec m, j \succ$ considered opens a "new" axis k_m of splitting.

Case A. $j > 0$.

Then $n' = \prec m, j-1 \succ < n$, thus m is "old". Moreover, $L(n) = L(n-1)$. We have to define p_{mj} but needn't to define any new k_r.

Stage 1. Fix an arbitrary sequence $\sigma \in 2^n$; for instance this can be the sequence 0^n of n zeros. Consider an arbitrary $x \in X_\sigma$. Then $x \in H_{mk_m}$ by **3°**, and hence there exist points $y_0, z_0 \in X_\sigma$ and a number $p_{mj} > \mu$ such that $y_0 \equiv_{m-1} z_0$ and $y_0(k_m)(p_{mj}) = 0$ but $z_0(k_m)(p_{mj}) = 1$. Easily $p_{mj} > p_{m,j-1}$: indeed $p_{m,j-1} \le \mu$ by the choice of μ.

Stage 2. Define $g \in (2^{\mathbb{N}})^{\mathbb{N}}$ so that $g(r)(q) = 1$ iff both $m \le r \le k_{L(n)}$ and $y_0(r)(q) \ne z_0(r)(q)$. Then $g \in \mathbf{F}$ since $y \mathsf{E}_3 z$. Moreover we have $\operatorname{supp} g \subseteq [m, k_{L(n)}]$, in other words, $g \in \mathbf{F}_{\ge m} \cap \mathbf{F}_{\le k_{L(n)}}$. In addition $g(k_m)(p_{mj}) = 1$.

We observe that by definition $g \cdot y_0 \equiv_{k_{L(n)}} z_0$. Thus the Σ^1_1 sets

$$\begin{aligned} Y &= \{y \in X_\sigma : y(k_m)(p_{mj}) = 0 \wedge \exists z \in Z\,(z(k_m)(p_{mj}) = 1 \wedge g \cdot y \equiv_{k_{L(n)}} z)\}, \\ Z &= \{z \in X_\sigma : z(k_m)(p_{mj}) = 1 \wedge \exists y \in Y\,(y(k_m)(p_{mj}) = 0 \wedge g \cdot y \equiv_{k_{L(n)}} z)\} \end{aligned}$$

are still non-empty (containing, resp. y_0, z_0) and satisfy $g \cdot Y \equiv_{k_{L(n)}} Z$; in addition $y(k_m)(p_{mj}) = 0$ and $z(k_m)(p_{mj}) = 1$ for all $y \in Y$ and $z \in Z$.

As a matter of fact, w.l.o.g. we can assume that $Y \cup Z = X_\sigma$: indeed otherwise put $P = Y \cup Z$ and apply Lemma 14.8.1.

Stage 3. Put $X_{\sigma^\frown 0} = Y$ and $X_{\sigma^\frown 1} = Z$, thus

(a) $g \cdot X_{\sigma^\frown 0} \equiv_{k_{L(n)}} X_{\sigma^\frown 1}$,

and then

$$X_{s^\frown \xi} = \{x \in X_s : \exists y \in X_{\sigma^\frown \xi}\,(g_{\sigma s}(y) \equiv_{k_{L(n)}} x)\}$$

for all $s \in 2^n$ and $\xi = 0, 1$. It follows, by **7°** at the level n, that

(b) $X_{s^\frown \xi} \equiv_{k_{L(n)}} g_{\sigma s} \cdot X_{\sigma^\frown \xi}$ for all $s \in 2^n$ and $\xi = 0, 1$.

Put $g_{s^\frown \xi, t^\frown \xi} = g_{st}$ but $g_{s^\frown \xi, t^\frown (1-\xi)} = g_{st} \cdot g$ for all $s, t \in 2^n$ and $\xi = 0, 1$,[3] or saying it differently

[3] In the definition of g_{st}, we make use of the fact that $\langle (2^{\mathbb{N}})^{\mathbb{N}} ; \cdot \rangle$ is an abelian group. In the non-abelian case, we would have to define $g_{s^\frown i, t^\frown (1-i)} = g_{\sigma t} \cdot g \cdot g_{s\sigma}$ and, accordingly, change some other related definitions in a somewhat more complicated way.

(c) $g_{s^\frown\xi\,,\,t^\frown\eta} = g_{st} \cdot g^{\xi-\eta}$ for all $s, t \in 2^n$ and $\xi, \eta = 0, 1$,

where $g^1 = g^{-1} = g$ while $g^0 = \mathbf{0}^{\mathbb{N}}$ is the neutral element in $\langle (2^{\mathbb{N}})^{\mathbb{N}} ; \cdot \rangle$.

Stage 4. Lemma 14.8.1 allows us to reduce the sets X_s, $s \in 2^{n+1}$, in several rounds to make sure that conditions $\mathbf{1}^\circ$ and $\mathbf{2}^\circ$ are satisfied at level $n+1$; the resulting Σ^1_1 sets are still denoted by X_s.

This ends the transition from n to $n+1$. It remains for us to show that conditions $\mathbf{1}^\circ$–$\mathbf{9}^\circ$ of Section 14.5 are satisfied in the extended $(\le n+1)$-domain.

Verification. As $\mathbf{1}^\circ$ and $\mathbf{2}^\circ$ are explicitly fulfilled, $\mathbf{3}^\circ$ in Case 1 is vacuous, and $\mathbf{4}^\circ$ and $\mathbf{5}^\circ$ clearly hold by definition, we begin with $\mathbf{6}^\circ$. We have to prove that

$$g_{s^\frown\xi\,,\,u^\frown\zeta} = g_{t^\frown\eta\,,\,u^\frown\zeta} \cdot g_{s^\frown\xi\,,\,t^\frown\eta}$$

for all $s, t, u \in 2^n$ and $\xi, \eta, \zeta = 0, 1$. By definition this equality is equivalent to $g_{su} \cdot g^{\xi-\zeta} = g_{tu} \cdot g^{\eta-\zeta} \cdot g_{st} \cdot g^{\xi-\eta}$. However obviously $g^{\xi-\zeta} = g^{\eta-\zeta} \cdot g^{\xi-\eta}$, and on the other hand in our assumptions $g_{su} = g_{tu} \cdot g_{st}$ by $\mathbf{6}^\circ$ at level n.

Let us check $\mathbf{7}^\circ$, that is, $g_{s^\frown\xi\,,\,t^\frown\eta} \cdot X_{s^\frown\xi} \equiv_{k_{L(n)}} X_{t^\frown\eta}$ for all $s, t \in 2^n$ and $\xi, \eta = 0, 1$. It follows from (b) that the left-hand side is $\equiv_{k_{L(n)}}$-equivalent to $g_{st} \cdot g^{\xi-\eta} \cdot g_{\sigma s} \cdot X_{\sigma^\frown\xi}$ while the right-hand side is $\equiv_{k_{L(n)}}$-equivalent to $g_{\sigma t} \cdot X_{\sigma^\frown\eta}$. On the other hand it follows from (a) that $g^{\xi-\eta} \cdot X_{\sigma^\frown\xi} \equiv_{k_{L(n)}} X_{\sigma^\frown\eta}$. This allows us to easily get the result required.

Let us check $\mathbf{8}^\circ$. Suppose that s, t, ℓ, n', s', t' are as indicated in $\mathbf{8}^\circ$. Then $s = \bar{s}^\frown\xi$ and $t = \bar{t}^\frown\eta$, where $\bar{s}, \bar{t} \in 2^n$ while $\xi = s(n)$ and $\eta = t(n)$ are numbers in $\{0, 1\}$. Then $g_{\bar{s}\bar{t}}(i) = g_{s't'}(i)$ for all $i \le \ell$ by $\mathbf{8}^\circ$ in the domain 2^n. Thus if $\xi = \eta$, then the result holds immediately because then $g_{st} = g_{\bar{s}\bar{t}}$ by (c). Assume that, e.g., $\xi = 0$ and $\eta = 1$. Then $\ell < m$ in the assumptions of $\mathbf{8}^\circ$, and hence the set $\operatorname{supp} g$ does not contain numbers $i \le \ell$; in other words, $g(i) = \mathbf{0}$ for all $i \le \ell$. It follows that $g_{st}(i) = g_{\bar{s}\bar{t}}(i)$ for every $i \le \ell$, as required.

We finally check $\mathbf{9}^\circ$. Suppose that $s^\frown\xi$ and $t^\frown\eta$ belong to 2^{n+1} and $\xi \ne \eta$, say $\xi = 0 \ne 1 = \eta$. We have to prove that $x(k_m)(p_{mj}) = \xi$ for all $\xi = 0, 1$ and $x \in X_{s^\frown\xi}$. First of all note that by definition $x(k_m)(p_{mj}) = \xi$ for all $x \in X_{\sigma^\frown\xi}$. On the other hand $g_{\sigma s}(k_m)(p_{mj}) = 0$ since $p_{mj} > \mu$ by construction. Thus $(g_{\sigma s} \cdot x)(k_m)(p_{mj}) = \xi$ for all $x \in X_{\sigma^\frown\xi}$. It remains to use (b).

Case B. $j > 0$.

Then there is no number $n' = \prec m', j' \succ < n$ such that $m' = m$ — in other words, m is "new". Obviously $m = L(n-1) + 1 = L(n)$ in this case.

Stage 1. The first goal is to appropriately choose a number k_m. Let us fix an arbitrary $\sigma \in 2^n$. Consider any $x \in X_\sigma$. As $X_\sigma \subseteq X_\Lambda \subseteq H = \bigcap_n \bigcup_{k>n} H_{nk}$, it follows from 14.4.2 that $x \in H_{(k_{L(n-1)}+1)\,k_m}$ for some $k_m > k_{L(n-1)} + 1$. In particular $k_m > k_{m-1}$, $k_m > L(n)$, and $x \in H_{L(n)\,k_m}$. Fix such a number k_m. It can be assumed *w.l.o.g.* that $X_\sigma \subseteq H_{L(n)\,k_m}$. (Indeed otherwise we can replace the set X_σ by $X'_\sigma = X_\sigma \cap H_{L(n)k_m}$, still a non-empty Σ^1_1 set, and apply Lemma 14.8.1 to shrink all sets X_s, $s \in 2^n$, accordingly.)

LEMMA 14.8.2. *In this assumption,* $X_s \subseteq H_{L(n)\,k_m}$ *for all* $s \in 2^n$.

PROOF. Consider an arbitrary point $x_0 \in X_s$ and prove that $x_0 \in H_{L(n)\,k_m}$. Fix an arbitrary number p and a Δ^1_1 set $A \subseteq (2^{\mathbb{N}})^{\mathbb{N}}$ containing x_0. We have to show that A contains a pair of points x, x' such that $x \equiv_{L(n)} x'$ but $x'(k_m)(j) \neq x(k_m)(j)$ for some $j > p$.

Recall that $X_s \equiv_{k_{L(n-1)}} g_{\sigma s} \cdot X_\sigma$ by **7°** in the domain 2^n, therefore there is a point $y_0 \in X_\sigma$ satisfying $x_0 \equiv_{k_{L(n-1)}} g_{\sigma s} \cdot y_0$. Then there exists an element $g \in \mathbf{F}$ with $\operatorname{supp} g \subseteq [0, k_m]$ such that $x_0 \equiv_{k_m} g \cdot y_0$. And it is clear that g extends $g_{\sigma s}$ in the sense that $g(r) = g_{\sigma s}(r)$ for all $r \le L(n-1)$.

The set $B = \{y \in (2^{\mathbb{N}})^{\mathbb{N}} : \exists x \in A\, (g \cdot y \equiv_{k_m} x)\}$ is then a Σ^1_1 set containing y_0. But in our assumptions $y_0 \in X_\sigma \subseteq H_{L(n)\,k_m}$, and hence there exist points $y, y' \in B$ and a number $j > p$ such that $y \equiv_{L(n)} y'$ but $y'(k_m)(j) \neq y(k_m)(j)$. By definition there exist points $x, x' \in A$ satisfying $g \cdot y \equiv_{k_m} x$ and $g \cdot y' \equiv_{k_m} x'$. In particular $x(r) = g(r) \cdot y(r)$ and $x'(r) = g(r) \cdot y'(r)$ for all $r \le k_n$. We conclude that $x \equiv_{L(n)} x'$ but $x'(k_m)(j) \neq x(k_m)(j)$, as required. □ (Lemma)

Stage 2. It follows from 14.4.3 that there exist points $y_0, z_0 \in X_\sigma$ and a number $p_{m0} \in \mathbb{N}$ such that $y_0 \equiv_{L(n)} z_0$ and $y_0(k_m)(p_{m0}) = 0 \neq 1 = z_0(k_m)(p_{m0})$. Following the construction in Case A, define $g \in \mathbf{F}_{\ge m} \cap \mathbf{F}_{\le k_{L(n)}}$ so that $g \cdot y_0 \equiv_{k_{L(n)}} z_0$, in particular, $g(k_m)(p_{m0}) = 1$. Then the Σ^1_1 sets

$$\begin{aligned} Y &= \{y \in X_\sigma : y(k_m)(p_{m0}) = 0 \wedge \exists z \in Z\, (z(k_m)(p_{m0}) = 1 \wedge g \cdot y \equiv_{k_{L(n)}} z)\}, \\ Z &= \{z \in X_\sigma : z(k_m)(p_{m0}) = 1 \wedge \exists y \in Y\, (y(k_m)(p_{m0}) = 0 \wedge g \cdot y \equiv_{k_{L(n)}} z)\} \end{aligned}$$

are still non-empty sets containing y_0, z_0, respectively, and satisfying $g \cdot Y \equiv_{k_{L(n)}} Z$. In addition $y(k_m)(p_{m0}) = 0$ and $z(k_m)(p_{m0}) = 1$ for all $y \in Y$ and $z \in Z$. And still *w.l.o.g.* we can assume that $Y \cup Z = X_\sigma$.

Stage 3. We define the sets $X_{s^\frown\xi} \subseteq X_s$ and elements $g_{s^\frown\xi, t^\frown\eta}$ ($s, t \in 2^n$ and $\xi = 0, 1$) exactly as in Stage 3 of Case A. Conditions (a), (b), and (c) still hold and by the same reasons.

Stage 4. Shrink the sets X_s, $s \in 2^{n+1}$, with the help of Lemma 14.8.1, in several rounds, so that the resulting Σ^1_1 sets, still denoted by X_s, satisfy **1°** and **2°** in the domain 2^{n+1}. This completes the transition from n to $n+1$.

Verification. A new feature here in comparison to Case A is the non-vacuous character of condition **3°**. It suffices to show that $X_{s^\frown\xi} \in H_{L(n)\,k_m}$ for all $s \in 2^n$ and $\xi = 0, 1$, or, which is sufficient, $X_s \in H_{L(n)\,k_m}$ for all $s \in 2^n$ — but this follows from Lemma 14.8.2. The verification of **4°–9°** is quite similar to the verification in Case A. We leave it to the reader. □ (The construction)

This finally accomplishes the construction of a system of sets X_s and g_{st} and numbers k_m and p_{mj} satisfying conditions **1°–9°** of Section 14.5, and the proof of Theorems 14.4.1, 14.2.1, and 5.7.6; see the end of Section 14.6.

14.9. A forcing notion associated with E_3

Following Section 11.7, define a forcing notion $\mathbb{P}_{\mathsf{E}_0\mathsf{E}_3}$ that consists of all $\mathbf{\Sigma}^1_1$ sets $X \subseteq (2^{\mathbb{N}})^{\mathbb{N}}$ such that $\mathsf{E}_3 \restriction X \sim_{\mathrm{B}} \mathsf{E}_3$. Theorem 14.2.1 implies that the associated ideal $\mathscr{I}_{\mathsf{E}_0\mathsf{E}_3}$ consists of all $\mathbf{\Sigma}^1_1$ sets $X \subseteq (2^{\mathbb{N}})^{\mathbb{N}}$ satisfying $\mathsf{E}_3 \restriction X \le_{\mathrm{B}} \mathsf{E}_0$. Thus for a Borel set $X \subseteq (2^{\mathbb{N}})^{\mathbb{N}}$ to be in $\mathscr{I}_{\mathsf{E}_0\mathsf{E}_3}$, it is necessary and sufficient that $\mathsf{E}_3 \restriction X$ is a hyperfinite equivalence relation. $\mathscr{I}_{\mathsf{E}_0\mathsf{E}_3}$ is a σ-ideal by Corollary 7.3.2.

EXERCISE 14.9.1. Using Theorems 14.3.1 and 14.4.1, prove that for a Σ^1_1 set $D \subseteq (2^{\mathbb{N}})^{\mathbb{N}}$ to belong to $\mathbb{P}_{\mathsf{E}_0\mathsf{E}_3}$ it is necessary and sufficient that $D \not\subseteq \mathbb{S}$. □

The forcing $\mathbb{P}_{\mathsf{E}_0\mathsf{E}_3}$ has properties roughly similar to those of $\mathbb{P}_{\mathsf{E}_0\mathsf{E}_1}$ in Section 11.7, in partucular:

LEMMA 14.9.2. *Any set $X \in \mathbb{P}_{\mathsf{E}_0\mathsf{E}_3}$ contains a closed subset $Y \in \mathbb{P}_{\mathsf{E}_0\mathsf{E}_3}$, $Y \subseteq X$.*

PROOF. Suppose that X is a lightface Σ^1_1. Then $D \not\subseteq \mathbf{S}$ by the result of Exercise 14.9.1, and hence $X \in \mathbb{P}_{\mathsf{E}_0\mathsf{E}_3}$ contains a closed subset $Y \subseteq X$ still in $\mathbb{P}_{\mathsf{E}_0\mathsf{E}_3}$ by Theorem 14.4.1. □

The next theorem is similar to Theorem 11.7.4.

THEOREM 14.9.3. *$\mathbb{P}_{\mathsf{E}_0\mathsf{E}_3}$ forces the existence of a countable set $X \subseteq 2^{\mathbb{N}}$ of "old" elements not covered by any "old" set Y countable in the ground universe.*

PROOF. It follows from Lemma 14.9.2 that $\mathbb{P}_{\mathsf{E}_0\mathsf{E}_3}$ forces a sequence $\mathbf{x} \in (2^{\mathbb{N}})^{\mathbb{N}}$. We claim that the terms $\mathbf{x}(k)$ of this sequence are forced to be "old" reals. This is an immediate consequence (by Lemma 14.9.2) of the following lemma (in the ground universe):

LEMMA 14.9.4. *Assume that $D \in \mathbb{P}_{\mathsf{E}_0\mathsf{E}_3}$. Then for every k there exists a set $Y \in \mathbb{P}_{\mathsf{E}_0\mathsf{E}_3}$, $Y \subseteq D$ such that $x(k) = y(k)$ for all $x, y \in Y$.*

PROOF. As usual we assume that D is lightface Σ^1_1. Then Case 2 of Section 14.2 holds for D, since otherwise $\mathsf{E}_3 \restriction D \le_{\mathrm{B}} \mathsf{E}_0$; see Section 14.3. In other words, the Σ^1_1 set $H = D \setminus \mathbf{S}$ is non-empty. Then (see Lemma 14.6.1 and Remark 14.6.2) there exists a continuous injection $f : (2^{\mathbb{N}})^{\mathbb{N}} \to H$ satisfying $x \mathrel{\mathsf{E}_3} y \Longleftrightarrow f(x) \mathrel{\mathsf{E}_3} f(y)$ for all $x, y \in (2^{\mathbb{N}})^{\mathbb{N}}$, and also satisfying $f(x)(k) = f(y)(k)$ whenever $x, y \in (2^{\mathbb{N}})^{\mathbb{N}}$ satisfy $x(r) = y(r)$ for all $r \le k$.

Let us fix any $z \in (2^{\mathbb{N}})^{\mathbb{N}}$. Put $X = \{x \in (2^{\mathbb{N}})^{\mathbb{N}} : \forall r \le k\,(x(r) = z(r))\}$. Clearly, $\mathsf{E}_3 \restriction X \sim_{\mathrm{B}} \mathsf{E}_3$, and hence the set $Y = f[X]$ (still a compact subset of D) satisfies $\mathsf{E}_3 \restriction Y \sim_{\mathrm{B}} \mathsf{E}_3$ by the choice of f, thus $Y \in \mathbb{P}_{\mathsf{E}_0\mathsf{E}_3}$. Finally, it follows from the second property of f that $y(k) = x(k)$ for all $x, y \in Y$. □ (Lemma)

Now to accomplish the proof of the theorem, it suffices to show that for every countable set $W = \{a_k : k \in \mathbb{N}\} \subseteq 2^{\mathbb{N}}$ in the ground universe, $\mathbb{P}_{\mathsf{E}_0\mathsf{E}_3}$ forces that there is k such that $\mathbf{x}(k) \notin W$. The set $X = \{x \in (2^{\mathbb{N}})^{\mathbb{N}} : \forall n\,(x(n) \in W)\}$ is Borel, more exactly, lightface Π^0_3. We claim that X does not belong to $\mathbb{P}_{\mathsf{E}_0\mathsf{E}_3}$. Indeed otherwise $X \not\subseteq \mathbf{S}$ by the result of Exercise 14.9.1. Then (Lemma 14.6.1) there exists a continuous injection $f : (2^{\mathbb{N}})^{\mathbb{N}} \to X \setminus \mathbf{S}$ satisfying $x \mathrel{\mathsf{E}_3} y \Longleftrightarrow f(x) \mathrel{\mathsf{E}_3} f(y)$ for all $x, y \in (2^{\mathbb{N}})^{\mathbb{N}}$, and also such that

$$\neg\, x(0) \mathrel{\mathsf{E}_0} y(0) \Longrightarrow \neg\, f(x)(k_0) \mathrel{\mathsf{E}_0} f(y)(k_0) \Longrightarrow f(x)(k_0) \ne f(y)(k_0)$$

for all $x, y \in (2^{\mathbb{N}})^{\mathbb{N}}$. It follows that $\{x(k_0) : x \in Z\}$ is uncountable, where $Z = \operatorname{ran} f \subseteq X \setminus \mathbf{S}$, yet $\{x(k_0) : x \in X\} \subseteq W$ is countable, a contradiction.

Thus $X \notin \mathbb{P}_{\mathsf{E}_0\mathsf{E}_3}$. Therefore for every set $Y \in \mathbb{P}_{\mathsf{E}_0\mathsf{E}_3}$ the difference $Y \setminus X$ still belongs to $\mathbb{P}_{\mathsf{E}_0\mathsf{E}_3}$. We leave it to the reader to verify that then every $Y \in \mathbb{P}_{\mathsf{E}_0\mathsf{E}_3}$ forces that $\mathbf{x}$ does not belong to the set $X = \{x \in (2^{\mathbb{N}})^{\mathbb{N}} : \forall n\,(x(n) \in W)\}$ (as defined in the generic extension), and hence forces $\exists k\,(\mathbf{x}(k) \notin W)$, as required.

□ (Theorem 14.9.3)

REMARK 14.9.5. ZAPLETAL has considerably strengthened Theorem 14.9.3 by showing that $\mathbb{P}_{\mathsf{E}_0\mathsf{E}_3}$ forces the collapse of the old continuum $\mathfrak{c}$ to $\aleph_0$.

To prove the result, begin with a sequence of $\mathfrak{c}$ mutually disjoint BERNSTEIN[4] sets $X_\alpha \subseteq 2^{\mathbb{N}}$, where $\alpha < \mathfrak{c}$. Coming back to the beginning of the proof of Theorem 14.9.3, define, in the extension, α_n to be equal to the first index $\alpha < \mathfrak{c}$ such that X_α contains $\mathbf{x}(n)$. (Recall that $\mathbf{x}(n)$ is an "old" real by Lemma 14.9.4.)

We claim that the sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ contains all ordinals $\alpha < \mathfrak{c}$.

Suppose toward the contrary that a condition $D \in \mathbb{P}_{\mathsf{E}_0\mathsf{E}_3}$ forces that some $\alpha < \mathfrak{c}$ does not belong to the sequence. Arguing as in the proof of Lemma 14.9.4, we can find an index k and a perfect set $A \subseteq 2^{\mathbb{N}}$ such that for every $a \in X$ the set $P_x = \{x \in D : x(k) = a\}$ is still a condition in $\mathbb{P}_{\mathsf{E}_0\mathsf{E}_3}$. Choose any $x \in X \cap X_\alpha$. Then D_x forces $\alpha_k = \alpha$, a contradiction.

Unfortunately this method does not work for the forcing notion $\mathbb{P}_{\mathsf{E}_0\mathsf{E}_1}$. (See Theorem 11.7.4 with a much weaker result.) □

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[4] A Bernstein set is a set that a has non-empty intersection with every perfect set. The construction of $\mathfrak{c}$-many mutually disjoint Bernstein sets is a rather standard application of the axiom of choice.

CHAPTER 15

Summable equivalence relations

Recall that given a sequence of non-negative reals r_n, the summable ideal $\mathscr{S}_{\{r_n\}}$ consists of all sets $x \subseteq \mathbb{N}$ such that $\mu_{\{r_n\}}(x) = \sum_{n\in x} r_n < +\infty$. The corresponding equivalence relation $\mathsf{E}_{\{r_n\}}$ is defined on $\mathscr{P}(\mathbb{N})$ so that $x \mathrel{\mathsf{E}_{\{r_n\}}} y$ iff $x \mathbin{\triangle} y \in \mathscr{S}_{\{r_n\}}$, and on $2^{\mathbb{N}}$ the same way, with $a \mathbin{\triangle} b = \{n : a(n) \neq b(n)\}$ for a, $b \in 2^{\mathbb{N}}$. In particular, these families contain the ideal

$$\mathscr{I}_2 = \mathscr{S}_{\{1/n\}} = \{x \subseteq \mathbb{N} : \textstyle\sum_{n\in x,\, n\geq 1} n^{-1} < +\infty\}$$

and the equivalence relation $\mathsf{E}_2 = \mathsf{E}_{\mathscr{I}_2}$ defined on $2^{\mathbb{N}}$ so that

$$x \mathrel{\mathsf{E}_{\{r_n\}}} y \quad \text{iff} \quad \sum_{n\in x\triangle y,\, n\geq 1} n^{-1} < +\infty\,.$$

This chapter is mainly devoted to the proof of Theorem 5.7.4 of HJORTH, the 4th dichotomy theorem, saying that if $\mathsf{E} \leq_{\mathrm{B}} \mathsf{E}_2$, then either E is essentially countable or $\mathsf{E} \sim_{\mathrm{B}} \mathsf{E}_2$. Two results related to the forcing by E_2-large sets are proved in the end of the chapter.

We begin with a few remarks related to connections between different families of summable ideals and equivalence relations.

15.1. Classification of summable ideals and equivalence relations

Suppose that r_n, $n \in \mathbb{N}$, are non-negative reals. If $\sum_n r_n < +\infty$, then obviously $\mathscr{S}_{\{r_n\}} = \mathscr{P}(\mathbb{N})$, therefore $\mathsf{E}_{\{r_n\}}$ makes everything equivalent. This allows us to concentrate on the non-trivial case $\sum_n r_n = +\infty$. FARAH [**Far00**, § 1.12] gives the following classification of summable ideals based on the distribution of reals r_n satisfying $\sum_{n=0}^{\infty} r_n = +\infty$ as the blanket assumption:

(S1) *Atomic* ideals: there is $\varepsilon > 0$ such that the set $A_\varepsilon = \{n : r_n \geq \varepsilon\}$ is infinite and satisfies $\mu_{\{r_n\}}(\mathbb{N} \smallsetminus A_\varepsilon) < +\infty$.

In this case $\mathscr{S}_{\{r_n\}} = \{x : x \cap A_\varepsilon \in \mathsf{Fin}\}$, and hence the corresponding equivalence relation $\mathsf{E}_{\{r_n\}}$ is $\sim_{\mathrm{B}}$-equivalent to E_0. KECHRIS [**Kec98**] calls ideals of this type *trivial variations of* Fin; see footnote 1 in Chapter 11.

(S2) *Dense* (summable) ideals: $r_n \to 0$ (and $\sum_n r_n = +\infty$).

In this case $\mathsf{E}_{\{r_n\}} \sim_{\mathrm{B}} \ell^1$ by Lemma 6.2.4; therefore, all equivalence relations in this subfamily are $\sim_{\mathrm{B}}$-equivalent to each other and to $\mathsf{E}_{\{1/n\}}$, that is, to E_2.

(S3) There is a decreasing sequence of positive reals $\varepsilon_n \to 0$ such that all sets $\Delta_n = A_{\varepsilon_{n+1}} \smallsetminus A_{\varepsilon_n}$ are infinite.

In this case still $\mathsf{E}_{\{r_n\}} \sim_{\mathrm{B}} \mathsf{E}_2$. Indeed, to prove $\mathsf{E}_{\{r_n\}} \le_{\mathrm{B}} \mathsf{E}_2$, it suffices to associate with any n a finite set $u(n) \subseteq \mathbb{N}$ such that $|r_n - \sum_{k \in u(n)} \frac{1}{k}| < 2^{-n}$ and $u(n) \cap u(m) = \varnothing$ whenever $m \ne n$. The map $x \mapsto \bigcup_{n \in x} u(n)$ reduces $\mathsf{E}_{\{r_n\}}$ (as an equivalence relation on $\mathscr{P}(\mathbb{N})$) to E_2. To prove $\mathsf{E}_2 \le_{\mathrm{B}} \mathsf{E}_{\{r_n\}}$, note that under the assumptions of (S3) a finite set $u(n) \subseteq \mathbb{N}$ can be associated with an arbitrary n so that $|\frac{1}{n} - \sum_{k \in u(n)} r_k| < 2^{-n}$ and $u(n) \cap u(m) = \varnothing$ whenever $m \ne n$. The map $x \mapsto \bigcup_{n \in x} u(n)$ reduces E_2 to $\mathsf{E}_{\{r_n\}}$.

(S4) Ideals of the form $\mathtt{Fin} \oplus \mathrm{dense}$: there is a real $\varepsilon > 0$ such that the set A_ε is infinite, $\mu_{\{r_n\}}(\mathbb{N} \smallsetminus A_\varepsilon) = +\infty$, and $\mathtt{lim}_{n \to \infty,\, n \notin A_\varepsilon} r_n = 0$.

In this case $\mathsf{E}_{\{r_n\}} \sim_{\mathrm{B}} \mathsf{E}_2 \times \mathsf{E}_0$ by the above, thus in fact still $\mathsf{E}_{\{r_n\}} \sim_{\mathrm{B}} \mathsf{E}_2$ because $\mathsf{E}_0 \le_{\mathrm{B}} \mathsf{E}_2 \sim_{\mathrm{B}} \mathsf{E}_2 \times \mathsf{E}_2$.

To conclude, every summable equivalence relation $\mathsf{E}_{\{r_n\}}$ (where $r_n \ge 0$ and $\sum_n r_n = +\infty$) is $\sim_{\mathrm{B}}$-equivalent either to E_0 or to E_2. Let us concentrate on the summable equivalence relation E_2, or $\mathsf{E}_{\{1/n\}}$, which is the same.

15.2. Grainy sets and the two cases

Beginning the *proof of Theorem* 5.7.4, note that the same simplifying argument based on Corollary 5.2.2 as in Section 11.3 and Section 14.2 enables us to reduce Theorem 5.7.4 to the following particular form.

THEOREM 15.2.1. *Suppose that* $D \subseteq 2^{\mathbb{N}}$ *is a* $\mathbf{\Sigma}^1_1$ *set. Then* either $\mathsf{E}_2 \restriction D$ *is essentially countable*[1] or $\mathsf{E}_2 \sqsubseteq_{\mathrm{B}} \mathsf{E}_2 \restriction D$, *and hence* $\mathsf{E}_2 \restriction D \sim_{\mathrm{B}} \mathsf{E}_2$.

The two cases of the theorem are incompatible because E_2 is not essentially countable by Corollary 13.9.2. It is a major open problem (see Question 5.7.5) to figure out whether the "either" case can be strengthened to $\mathsf{E}_2 \restriction D \le_{\mathrm{B}} \mathsf{E}_0$.

PROOF (Theorem 15.2.1). As usual we can assume that $D \subseteq 2^{\mathbb{N}}$ is a lightface Σ^1_1 set. The continuation of the proof below in this chapter will be accomplished in Section 15.5.

Several new definitions are involved in the proof of Theorem 15.2.1. For a, $b \in 2^{\mathbb{N}}$ put[2] $\boldsymbol{\delta}(a,b) = \sum_{n \in a \triangle b} \frac{1}{n}$, where $a \triangle b = \{n : a(n) \ne b(n)\}$. The value of $\boldsymbol{\delta}(a,b)$ is a non-negative real or $+\infty$, and $a \mathrel{\mathsf{E}_2} b$ is equivalent to $\boldsymbol{\delta}(a,b) < +\infty$.

The following notation will be useful below. Put $\boldsymbol{\delta}^m_k(a,b) = \sum_{n \in a \triangle b,\, k \le n < m} \frac{1}{n}$ for $1 \le k \le m$, and accordingly $\boldsymbol{\delta}^\infty_k(a,b) = \sum_{n \in a \triangle b,\, k \le n < \infty} \frac{1}{n}$. Define $\boldsymbol{\delta}(a) = \sum_{a(n)=1} \frac{1}{n}$ and similarly $\boldsymbol{\delta}^m_k(a)$ and $\boldsymbol{\delta}^\infty_k(a)$. The next definition introduces two key notions: galaxies and grainy sets.

DEFINITION 15.2.2. If $a \in A \subseteq 2^{\mathbb{N}}$ and $q \in \mathbb{Q}^+$, then $\mathtt{Gal}^q_A(a)$, the *q-galaxy of a in A*, is the set of all $b \in A$ such that there is a finite chain $a = a_0, a_1, \dots, a_n = b$ of reals $a_i \in A$ satisfying $\boldsymbol{\delta}(a_i, a_{i+1}) < q$ for all i.

A set $A \subseteq 2^{\mathbb{N}}$ is *q-grainy*, where $q \in \mathbb{Q}^+$, iff $\boldsymbol{\delta}(a,b) < 1$ for all $a \in A$ and $b \in \mathtt{Gal}^q_A(a)$. A set is *grainy* if it is q-grainy for some $q \in \mathbb{Q}^+$. □

For instance, all sets of $\boldsymbol{\delta}$-diameter < 1 are q-grainy for all q.

[1] Recall that essentially countable means Borel reducible to a (Borel) countable equivalence relation, or, equivalently, Borel reducible to $\mathsf{E}_\infty = \mathbf{E}(F_2, 2)$.

[2] As $\frac{1}{n}$ is not defined for $n = 0$, the default domain of summation will be $[1, \infty)$ in this chapter. For instance, $\sum_{n \in a \triangle b} \frac{1}{n}$ is understood to be $\sum_{n \in a \triangle b,\, n \ge 1} \frac{1}{n}$.

LEMMA 15.2.3. *Suppose that* $A \subseteq 2^{\mathbb{N}}$ *and* $q \in \mathbb{Q}^+$. *If* $a, b \in A$, *then the galaxies* $\mathtt{Gal}^q_A(a)$ *and* $\mathtt{Gal}^q_A(b)$ *either coincide or are disjoint.*

If $\delta(a,b) < +\infty$ *for all* $a, b \in A$, *then the number of different galaxies* $\mathtt{Gal}^q_A(a)$, $a \in A$, *is at most countable.*

PROOF. To prove the last claim, suppose toward the contrary that there is an uncountable set $B \subseteq A$ such that $\mathtt{Gal}^q_A(a) \cap \mathtt{Gal}^q_A(b) = \varnothing$ for any two different a, $b \in B$. Fix an arbitrary $a \in A$. Note that $b \notin \mathtt{Gal}^q_A(a)$ and $\delta(b,a) \geq q$ for all $b \neq a$ in B. On the other hand, as $\delta(a,b) < +\infty$ for all $b \in B$, there is m and a still uncountable set $B' \subseteq B$ such that $\delta^{\infty}_m(a,b) < q/2$ for all $b \in B'$. Now take a pair of $b \neq b' \in B'$ with $b \restriction [0,m) = b' \restriction [0,m)$. Then $\delta(b,b') < q$, a contradiction. □

CLAIM 15.2.4. *Any* q*-grainy* Σ^1_1 *set* $A \subseteq 2^{\mathbb{N}}$ *is covered by a* q*-grainy* Δ^1_1 *set.*

PROOF. [3] The set $C_0 = \{b \in 2^{\mathbb{N}} : A \cup \{b\}$ is q-grainy$\}$ is Π^1_1 and $A \subseteq C_0$, hence, there is a Δ^1_1 set B_1 with $A \subseteq B_1 \subseteq C_0$. Note that $A \cup \{a\}$ is q-grainy for all $a \in B_1$. It follows that the Π^1_1 set

$$C_1 = \{b \in B_1 : A \cup \{a, b\} \text{ is } q\text{-grainy for all } a \in B_1\}$$

still contains A, hence, there is a Δ^1_1 set B_2 with $A \subseteq B_2 \subseteq C_1 \subseteq B_1$. Note that $A \cup \{a_1, a_2\}$ is q-grainy for all $a_1, a_2 \in B_2$. In general, as soon as we have a Δ^1_1 set B_n with $A \subseteq B_n$ and such that $A \cup \{a_1, \ldots, a_n\}$ is q-grainy for all $a_1, \ldots, a_n \in B_n$, then the Π^1_1 set

$$C_n = \{b \in B_n : A \cup \{a_1, \ldots, a_n, b\} \text{ is } q\text{-grainy for all } a_1, \ldots, a_n \in B_n\}$$

contains A, hence, there is a Δ^1_1 set B_{n+1} with $A \subseteq B_{n+1} \subseteq C_n \subseteq B_n$.

As usual in similar cases, the choice of the sets B_n can be made effective enough for the set $B = \bigcap_n B_n$ to be still Δ^1_1, not merely Borel. On the other hand, $A \subseteq B$ and B is q-grainy. □ (Claim)

Let $\mathbb{S}$ be the union of all grainy Δ^1_1 sets, this is the structural domain here. Accordingly, $\mathbf{H} = 2^{\mathbb{N}} \setminus \mathbb{S}$ is the chaotic domain.

EXERCISE 15.2.5. Using ordinary computation, prove that the set $\mathbb{S}$ is Π^1_1 and the set $\mathbf{H}$ is Σ^1_1. □

We have two cases.

Case 1. The set D in Theorem 15.2.1 (assumed to be Σ^1_1) satisfies $D \subseteq \mathbb{S}$. We will prove that then $\mathsf{E}_2 \restriction D$ is essentially countable.

Case 2. $D \cap \mathbf{H} \neq \varnothing$. It will be shown that in this case $\mathsf{E}_2 \sim_{\mathrm{B}} \mathsf{E}_2 \restriction D$.

If D is not lightface Σ^1_1, then it is $\Sigma^1_1(p)$ for some parameter $p \in 2^{\mathbb{N}}$. In this case we should consider the union $\mathbb{S}(p)$ of all grainy $\Delta^1_1(p)$ sets and its complement $\mathbf{H}(p)$ instead of $\mathbb{S}$ and $\mathbf{H}$, but the arguments would remain essentially the same.

15.3. Case 1

The next theorem shows that the Case 1 assumption leads to the "either" case of Theorem 15.2.1.

[3] The result can be achieved as a routine application of a reflection principle, yet we would like to show how it works with a low level technique.

THEOREM 15.3.1. *If* $D \subseteq 2^{\mathbb{N}}$ *is an arbitrary* Σ^1_1 *set and* $D \subseteq \mathbb{S}$, *then* $\mathsf{E}_2 \restriction D$ *is essentially countable.*

PROOF. By Separation (Theorem 2.3.2) there is a Δ^1_1 set D' such that $D \subseteq D'$ and still $D' \subseteq \mathbb{S}$. Thus it can be assumed that D is Δ^1_1.

Fix a standard enumeration $(W)_e$, $e \in \mathtt{Cod}(\Delta^1_1)$, of all Δ^1_1 subsets of $2^{\mathbb{N}}$, where, as usual, $\mathtt{Cod}(\Delta^1_1) \subseteq \mathbb{N}$ is a Π^1_1 set; see Section 2.8. By Kreisel Selection (Theorem 2.4.5), there exist Δ^1_1 functions $a \longmapsto e(a)$ and $a \longmapsto q(a)$, defined on D and with values in $\mathtt{Cod}(\Delta^1_1)$ and $\mathbb{Q}^+$, respectively, such that given $a \in D$ the Δ^1_1 set $W(a) = (W)_{e(a)}$ contains a and is $q(a)$-grainy. The final point of our argument will be an application of Lemma 7.6.1, with ρ being a derivate of the function $G(a) = \mathtt{Gal}^{q(a)}_{W(a)}(a)$.

CLAIM 15.3.2. *If* $a \in D$, *then* $\gamma_a = \{G(b) : b \in [a]_{\mathsf{E}_2} \cap D\}$ *is at most countable.*

PROOF. Otherwise there is a pair of elements $e \in \mathtt{Cod}(\Delta^1_1)$ and $q \in \mathbb{Q}^+$ and an uncountable set $B \subseteq [a]_{\mathsf{E}_2} \cap D$ such that $q(b) = q$ and $e(b) = e$ for all $b \in B$ and $G(b') \neq G(b)$ for all different elements $b \neq b'$ in B. Then the sets $G(b)$, $b \in B$ are q-galaxies in one and the same set $W(a) = W(b) = (W)_e$. Lemma 15.2.3 ends the proof. □ (Claim)

It follows that $a \mapsto G(a)$ maps every E_2-class in D onto a countable set of galaxies $G(a)$. Note that the sets $G(a)$ are galaxies in different sets $W(a)$ and with different steps $q(a)$; therefore, different $G(a)$ are not necessarily disjoint. Yet a weaker result holds. Put $\varphi(a) = \bigcup_m \{b \restriction m : b \in G(a)\}$. Thus $\varphi(a) \subseteq 2^{<\omega}$ codes the Polish topological closure of the galaxy $G(a)$.

CLAIM 15.3.3. *If* $a, b \in D$ *and* $\neg\, a \,\mathsf{E}_2\, b$, *then* b *does not belong to the topological closure of* $G(a)$, *in particular,* $b \restriction m \notin \varphi(a)$ *for some* m.

PROOF. Take m big enough for $\delta^m_0(a,b) \geq 2$. Then $u = b \restriction m$ does not belong to $\varphi(a)$ because every $a' \in G(a)$ satisfies $\delta(a,a') < 1$. □ (Claim)

EXERCISE 15.3.4. Prove that the following sets belong to Σ^1_1:

$$\Gamma = \{\langle a,b\rangle : a \in D \wedge b \in G(a)\} \quad \text{and} \quad \Phi = \{\langle a,u\rangle : a \in D \wedge u \in \varphi(a)\}. \qquad \square$$

This result does not imply that $a \mapsto \varphi(a)$ is a Borel map. Yet we can change it appropriately to get a Borel map with similar properties. It follows from Claim 15.3.3 and Kreisel Selection that there is a Δ^1_1 function $\mu : D \times D \to \mathbb{N}$ such that $b \restriction \mu(a,b) \notin \varphi(a)$ for every pair of $a, b \in D$ with $\neg\, a \,\mathsf{E}_2\, b$. Define the following Σ^1_1 equivalence relation on D:

$$a \,\mathsf{F}\, b \quad \text{iff} \quad e(a) = e(b) \wedge q(a) = q(b) \wedge G(a) = G(b).$$

(To see that F is Σ^1_1, note that $G(a) = G(b)$ is equivalent to $b \in G(a)$, and that Γ is Σ^1_1 by Exercise 15.3.4.) Then for every $a \in D$, the set

$$\psi(a) = \{b \restriction \mu(a',b) : a', b \in D \wedge a \,\mathsf{F}\, a' \wedge \neg\, a' \,\mathsf{E}_2\, b)\} \subseteq 2^{<\omega}$$

does not intersect $\varphi(a)$, and hence the Σ^1_1 set

$$\Psi = \{\langle a,h\rangle : a \in D \wedge h \in \psi(a)\}$$

does not intersect Φ. Note that by definition Ψ is F-invariant *w.r.t.* the first argument; that is, if $a, a' \in D$ satisfy $a \,\mathsf{F}\, a'$, then $\psi(a) = \psi(a')$. It follows from

Lemma 10.4.2 that there exists a Δ^1_1 set $\Theta \subseteq D \times 2^{<\omega}$ satisfying $\Phi \subseteq \Theta$ but $\Psi \cap \Theta = \varnothing$, and F-invariant in the same sense. Then the map

$$a \mapsto \vartheta(a) = \{h : \Theta(a,h)\}$$

is obviously Δ^1_1.

CLAIM 15.3.5. *Suppose that $a, b \in D$. Then $a \mathsf{F} b$ implies $\vartheta(a) = \vartheta(b)$, while $\neg\, a \mathsf{E}_2 b$ implies $\vartheta(a) \neq \vartheta(b)$.*

PROOF. The first statement holds just because Θ is F-invariant. Now suppose that $\neg\, a \mathsf{E}_2 b$. Then by definition $h = b \restriction \mu(a,b) \in \psi(a)$, therefore $h \notin \vartheta(a)$. On the other hand, $h \in \varphi(b) \subseteq \vartheta(b)$. □ (Claim)

CLAIM 15.3.6. *If $a \in D$, then the set $T_a = \{\vartheta(b) : b \in [a]_{\mathsf{E}_2} \cap D\}$ is at most countable.*

PROOF. Suppose that $b, c \in [a]_{\mathsf{E}_2} \cap D$. It follows from Claim 15.3.5 that if $G(b) = G(c)$, $e(b) = e(c)$, and $q(b) = q(c)$, then $\vartheta(b) = \vartheta(c)$. It remains to note that G takes only countably many values on $[a]_{\mathsf{E}_2} \cap D$ by Claim 15.3.2. □

Finally, note that if $a, b \in D$ and $\neg\, a \mathsf{E}_2 b$, then $\vartheta(a) \neq \vartheta(b)$ by Claim 15.3.5. Thus ϑ witnesses that $\mathsf{E}_2 \restriction D$ is essentially countable by Lemma 7.6.1.

□ (Theorem 15.3.1 and case 1 in Theorem 15.2.1)

15.4. Case 2

Accordingly, we obtain the "or" case in Theorem 15.2.1 in the assumptions of Case 2. We prove:

THEOREM 15.4.1. *If $D \subseteq 2^{\mathbb{N}}$ is an arbitrary Σ^1_1 set and $D \cap \mathbf{H} \neq \varnothing$, then $\mathsf{E}_2 \sqsubseteq_{\mathsf{B}} \mathsf{E}_2 \restriction D$.*

PROOF. Thus let us assume that the Σ^1_1 set $H = D \cap \mathbf{H}$ is non-empty. Note that there is no non-empty Σ^1_1 grainy set $A \subseteq H$ by Claim 15.2.4.

Put $\mathscr{B}_s = \{a \in 2^{\mathbb{N}} : s \subset a\}$ for $s \in 2^{<\omega}$, a basic open nbhd in $2^{\mathbb{N}}$.

DEFINITION 15.4.2. Suppose that $X, Y \subseteq 2^{\mathbb{N}}$. We write $\delta^{\infty}_k(X,Y) \leq \varepsilon$ iff

$$\forall a \in X\, \exists b \in Y\, (\delta^{\infty}_k(a,b) \leq \varepsilon) \quad \text{and} \quad \forall b \in Y\, \exists a \in X\, (\delta^{\infty}_k(a,b) \leq \varepsilon).$$

Inequalities like $\delta^{k'}_k(X,Y) \leq \varepsilon$ will be understood accordingly. □

To prove that $\mathsf{E}_2 \sqsubseteq_{\mathsf{B}} \mathsf{E}_2 \restriction H$, we define an increasing sequence of natural numbers $1 = k_0 < k_1 < k_2 < \cdots$, and also objects A_s, g_s for every $s \in 2^{<\omega}$, which satisfy the following list of requirements **1°–6°**.

1°. If $s \in 2^m$, then $g_s \in 2^{k_m}$, and $s \subset t \Longrightarrow g_s \subset g_t$.

2°. $\varnothing \neq A_s \subseteq H \cap \mathscr{B}_{g_s}$, A_s is Σ^1_1, and $s \subset t \Longrightarrow A_t \subseteq A_s$.

3°. If $s \in 2^n$, then $\delta^{\infty}_{k_n}(A_{0^n}, A_s) \leq 2^{-n-2}$, where 0^n is the sequence of n zeros.

4°. If $s \in 2^n$, $1 \leq m < n$, $s(m) = 0$, then $\delta^{k_{m+1}}_{k_m}(g_s, g_{0^m}) \leq 2^{-m-1}$.

5°. If $s \in 2^n$, $1 \leq m < n$, $s(m) = 1$, then $|\delta^{k_{m+1}}_{k_m}(g_s, g_{0^m}) - \frac{1}{m}| \leq 2^{-m-1}$.

6°. For every n, a certain condition, in terms of the Gandy–Harrington forcing and a fixed countable transitive model $\mathfrak{M}$ of **ZFC**$^-$, similar to **2°** in Section 10.2 or **2°** in Section 10.6, related to all sets A_s, $s \in 2^n$, so that, as a consequence, $\bigcap_n A_{a \restriction n}$ is a singleton for all $a \in 2^{\mathbb{N}}$.

It obviously follows from **4°** and **5°** that

7°. If $s, t \in 2^n$, $1 \le m < n$, $s(m) = t(m)$, then $|\delta_{k_m}^{k_m+1}(g_s, g_t)| \le 2^{-m}$, but if $s(m) \ne t(m)$, then $|\delta_{k_m}^{k_m+1}(g_s, g_t) - \frac{1}{m}| \le 2^{-m}$.

With such a splitting system, we can accomplish the proof of Theorem 15.4.1 as follows. For every $a \in 2^{\mathbb{N}}$, define $\varphi(a) = \bigcup_n g_{a\restriction n}$, so that $\varphi(a) \in 2^{\mathbb{N}}$ is the only element satisfying $g_{a\restriction n} \subset \varphi(a)$ for all n. It follows by **6°** that $\varphi(a) = \bigcap_n A_{a\restriction n}$, and hence $\varphi : 2^{\mathbb{N}} \to A_\Lambda \subseteq H \subseteq D$ is a continuous 1-to-1 map.

LEMMA 15.4.3. *The map φ is an embedding of E_2 into $\mathsf{E}_2 \restriction H$; that is, the equivalence $a \,\mathsf{E}_2\, b \iff \varphi(a) \,\mathsf{E}_2\, \varphi(b)$ holds for all $a, b \in 2^{\mathbb{N}}$.*

PROOF. By definition $\delta(\varphi(a), \varphi(b)) = \lim_{n\to\infty} \delta_1^{k_n}(g_{a\restriction n}, g_{b\restriction n})$. On the other hand

$$|\delta_1^{k_n}(g_{a\restriction n}, g_{b\restriction n}) - \delta_1^n(a \restriction n, b \restriction n)| \le \textstyle\sum_{m<n} 2^{-m} < 2$$

by **7°**. Therefore $|\delta(\varphi(a), \varphi(b)) - \delta(a, b)| \le 2$, as required. □ (Lemma)

□ (Theorem 15.4.1 modulo the construction **1°**–**6°**)

This accomplishes case 2 in the proof of Theorem 15.2.1.

□ (Theorems 15.2.1 and 5.7.4 modulo the construction **1°**–**6°**)

15.5. The construction of a splitting system

The construction of a system of numbers, sets, and sequences satisfying **1°**–**6°** goes on by induction. To begin with, we set $k_0 = 0$, $g_\Lambda = \Lambda$, and $A_\Lambda = H$. Suppose that, for some n, we have the objects as required for all previous levels $n' \le n$, and extend the construction to the next level $n + 1$.

The following shrinking method works. Suppose that $\sigma \in 2^n$, and $\varnothing \ne A \subseteq A_\sigma$ is a Σ_1^1 set. Put

$$A'_{0^n} = \{a \in A_{0^n} : \exists\, b \in A\, (\delta_{k_n}^\infty(a, b) \le 2^{-n-2})\}, \tag{1}$$

and then

$$A'_s = \{a \in A_s : \exists\, b \in A'_{0^n}\, (\delta_{k_n}^\infty(a, b) \le 2^{-n-2})\}$$

for all $s \in 2^n$ except for $s = \sigma$, where we put $A'_\sigma = A$. Obviously all sets A'_s are non-empty Σ_1^1, and the system of those sets still satisfies **3°**. Applying this construction 2^n times, we get a system of non-empty Σ_1^1 sets $A'_s \subseteq A_s$ still satisfying the same requirements, and in addition satisfying **6°** already for the step $n + 1$. Redenote by A_s the resulting sets A'_s, $s \in 2^n$.

Now carry out the splitting step. Recall that every non-empty Σ_1^1 subset of H, in particular, A_{0^n}, is not grainy, in particular, not 2^{-n-2}-grainy. It follows that there is a chain $a_0, a_1, \ldots, a_l$ in A_{0^n} such that $\delta(a_0, a_l) \ge 1$ while $\delta(a_i, a_{i+1}) < 2^{-n-2}$ for all i. Then we have $\delta(a_0, a_i) > \frac{1}{n}$ but $\delta(a_0, a_i) - \frac{1}{n} < 2^{-n-2}$ for some i. Put $a^0 = a_0$ and $a^1 = a_i$. (If $n = 0$ in the inductive step $0 \to 1$, then $\frac{1}{n}$ could be replaced by 1.) Note that $a^0 \restriction k_n = a^1 \restriction k_n$ by **1°** and **2°**; therefore, there is

$k_{n+1} > k_n$ still satisfying $|\delta_{k_n}^{k_{n+1}}(a^0, a^1) - \frac{1}{n}| < 2^{-n-2}$. According to **3°**, for every $s \in 2^n$ there exist $b_s^0, b_s^1 \in A_s$ with $\delta_{k_n}^{\infty}(a^i, b_s^i) \le 2^{-n-2}$ for $i = 0, 1$. We can, of course, assume that $b_{0^n}^i = a^i$. Moreover, the number k_{n+1} can be chosen large enough for the following to hold:

$$(2) \qquad \delta_{k_{n+1}}^{\infty}(b_s^i, a^i) \le 2^{-n-3} \quad \text{for all} \quad s \in 2^n \quad \text{and} \quad i = 0, 1.$$

Put $g_{s^\frown i} = b_s^i \restriction k_{n+1}$ for all $s^\frown i \in 2^{n+1}$. This definition preserves **1°**. To check **4°** for $s' = s^\frown 0 \in 2^{n+1}$ and $m = n$, note that

$$\delta_{k_n}^{k_{n+1}}(g_{s'}, g_{0^{n+1}}) = \delta_{k_n}^{k_{n+1}}(b_s^0, a^0) \le 2^{-n-2}.$$

To check **5°** for $s' = s^\frown 1 \in 2^{n+1}$ and $m = n$, note that

$$|\delta_{k_n}^{k_{n+1}}(g_{s'}, g_{0^{n+1}}) - \tfrac{1}{n}| \le \delta_{k_n}^{k_{n+1}}(b_s^1, a^1) + |\delta_{k_n}^{k_{n+1}}(a^0, a^1) - \tfrac{1}{n}| \le 2^{-n-1}.$$

Finally, let us define the sets $A_{s'} \subseteq A_s$, for all $s' = s^\frown i \in 2^{n+1}$ (where $s \in 2^n$ and $i = 0, 1$). To fulfill **2°**, we begin with $A'_{s^\frown i} = A_s \cap \mathscr{B}_{g_{s^\frown i}}$. This is a Σ_1^1 subset of A_s, containing b_s^i. To fulfill **3°**, define $A_{0^{n+1}}$ to be the set of all $a \in A'_{0^{n+1}}$ such that

$$\forall\, s' = s^\frown i \in 2^{n+1}\ \exists\, b \in A'_{s'}\ (\delta_{k_{n+1}}^{\infty}(a, b) \le 2^{-n-3})\ ;$$

this is still a Σ_1^1 set containing $b_{0^n}^0 = a^0$ by (2). Finally, we put

$$A_{s^\frown i} = \{b \in A'_{s^\frown i} : \exists\, b \in A_{0^{n+1}}\ (\delta_{k_{n+1}}^{\infty}(a, b) \le 2^{-n-3})\}$$

for all $s^\frown i \ne 0^{n+1}$. □ (The construction)

This accomplishes the construction of a system of sets A_s, sequences g_s, and numbers k_m satisfying conditions **1°**–**6°** of Section 15.4, and the proof of Theorems 15.2.1, 15.4.1, and 5.7.4; see the end of Section 15.4.

15.6. A forcing notion associated with E_2

Following Section 11.7 and Section 14.9, consider a forcing notion $\mathbb{P}_{\mathsf{E}_0\mathsf{E}_2}$ that consists of all $\mathbf{\Sigma}_1^1$ sets $X \subseteq 2^{\mathbb{N}}$ such that $\mathsf{E}_2 \restriction X \sim_{\mathsf{B}} \mathsf{E}_2$. By Theorem 15.2.1, the associated ideal $\mathscr{I}_{\mathsf{E}_0\mathsf{E}_2}$ consists of all Borel sets $X \subseteq 2^{\mathbb{N}}$ such that $\mathsf{E}_2 \restriction X$ is essentially countable. Note that $\mathscr{I}_{\mathsf{E}_0\mathsf{E}_2}$ is a σ-ideal by Corollary 7.3.2.

EXERCISE 15.6.1. Prove following Exercise 14.9.1 and Lemma 14.9.2:

(a) A Σ_1^1 set $R \subseteq 2^{\mathbb{N}}$ belongs to $\mathbb{P}_{\mathsf{E}_0\mathsf{E}_2}$ iff $R \not\subseteq \mathbb{S}$.

(b) Every set $X \in \mathbb{P}_{\mathsf{E}_0\mathsf{E}_2}$ contains a closed subset $Y \in \mathbb{P}_{\mathsf{E}_0\mathsf{E}_2}$, $Y \subseteq X$.

(c) Moreover, if $X \in \mathbb{P}_{\mathsf{E}_0\mathsf{E}_2}$ and $\varepsilon > 0$, then there exists a continuous injection $\varphi : 2^{\mathbb{N}} \to X$ satisfying $|\delta(x,y) - \delta(\varphi(x), \varphi(y))| \le \varepsilon$ for all $x, y \in 2^{\mathbb{N}}$. Every such a map is obviously a reduction of E_2 to $\mathsf{E}_2 \restriction X$.

To prove (c), assume as usual that X is a lightface Σ_1^1 set. Then $X \not\subseteq \mathbb{S}$, and hence we may assume that $X \subseteq \mathbf{H}$. Then carry out the construction of A_s as in Sections 15.4 and 15.5, but with all bounds such as 2^{-n-1} uniformly replaced everywhere by $\frac{\varepsilon}{2} \cdot 2^{-n-1}$. The inequality $|\delta(\varphi(a), \varphi(b)) - \delta(a, b)| \le 2$ in the proof of Lemma 15.4.3 strengthens to $|\delta(\varphi(a), \varphi(b)) - \delta(a, b)| \le \varepsilon$. □

Yet there is a substantial difference between $\mathbb{P}_{\mathsf{E}_0\mathsf{E}_2}$ on the one hand and $\mathbb{P}_{\mathsf{E}_0\mathsf{E}_1}$, $\mathbb{P}_{\mathsf{E}_0\mathsf{E}_3}$ on the other hand: $\mathbb{P}_{\mathsf{E}_0\mathsf{E}_2}$ preserves $\aleph_1$, and moreover, THEOREM 15.6.2 (joint with ZAPLETAL). $\mathbb{P}_{\mathsf{E}_0\mathsf{E}_2}$ *forces that every countable set* $X \subseteq 2^{\mathbb{N}}$ *of "old" elements in the extension is covered by an "old" set* $Y \subseteq 2^{\mathbb{N}}$ *countable in the ground universe.*

PROOF (sketch). Suppose that, in the ground universe, $\{D_n\}_{n\in\mathbb{N}}$ is a sequence of dense subsets $D_n \subseteq \mathbb{P}_{\mathsf{E}_0\mathsf{E}_2}$. We may assume that each D_n consists of closed sets. Fix $X_0 \in \mathbb{P}_{\mathsf{E}_0\mathsf{E}_2}$. Carry out the construction, as in Section 15.5, of objects satisfying the list of requirements **1°–6°** in Section 15.4, in which conditions **2°** and **6°** are amended as follows:

2′. $A_s \subseteq \mathscr{B}_{g_s} \cap X_0$, $A_s \in \mathbb{P}_{\mathsf{E}_0\mathsf{E}_2}$, A_s is closed, and $s \subset t \Longrightarrow A_t \subseteq A_s$.

6′. If $s \in 2^n$, then $A_s \in D_n$.

There is no need here in Gandy–Harrington style conditions because the non-emptiness of intersections of the form $\bigcap_n A_{a\restriction n}$, $a \in 2^{\mathbb{N}}$ follows from the assumption that the sets A_s are closed.

The shrinking method of Section 15.5 (in the step from n to $n+1$) works in the following slightly changed form.

Note first of all that if all A_s, $s \in 2^n$, are closed sets in $\mathbb{P}_{\mathsf{E}_0\mathsf{E}_2}$ satisfying **3°** in Section 15.4, and $A \subseteq A_\sigma$ for some $\sigma \in 2^n$ is still a closed set in $\mathbb{P}_{\mathsf{E}_0\mathsf{E}_2}$, then the set A'_{0^n} defined by equation (1) in Section 15.5 is a closed non-empty subset of A_{0^n}. We claim that $A'_{0^n} \in \mathbb{P}_{\mathsf{E}_0\mathsf{E}_2}$.

Indeed, otherwise $\mathsf{E}_2 \restriction A'_{0^n}$ is an essentially countable Borel equivalence relation by Theorem 15.2.1. On the other hand, by definition for every $a \in A$, the set $F(a) = \{b \in A'_{0^n} : \boldsymbol{\delta}^{\infty}_{k_n}(a,b) \le 2^{-n-2}\}$ is a non-empty closed subset of A'_{0^n}. Let $f(a)$ denote the least, in the sense of the lexicographic ordering of $2^{\mathbb{N}}$, element of $F(a)$. Then f is a Borel map $A \to A'_{0^n}$ satisfying $\boldsymbol{\delta}^{\infty}_{k_n}(a, f(a)) \le 2^{-n-2}$, and hence $\boldsymbol{\delta}(a, f(a)) < \infty$ for all $a \in A$. It follows that f is a Borel reduction of $\mathsf{E}_2 \restriction A$ to $\mathsf{E}_2 \restriction A'_{0^n}$. Thus $\mathsf{E}_2 \restriction A$ is essentially countable since $\mathsf{E}_2 \restriction A'_{0^n}$ is also. However, $\mathsf{E}_2 \sim_{\mathsf{B}} \mathsf{E}_2 \restriction A$ by the choice of A. Therefore, E_2 is essentially countable, contrary to Corollary 13.9.2. Thus $A'_{0^n} \in \mathbb{P}_{\mathsf{E}_0\mathsf{E}_2}$.

The sets

$$A'_s = \{a \in A_s : \exists\, b \in A'_{0^n}\ (\boldsymbol{\delta}^{\infty}_{k_n}(a,b) \le 2^{-n-2})\}$$

are closed sets and conditions in $\mathbb{P}_{\mathsf{E}_0\mathsf{E}_2}$ by the same reasons. Thus the shrinking method works within the collection of closed sets in $\mathbb{P}_{\mathsf{E}_0\mathsf{E}_2}$.

Now let us review the splitting step in Section 15.5. Suppose that A_s, $s \in 2^n$, are closed sets in $\mathbb{P}_{\mathsf{E}_0\mathsf{E}_2}$ satisfying the requirements at the level n and below. Fix a parameter $p \in 2^{\mathbb{N}}$ such that all sets A_s, $s \in 2^n$, belong to $\Delta^1_1(p)$. Then $A_s \not\subseteq \mathbf{S}(p)$, where $\mathbf{S}(p)$ is the p-parametrized version of $\mathbf{S}$, since otherwise $\mathsf{E}_2 \restriction A_s$ would be essentially countable by the p-version of Theorem 15.3.1. We can assume *w.l.o.g.* that in fact $A_s \subseteq \mathbf{H}(p)$ for all s. (Otherwise apply the shrinking method modified as just above. This allows us to successively shrink the sets A_s, with all requirements preserved, so that the resulting sets are subsets of $\mathbf{H}(p)$.) In this assumption, the sets A_s are not grainy in the p-parametrized sense. This allows us to carry out the last part of the splitting construction (in the step $n \to n+1$) as in Section 15.5: the choice of elements $a^0, a^1 \in A_{0^n}$, *etc.*

Such a construction results in a closed set $X \in \mathbb{P}_{\mathsf{E}_0\mathsf{E}_2}$ such that for every n there is a finite subset $\{X_1, \dots, X_n \subseteq D_n\}$ satisfying $X \subseteq X_1 \cup \dots \cup X_n$. Now to prove the theorem, it suffices to define D_n to be the family of all conditions in

$\mathbb{P}_{\mathsf{E}_0\mathsf{E}_2}$ which decide the value of $t(n)$, where t is a given $\mathbb{P}_{\mathsf{E}_0\mathsf{E}_2}$-term forced to be a countable sequence of "old" reals. □

It is unclear whether the forcing $\mathbb{P}_{\mathsf{E}_0\mathsf{E}_2}$ preserves larger cardinals. On the other hand, $\mathbb{P}_{\mathsf{E}_0\mathsf{E}_2}$ has the minimality property!

THEOREM 15.6.3 (joint with ZAPLETAL). $\mathbb{P}_{\mathsf{E}_0\mathsf{E}_2}$ *forces the generic real to be minimal over the ground universe.*

PROOF (sketch). Since the preservation of $\aleph_1$ is established by Theorem 15.6.2, it suffices to prove that for every $X \in \mathbb{P}_{\mathsf{E}_0\mathsf{E}_2}$ and a continuous $f : X \to 2^{\mathbb{N}}$ there is a set $Y \in \mathbb{P}_{\mathsf{E}_0\mathsf{E}_2}$, $Y \subseteq X$, such that $f \restriction Y$ is either a bijection or a constant.

Assume as usual that f is Δ^1_1 while X is Σ^1_1. Then it can be assumed *w.l.o.g.* that $X \subseteq \mathbf{H}$.

Suppose that there is no $Y \in \mathbb{P}_{\mathsf{E}_0\mathsf{E}_2}$, $Y \subseteq X$, such that $f \restriction Y$ a constant.

LEMMA 15.6.4. *If* $X' \in \mathbb{P}_{\mathsf{E}_0\mathsf{E}_2}$, $X' \subseteq X$, *and* $r, \gamma > 0$, *then there exist points* $a, b \in X$ *such that* $|\boldsymbol{\delta}(a,b) - r| < \gamma$ *and* $f(a) \neq f(b)$.

PROOF. In view of the result of Exercise 15.6.1(c), we can suppose *w.l.o.g.* that in fact $X = X' = 2^{\mathbb{N}}$. It is even more convenient to assume that

$$X = X' = H, \quad \text{where} \quad H = \{a \in 2^{\mathbb{N}} : \forall\, k \le \gamma^{-1}\,(a(k) = 0)\}.$$

Prove first that there exist $a, b \in H$ with $\boldsymbol{\delta}(a,b) < +\infty$ and $f(a) \neq f(b)$. Indeed suppose toward the contrary that f satisfies

$$\boldsymbol{\delta}(a,b) < +\infty \Longrightarrow f(a) \neq f(b).$$

In other words, f is an $(\mathsf{E} \to \Delta_{2^{\mathbb{N}}})$-invariant map in the sense of Section 13.3, where $\mathsf{E} = \mathsf{E}_2 \restriction H$.

Now we have to come back to the discussion in Section 13.9. The ideal $\mathscr{I} = \{x \in \mathscr{I}_2 : \mathtt{min}\, x \ge \gamma^{-1}\}$ is clearly a non-trivial Borel special ideal. The equality $\Delta_{2^{\mathbb{N}}}$ obviously belongs to the family $\mathscr{F}_0$ of Definition 13.5.2. It follows by Theorem 13.9.1 that E is gen. $\Delta_{2^{\mathbb{N}}}$-ergodic. Therefore f is a gen. $\Delta_{2^{\mathbb{N}}}$-constant, that is, $f(a) = f(b)$ for all a, b in a comeager set $C \subseteq H$. However, every comeager $C = \bigcap_n U_n \subseteq 2^{\mathbb{N}}$, where each $U_n \subseteq 2^{\mathbb{N}}$ is open dense, contains a subset $Y \in \mathbb{P}_{\mathsf{E}_0\mathsf{E}_2}$, $Y \subseteq C$. We leave it as an exercise for the reader to get such a Y by means of the construction in Section 15.5, modified so that all sets A_s are Baire intervals in $2^{\mathbb{N}}$ satisfying $A_s \subseteq U_n$ whenever $s \in 2^n$. Yet the existence of such a set Y contradicts the assumption in the beginning of the proof of Theorem 15.6.3.

Thus there there exist $a, b \in H$ with $\boldsymbol{\delta}(a,b) < +\infty$ and $f(a) \neq f(b)$. We may assume *w.l.o.g.* that $\boldsymbol{\delta}(a,b) < \gamma$. (If not, then connect a with b by a finite chain $a = a_0, a_1, \dots, a_n = b$ of elements $a_i \in H$ such that $\boldsymbol{\delta}(a_i, a_{i+1}) < \gamma$ for all i. Such a chain exists by the choice of H.) Now choose any $c \in H$ satisfying $\boldsymbol{\delta}(a,c) = r$. If $f(a) \neq f(c)$, then the pair a, c proves the lemma. Otherwise, the pair b, c works as required. □ (Lemma)

Come back to the theorem. Slightly modifying the construction in Section 15.5, note that it suffices to prove the following. Given a system of non-empty Σ^1_1 sets $A_s \subseteq X$, $s \in 2^n$, satisfying **3°** in Section 15.4, and $\varepsilon > 0$, there exists a system of non-empty Σ^1_1 sets $A'_s \subseteq A_s$ satisfying **3°** in a slightly weaker form,

$$(*) \qquad \boldsymbol{\delta}^{\infty}_{k_n}(A_{0^n}, A_s) \le 2^{-n-2} + \varepsilon \text{ for all } s \in 2^n,$$

and also satisfying the following condition:

(†) for every $\sigma \neq \tau \in 2^n$, there is a number $n = n(\sigma, \tau)$ such that $f(a)(n) = 0$ for all $a \in A'_\sigma$ but $f(a)(n) = 1$ for all $a \in A'_\tau$, or conversely $f(a)(n) = 1$ for all $a \in A'_\sigma$ but $f(a)(n) = 0$ for all $a \in A'_\tau$.

Since the construction can be carried out iteratively so that each single step (of $2^n(2^n - 1)$ total steps) takes care of a single pair of $\sigma \neq \tau$, it suffices to maintain the construction of sets A'_s so that, together with (∗), condition (†) is fulfilled only for one given pair of $\sigma \neq \tau \in 2^n$.

Choose, by the lemma, a pair of points $b', b'' \in A_\sigma$ such that $\delta(b', b'') < \varepsilon$ and $f(b') \neq f(b'')$. As the sets A_s satisfy **3°**, there exist points $a \in A_{0^n}$ and $c \in A_\tau$ such that $\delta(b', a) \leq 2^{-n-2}$ and $\delta(a, c) \leq 2^{-n-2}$. Let b be that one of the two points b', b'' which satisfies $f(c) \neq f(b)$ (or either of them if both satisfy), say $f(c)(n) = 0 \neq 1 = f(b)(n)$ for some n. Then obviously $\delta(a, b) < 2^{-n-2} + \varepsilon$. It remains to choose $a_s \in A_s$ for all $s \in 2^n$ so that $a_{0^n} = a$, $a_\sigma = b$, $a_\tau = c$, and $\delta(a_{0^n}, a_s) \leq 2^{-n-2}$ for all $s \notin \{b, c\}$. The construction of the A'_s can now be accomplished by the same method as at the end of Section 15.5, and we leave it as an exercise. □ (Theorem 15.6.3)

CHAPTER 16

$\mathbf{c}_0$-equalities

Recall that the equivalence relation $\mathbf{c}_0$ is defined on $\mathbb{R}^{\mathbb{N}}$ as follows: $x \mathbin{\mathbf{c}_0} y$ if and only if $x(n) - y(n) \to 0$ with $n \to \infty$. This definition admits a straightforward generalization leading to a family of Borel equivalence relations rather similar to $\mathbf{c}_0$. They are called $\mathbf{c}_0$*-equalities*, and we will prove that there is a family of continuum-many pairwise $\le_{\mathrm{B}}$-incomparable $\mathbf{c}_0$-equalities.

16.1. $\mathbf{c}_0$-equalities: definition

The letter D in this context is due to FARAH [**Far01b**]. There is not any association whatsoever with the diagonal, *i.e.*, the true equality.

DEFINITION 16.1.1 (FARAH [**Far01b**]). Suppose that K is a non-empty index set and $\langle X_k\,; d_k\rangle$ is a metric space for every index $k \in K$. An equivalence relation $\mathsf{D} = \mathsf{D}(\langle X_k; d_k\rangle_{k\in K})$ on the cartesian product $X = \prod_k X_k$ is defined so that $x \mathbin{\mathsf{D}} y$ iff $\mathtt{lim}\ d_k(x(k), y(k)) = 0$, where the limit is associated with the filter of all cofinite subsets of K. In other words $\mathtt{lim}\ d_k(x(k), y(k)) = 0$ iff for every $\varepsilon > 0$ there exist only finitely many indices $k \in K$ such that $d_k(x(k), y(k)) > \varepsilon$.

If $K = \mathbb{N}$ (the most typical case below), then we write $\mathsf{D}(X_k; d_k)$ instead of $\mathsf{D}(\langle X_k; d_k\rangle_{k\in\mathbb{N}})$ for the sake of brevity.

We will be interested mostly in the case when

($*$) X_k are Borel sets in Polish spaces $\mathbb{X}_k$, and the distance functions d_k are Borel maps $X_k \times X_k \to \mathbb{R}^+$, not necessarily equal to the restrictions of Polish metrics of $\mathbb{X}_k$.

Then $\mathsf{D}(X_k; d_k)$ is obviously a Borel equivalence relation on $X = \prod_k X_k$.

The equivalence relation $\mathsf{D}(X_k; d_k)$ is *non-trivial* if $\mathtt{lim\,sup}_{k\to\infty}\,\mathtt{diam}(X_k) > 0$. (Otherwise $\mathsf{D}(X_k; d_k)$ obviously makes everything equivalent.)

A $\mathbf{c}_0$*-equality* is any equivalence relation of the form $\mathsf{D}(\langle X_k; d_k\rangle_{k\in\mathbb{N}})$, where all sets X_k are finite. □

Every $\mathbf{c}_0$-equality is easily a Borel equivalence relation, more exactly, of type $\mathbf{\Pi}^0_3$. The equivalence relation $\mathbf{c}_0$ itself is essentially a $\mathbf{c}_0$-equality (see below); this explains the meaning of the term "$\mathbf{c}_0$-equality".

The $\le_{\mathrm{B}}$-properties of these equivalence relations are largely unknown, except for the case of σ-compact metric spaces $\langle X_k\,; d_k\rangle$, easily reducible to the case of X_k finite ($=$ $\mathbf{c}_0$-equalities). This case is presented in this chapter. We prove that Borel reducibility of a $\mathbf{c}_0$-equality to another one implies a stronger additive reducibility of an infinitely generated $\mathbf{c}_0$-subequality (Theorem 16.3.2), prove that $\mathbf{c}_0$ is a $\le_{\mathrm{B}}$-largest $\mathbf{c}_0$-equality (Theorem 16.4.1), prove Theorem 16.5.1 on the turbulence of $\mathbf{c}_0$-equalities except those $\sim_{\mathrm{B}}$-equivalent to E_0 and E_3, and finally show that the $\le_{\mathrm{B}}$-structure of $\mathbf{c}_0$-equalities includes a substructure similar to $\langle \mathscr{P}(\mathbb{N}) ; \subseteq^* \rangle$ (Theorem 16.6.3).

16.2. Some examples and simple results

The following examples show that many typical equivalence relations can be defined in the form of $\mathbf{c}_0$-equalities.

EXAMPLE 16.2.1. (i) Let $X_k = \{0,1\}$ with $d_k(0,1) = 1$ for all k. Then clearly the relation $\mathsf{D}(X_k ; d_k)$ on $2^{\mathbb{N}} = \prod_k \{0,1\}$ is just E_0.

(ii) Let $X_{kl} = \{0,1\}$ with $d_{kl}(0,1) = k^{-1}$ for all $k, l \in \mathbb{N}$. Then the relation $\mathsf{D}(\langle X_{kl} ; d_{kl} \rangle_{k,l \in \mathbb{N}})$ on $2^{\mathbb{N}\times\mathbb{N}} = \prod_{k,l} \{0,1\}$ is exactly E_3.

(iii) Generally, if $0 = n_0 < n_1 < n_2 < \cdots$ and φ_i is a submeasure on $[n_i, n_{i+1})$, then let $X_i = \mathscr{P}([n_i, n_{i+1}))$ and $d_i(u,v) = \varphi_i(u \,\triangle\, v)$ for $u, v \subseteq [n_i, n_{i+1})$. Then $\mathsf{D}(X_i ; d_i)$ is isomorphic to $\mathsf{E}_{\mathscr{I}}$, where

$$\mathscr{I} = \mathtt{Exh}(\varphi) = \{x \subseteq \mathbb{N} : \lim_{n\to\infty} \varphi(x \cap [n, \infty)) = 0\}$$

and $\varphi(x) = \sup_i \varphi_i(x \cap [n_i, n_{i+1}))$.

(iv) Let, for all k, $X_k = \mathbb{R}$ with d_k being the usual distance on $\mathbb{R}$. Then the relation $\mathsf{D}(\langle X_k ; d_k \rangle_{k \in \mathbb{N}})$ on $\mathbb{R}^{\mathbb{N}}$ is just $\mathbf{c}_0$. □

LEMMA 16.2.2 (FARAH [**Far01b**] with a reference to HJORTH). *Every $\mathbf{c}_0$-equality $\mathsf{D} = \mathsf{D}(X_k ; d_k)$ is induced by a continuous action of a Polish group.*

The domain $\mathbb{X} = \prod_k X_k$ of D is considered with the product topology.

PROOF (sketch). For every k let S_k be the (finite) group of all permutations of X_k, with the distance $\rho_k(s,t) = \max_{x \in X_k} d_k(s(x), t(x))$. Then

$$\mathbb{G} = \{g \in \textstyle\prod_k S_k : \lim_{k\to\infty} \rho_k(g_k, e_k) = 0\}, \quad \text{where } e_k \in S_k \text{ is the identity},$$

is easily a subgroup of $\prod_k S_k$. Moreover, the distance $d(g,h) = \sup_k \rho_k(g_k, h_k)$ converts $\mathbb{G}$ into a Polish group, the natural action of which on $\mathbb{X}$, that is, $(g \cdot x)_k = g_k(x_k)$, $\forall\, k$, is continuous and induces D. □

Finally, let us show that the case of σ-compact spaces X_k does not give anything beyond the case of $\mathbf{c}_0$-equalities.

LEMMA 16.2.3. *Suppose that in the assumptions of Definition 16.1.1(*) $\langle X_k ; d_k \rangle$ are σ-compact spaces. Then $\mathsf{D}(X_k ; d_k)$ is $\sim_{\mathrm{B}}$-equivalent to a $\mathbf{c}_0$-equality.*

PROOF. Suppose that all spaces X_k are compact. Then for every k there exists a finite $\frac{1}{k}$-net $X'_k \subseteq X_k$. Given $x \in X = \prod_k X_k$, we define $\vartheta(x) \in X' = \prod_k X'_k$ so that $\vartheta(x)(k)$ is the d_k-closest to $x(k)$ element of X'_k (or the least, in the sense of a fixed ordering of X'_k, of such closest elements, whenever there exist two or more of them) for each k. Then ϑ is obviously a Borel reduction of $\mathsf{D}(X_k ; d_k)$ to the $\mathbf{c}_0$-equality $\mathsf{D}(X'_k ; d_k)$.

The general σ-compact case can be reduced to the compact case by the same trick as in the beginning of the proof of Lemma 6.2.2. □

16.3. $\mathbf{c}_0$-equalities and additive reducibility

The structure of $\mathbf{c}_0$-equalities tend to be connected more with the additive reducibility $\le_{\mathrm{A}}$ than with the general Borel reducibility.[1] In particular, we have

LEMMA 16.3.1. *If* $\mathsf{D} = \mathsf{D}(\langle X_k ; d_k\rangle_{k\in\mathbb{N}})$ *is a* $\mathbf{c}_0$*-equality and* D' *is a Borel equivalence relation on a set of the form* $\prod_k X'_k$ *with finite non-empty factors* X'_k *and* $\mathsf{D}' \le_{\mathrm{A}} \mathsf{D}$, *then* D' *itself is a* $\mathbf{c}_0$*-equality.*

PROOF. Let a sequence $0 = n_0 < n_1 < n_2 < \cdots$ and a collection of maps $H_i : X'_i \to \prod_{n_i\le k<n_{i+1}} X_k$ witness $\mathsf{D}' \le_{\mathrm{A}} \mathsf{D}$. For $x', y' \in X'_i$ put

$$d'_i(x', y') = \max_{n_i\le k<n_{i+1}} d_k(H_i(x')_k, H_i(y')_k)\,.$$

Then easily $\mathsf{D}' = \mathsf{D}(\langle X'_k ; d'_k\rangle_{k\in\mathbb{N}})$. □

It is perhaps not true that $\mathsf{D} \le_{\mathrm{B}} \mathsf{D}'$ implies $\mathsf{D} \le_{\mathrm{A}} \mathsf{D}'$ for any pair of $\mathbf{c}_0$-equalities. Yet a somewhat weaker statement holds by the next theorem of FARAH [**Far01b**].

THEOREM 16.3.2. *If* $\mathsf{D}= \mathsf{D}(\langle X_k ; d_k\rangle_{k\in\mathbb{N}})$ *and* $\mathsf{D}'= \mathsf{D}(\langle X'_k ; d'_k\rangle_{k\in\mathbb{N}})$ *are* $\mathbf{c}_0$*-equalities and* $\mathsf{D} \le_{\mathrm{B}} \mathsf{D}'$, *then there is an infinite set* $A \subseteq \mathbb{N}$ *such that the* $\mathbf{c}_0$*-equality* $\mathsf{D}_A= \mathsf{D}(\langle X_k ; d_k\rangle_{k\in A})$ *satisfies* $\mathsf{D}_A \le_{\mathrm{A}} \mathsf{D}'$.

PROOF. Define $X_C = \prod_{k\in C} X_k$ and $X'_C = \prod_{k\in C} X'_k$ for any set $C \subseteq \mathbb{N}$, and $d'_C(x, y) = \sup_{k\in C} d'_k(x(k), y(k))$ for all $x, y \in X'$. Suppose that

$$\vartheta : X = \prod_{k\in\mathbb{N}} X_k \to X' = \prod_{k\in\mathbb{N}} X'_k$$

is a Borel reduction of D to D'. Then there exists an infinite set $A' \subseteq \mathbb{N}$ such that $\mathsf{D}(\langle X_k ; d_k\rangle_{k\in A'}) \le_{\mathrm{C}} \mathsf{D}'$ (via a continuous reduction); this can be proved analogously to the second claim of Lemma 5.3.1. Thus it can be assumed from the beginning that ϑ is a continuous reduction of D to D'.

To extract an additive reduction, we employ a version of the construction used in the proof of Theorem 6.3.1(i). In fact our task here is somewhat simpler because the given countinuity of ϑ allows us to avoid the Cohen genericity arguments.

Put $[s] = \{x \in X : x \restriction u = s\}$ for any $u \subseteq \mathbb{N}$ and $s \in X_u$. Consider the closed set $W = \bigcap_{i\in\mathbb{N}}[s_i]$ of all points $x \in X$ such that $x \restriction [n_i, n_{i+1}) = s_i$ for all i. Arguing approximately as in the proof of Theorem 6.3.1(i), we can define an increasing sequence $0 = k_0 = n_0 < k_1 < n_1 < k_2 < n_2 < \cdots$ and elements $s_i \in X_{(n_i,n_{i+1})}$ such that for all $u, v \in X_{[0,n_i]}$ and all $x, y \in X_{[n_{i+1},\infty)}$ satisfying $x \restriction (n_j, n_{j+1}) = y \restriction (n_j, n_{j+1}) = s_j$ for all indices $j > i$ and $u \restriction (n_j, n_{j+1}) = v \restriction (n_j, n_{j+1}) = s_j$ for all indices $j < i$,[2] the following holds:

(a) $\vartheta(u \cup s_i \cup x) \restriction [0, k_{i+1}) = \vartheta(u \cup s_i \cup y) \restriction [0, k_{i+1})$, and

(b) $d_{[k_{i+1},\infty)}(\vartheta(u \cup s_i \cup x), \vartheta(v \cup s_i \cup x)) < \frac{1}{i}$.

Put $A = \{n_i : i \in \mathbb{N}\}$ and fix $z \in X_A$. For any i, if $\xi \in X_{n_i}$, then define $z^{i\xi} \in W$ so that $z^{i\xi}(n_i) = \xi$, $z^{i\xi}(n_j) = z(n_j)$ for all $j \ne i$, and $z^{i\xi} \restriction (n_j, n_{j+1}) = s_j$ for all j. If $x \in X_A$, then define $H(x) \in X'$ as follows:

(1) $H(x) \restriction [k_i, k_{i+1}) = \vartheta(z^{i,x(n_i)}) \restriction [k_i, k_{i+1})$ for every $i \in \mathbb{N}$.

[1] See Section 5.4 on $\le_{\mathrm{A}}$ and the associated relations $<_{\mathrm{A}}$ and $\sim_{\mathrm{A}}$.

[2] Under this assumption the points $u \cup s_i \cup x$, $u \cup s_i \cup y$, $v \cup s_i \cup x$ in (a), (b) belong to W.

Clearly, H is a continuous map from X_A to X' (in the sense of the Polish product topologies). Moreover for every i the value $H(x) \restriction [k_i, k_{i+1})$ obviously depends only on $x(n_i)$. Thus to accomplish the proof of the theorem, we need only prove that H is a reduction of D_A to D'.

For any $x \in X_A$, define $f(x) \in W$ so that $f(x) \restriction A = x$ and $f(x) \restriction (n_j, n_{j+1}) = s_j$ for all j. Then f is a reduction of D_A to D. Therefore it suffices to prove that $\vartheta(f(x))\ \mathsf{D}'\ H(x)$ for every $x \in X_A$. For an arbitrary $i \geq 1$, let us show that

(2) $d'_{[k_i, k_{i+1})}(\vartheta(f(x)), H(x)) \leq 1/i$.

The key fact is that by construction, the elements $a = f(x)$ and $b = z^{i, x(n_i)}$ of W satisfy $a \restriction (n_j, n_{j+1}) = b \restriction (n_j, n_{j+1}) = s_j$ for all j and in addition $a(n_i) = b(n_i) = x(n_i)$. Define an auxiliary element $c \in W$ by

$$c \restriction [0, n_i] = a \restriction [0, n_i] \quad \text{and} \quad c \restriction [n_{i+1}, \infty) = b \restriction [n_{i+1}, \infty).$$

Then $d'_{[k_i, k_{i+1})}(\vartheta(b), \vartheta(c)) \leq \frac{1}{i}$ by (b) and $\vartheta(a) \restriction [k_i, k_{i+1}) = \vartheta(c) \restriction [k_i, k_{i+1})$ by (a). (Note that (b) is applied in fact for the value $i-1$ instead of i.) It follows that $d'_{[k_i, k_{i+1})}(\vartheta(a), \vartheta(b)) \leq \frac{1}{i}$. However, $H(x) \restriction [k_i, k_{i+1}) = \vartheta(b) \restriction [k_i, k_{i+1})$ by (1). This proves (2), as required. □

16.4. A largest $\mathbf{c}_0$-equality

We define $\mathsf{c}_{\max} = \mathsf{D}(X_k; d_k)$, where $X_k = \{0, \frac{1}{k}, \frac{2}{k}, \dots, 1\}$ and d_k is the distance on X_k inherited from the real line $\mathbb{R}$. The next theorem says that $\mathsf{c}_{\max}$ is $\leq_{\mathrm{B}}$-largest among all $\mathbf{c}_0$-equalities. The proof will show that in fact $\mathsf{D} \leq_{\mathrm{A}} \mathsf{c}_{\max}$ in (ii), in the sense of the additive reducibility.

THEOREM 16.4.1 (FARAH [**Far01b**] with a reference to OLIVER).

(i) $\mathsf{c}_{\max} \sim_{\mathrm{B}} \mathbf{c}_0$;

(ii) *If* D *is a* $\mathbf{c}_0$*-equality, then* $\mathsf{D} \leq_{\mathrm{B}} \mathsf{c}_{\max}$.

It follows from (i) and Lemma 6.2.3 that $\mathsf{c}_{\max} \sim_{\mathrm{B}} \mathsf{Z}_0$.

PROOF. (i) It is clear that $\mathsf{c}_{\max}$ is the same as $\mathbf{c}_0 \restriction \mathbb{X}$, where $\mathbb{X} \subseteq \mathbb{R}^{\mathbb{N}}$ is defined as in the proof of Lemma 6.2.3, where it is also shown that $\mathbf{c}_0 \sim_{\mathrm{B}} \mathbf{c}_0 \restriction \mathbb{X}$.

(ii) To prove $\mathsf{D} \leq_{\mathrm{B}} \mathsf{c}_{\max}$, it suffices by (i) to show that $\mathsf{D} \leq_{\mathrm{B}} \mathbf{c}_0$. The proof is based on the following:

CLAIM 16.4.2. *Every finite* n*-element metric space* $\langle X\,; d\rangle$ *is isometric to an* n*-element subset of* $\langle \mathbb{R}^n\,; \rho_n\rangle$*, where* ρ_n *is the distance on* $\mathbb{R}^n$ *defined by* $\rho_n(x,y) = \max_{i<n} |x(i) - y(i)|$.

PROOF. Let $X = \{x_1, \dots, x_n\}$. It suffices to prove that for every $k \neq l$ there is a set of reals $\{r_1, \dots, r_n\}$ such that $|r_k - r_l| = d(x_k, x_l)$ and

(†) $|r_i - r_j| \leq d_{ij} = d(x_i, x_j)$: for all i, j.

We can assume that $k = 1$ and $l = n$.

Step 1. There is a least number $h_1 \geq 0$ such that (†) holds for the reals $\{r_i\} = \{\underbrace{0, 0, \dots, 0}_{n-1 \text{ times}}, h\}$ for all $0 \leq h \leq h_1$. Then, for some index k, $1 \leq k < n$, we have $h_1 - 0 = d_{kn}$ exactly. Suppose that $k \neq 1$; then it can be assumed that $k = n - 1$.

Step 2. Similarly, there is a least number $h_2 \geq 0$ such that (†) holds for the reals $\{r_i\} = \{\underbrace{0, 0, \dots, 0}_{n-2 \text{ times}}, h, h_1 + h\}$ for all $0 \leq h \leq h_2$. (For example, $h_2 = 0$ in the case when in Step 1 we have one more index $k' \neq k$ such that $h_1 = d_{k'n}$.) Then, for some k, ν, $1 \leq k < n - 1 \leq \nu \leq n$, we have $h_2 - 0 = d_{k\nu}$ exactly. Suppose that $k \neq 1$; then it can be assumed that $k = n - 2$.

Step 3. Similarly, there is a least number $h_3 \geq 0$ such that (†) holds for the reals $\{r_i\} = \{\underbrace{0, 0, \dots, 0}_{n-3 \text{ times}}, h, h_2 + h, h_1 + h_2 + h\}$ for all $0 \leq h \leq h_3$. Then again, for some k, ν, $1 \leq k < n - 2 \leq \nu \leq n$, we have $h_3 - 0 = d_{k\nu}$ exactly. Suppose that $k \neq 1$; then it can be assumed that $k = n - 3$.

Et cetera.

This process ends, after a number m $(m < n)$ steps, in such a way that the index k obtained at the final step is equal to 1. Then (†) holds for the numbers $\{\underbrace{0, 0, \dots, 0}_{n-m \text{ times}}, r_{n-m+1}, r_{n-m+1}, \dots, r_n\}$, where $r_{n-m+j} = h_m + h_{m-1} + \dots + h_{m-j+1}$ for each $j = 1, \dots, m$. Moreover, it follows from the construction that there is a decreasing sequence $n = k_0 > k_1 > k_2 > \dots > k_\mu = 1$ $(\mu \leq m)$ such that $r_{k_i} - r_{k_{i+1}} = d_{k_{i+1}, k_i}$ exactly for every i. Then $d_{1n} \leq \sum_i r_{k_i} - r_{k_{i+1}}$ by the triangle inequality. But the right-hand side is a part of the sum $r_n = h_1 + \dots + h_m$, and hence $r_n \geq d_{1n}$. On the other hand we have $r_n \leq d_{1n}$ by (†). We conclude that $r_n = d_{1n}$, as required. □ (Claim)

We come back to the proof of (ii), that is, $\mathsf{D} \leq_{\mathrm{B}} \mathbf{c}_0$ for an arbitrary $\mathbf{c}_0$-equality $\mathsf{D} = \mathsf{D}(X_k; d_k)$ on $\mathbb{X} = \prod_{k \in \mathbb{N}} X_k$, where each $\langle X_k ; d_k \rangle$ is a finite metric space. Let n_k be the number of elements in X_k. By the claim, let $\eta_k : X_k \to \mathbb{R}^{n_k}$ be an isometric embedding of $\langle X_k ; d_k \rangle$ into $\langle \mathbb{R}^{n_k} ; \rho_{n_k} \rangle$. It easily follows that the map $\vartheta(x) = \eta_0(x_0)^\frown \eta_1(x_1)^\frown \eta_2(x_2)^\frown \cdots$ (from $\mathbb{X}$ to $\mathbb{R}^{\mathbb{N}}$) reduces D to $\mathbf{c}_0$.

□ (Theorem 16.4.1)

16.5. Classification

Recall that for a metric space $\langle A ; d \rangle$, a rational $q > 0$, and $a \in A$, the galaxy $\mathtt{Gal}^q_A(a)$ is the set of all $b \in A$ which can be connected with a by a finite chain $a = a_0, a_1, \dots, a_n = b$ with $d(a_i, a_{i+1}) < q$ for all i. Define, for $r > 0$,

$$\delta(r, A) = \mathtt{inf}\,\{q \in \mathbb{Q}^+ : \exists\, a \in A\,(\mathtt{diam}\,(\mathtt{Gal}^q_A(a)) \geq r)\}$$

(with the understanding that here $\mathtt{inf}\,\varnothing = +\infty$), and

$$\Delta(A) = \{d(a, b) : a \neq b \in A\}\,, \quad \text{so that} \quad \mathtt{diam}\, A = \sup(\Delta(A) \cup \{0\})\,.$$

Now suppose that $\mathsf{D} = \mathsf{D}(X_k; d_k)$ is a $\mathbf{c}_0$-equality on a set of the form $\mathbb{X} = \prod_{k \in \mathbb{N}} X_k$. The next theorem of FARAH [**Far01b**] shows that basic properties of D in the $\leq_{\mathrm{B}}$-structure of Borel equivalence relations are determined by the following two conditions:

(co1) $\mathtt{liminf}_{k \to \infty}\, \delta(r, X_k) = 0$ for some $r > 0$.

(co2) $\forall\, \varepsilon > 0\; \exists\, \varepsilon' \in (0, \varepsilon)\; \exists^\infty k\; (\Delta(X_k) \cap [\varepsilon', \varepsilon) \neq \varnothing)$.

Clearly (co1) implies both the non-triviality of $\mathsf{D}(X_k; d_k)$ and (co2).

THEOREM 16.5.1. *Let* $\mathsf{D} = \mathsf{D}(\langle X_k; d_k\rangle_{k\in\mathbb{N}})$ *be a non-trivial* $\mathbf{c}_0$*-equality. Then*:

(i) *if* (co2) *fails* (then (co1) also fails), *then* $\mathsf{D} \sim_{\mathrm{B}} \mathsf{E}_0$;

(ii) *if* (co1) *fails but* (co2) *holds, then* $\mathsf{D} \sim_{\mathrm{B}} \mathsf{E}_3$;

(iii) *if* (co1) *holds* (then (co2) also holds), *then there exists a turbulent* $\mathbf{c}_0$*-equality* D' *satisfying* $\mathsf{E}_0 <_{\mathrm{B}} \mathsf{D}'$ *and* $\mathsf{D}' \le_{\mathrm{B}} \mathsf{D}$.

Thus every non-trivial $\mathbf{c}_0$-equality D $\le_{\mathrm{B}}$-contains a turbulent $\mathbf{c}_0$-equality D' with $\mathsf{E}_3 <_{\mathrm{B}} \mathsf{D}'$ unless D is $\sim_{\mathrm{B}}$-equivalent to either E_0 or E_3. In addition, (co1) is necessary for the turbulence of D itself and sufficient for a turbulent $\mathbf{c}_0$-equality $\mathsf{D}' \le_{\mathrm{B}} \mathsf{D}$ to exist. The proof will show that in fact $\le_{\mathrm{B}}$ can be improved to $\le_{\mathrm{A}}$ in the theorem.

PROOF. (i) To show that $\mathsf{E}_0 \le_{\mathrm{B}} \mathsf{D}$, note that, by the non-triviality of D, there exist a number $p > 0$, an increasing sequence $0 = n_0 < n_1 < n_2 < \cdots$, and, for every i, a pair of elements $x_{n_i}, y_{n_i} \in X_{n_i}$ with $d_{n_i}(x_{n_i}, y_{n_i}) \ge p$. For n not of the form n_i, fix an arbitrary $z_n \in X_n$. Now, if $a \in 2^{\mathbb{N}}$, then define $\vartheta(a) \in \prod_k X_k$ so that $\vartheta(a)(n) = z_n$ for n not of the form n_i, while $\vartheta(a)(n_i) = x_{n_i}$ or $= y_{n_i}$ if resp. $a_i = 0$ or $= 1$. This map ϑ witnesses $\mathsf{E}_0 \le_{\mathrm{B}} \mathsf{D}$.

Now prove that $\mathsf{D} \le_{\mathrm{B}} \mathsf{E}_0$. As (co2) fails, there is $\varepsilon > 0$ such that for each ε' with $0 < \varepsilon' < \varepsilon$, we have only finitely many k with the propery that $\varepsilon' \le d_k(\xi, \eta) < \varepsilon$ for some $\xi, \eta \in X_k$. Let G_k be the (finite) set of all $\frac{\varepsilon}{2}$-galaxies in X_k, and let $\vartheta : \mathbb{X} = \prod_k X_k \to G = \prod_k G_k$ be defined as follows. For every k, $\vartheta(x)(k)$ is that galaxy in G_k to which $x(k)$ belongs. Let E be the G-version of E_0, that is, if g, $h \in G$, then $g \mathrel{\mathsf{E}} h$ iff $g(k) = h(k)$ for all but finite k. As easily $\mathsf{E} \le_{\mathrm{B}} \mathsf{E}_0$, it suffices to demonstrate that $\mathsf{D} \le_{\mathrm{B}} \mathsf{E}$ via ϑ.

Suppose that $x, y \in \mathbb{X}$ and $\vartheta(x) \mathrel{\mathsf{E}} \vartheta(y)$, and prove $x \mathrel{\mathsf{D}} y$ (the non-trivial direction). Suppose toward the contrary that $x \not\mathrel{\mathsf{D}} y$, so that there is a number $p > 0$ with $d_k(x(k), y(k)) > p$ for infinitely many k. We can assume that $p < \frac{\varepsilon}{2}$. On the other hand, as $\vartheta(x) \mathrel{\mathsf{E}} \vartheta(y)$, there is k_0 such that $x(k)$ and $y(k)$ belong to one and the same $\frac{\varepsilon}{2}$-galaxy in X_k for all $k > k_0$. Then, for every $k > k_0$ with $d_k(x(k), y(k)) > p$ (and hence for infinitely many indices k) there exists an element $z_k \in X_k$ in the same galaxy such that $p < d_k(x(k), z_k) < \varepsilon$, but this is a contradiction to the choice of ε (indeed, take $\varepsilon' = p$).

(ii) First prove that if (co2) holds, then $\mathsf{E}_3 \le_{\mathrm{B}} \mathsf{D}$. It follows from (co2) that there exist an infinite sequence $\varepsilon_1 > \varepsilon_2 > \varepsilon_3 > \cdots > 0$, for every i an infinite set $J_i \subseteq \mathbb{N}$, and for every $j \in J_i$ a pair of elements $x_{ij}, y_{ij} \in X_j$ with $d_j(x_{ij}, y_{ij}) \in [\varepsilon_{i+1}, \varepsilon_i)$. We may assume that the sets J_i are pairwise disjoint. Then the $\mathbf{c}_0$-equality $\mathsf{D}' = \mathsf{D}(\langle\{x_{ij}, y_{ij}\}; d_j\rangle_{i\in\mathbb{N},\, j\in J_i})$ satisfies both $\mathsf{D}' \le_{\mathrm{B}} \mathsf{D}$ and $\mathsf{D}' \cong \mathsf{E}_3$ (an isomorphism via a Borel bijection between the underlying sets).

Now, assuming that, in addition, (co1) fails, we show that $\mathsf{D} \le_{\mathrm{B}} \mathsf{E}_3$. For all k, $n \in \mathbb{N}$ let G_{kn} be the (finite) set of all $\frac{1}{n}$-galaxies in X_k. For every $x \in \mathbb{X} = \prod_i X_i$ define $\vartheta(x) \in G = \prod_{k,n} G_{kn}$ so that for all k, n, $\vartheta(x)(k,n)$ is that $\frac{1}{n}$-galaxy in G_{kn} to which $x(k)$ belongs (for all k, n). The equivalence relation

$$g \mathrel{\mathsf{E}} h \quad \text{iff} \quad \forall n\, \forall^\infty k\, (g(k,n) = h(k,n)) \qquad (g, h \in G),$$

where $\forall^\infty k$ means *for all but finitely many* k, is obviously $\le_{\mathrm{B}} \mathsf{E}_3$, so it suffices to show that $\mathsf{D} \le_{\mathrm{B}} \mathsf{E}$ via ϑ. Suppose that $x, y \in \mathbb{X}$ and $\vartheta(x) \mathrel{\mathsf{E}} \vartheta(y)$, and prove $x \mathrel{\mathsf{D}} y$

(the non-trivial direction). Otherwise there is some $r > 0$ with $d_k(x(k), y(k)) > r$ for infinitely many indices k. As (co1) fails for this r, there is n big enough for $\delta(r, X_k) > \frac{1}{n}$ to hold for almost all k. Then, by the choice of r, we have $\vartheta(x)(k,n) \neq \vartheta(y)(k,n)$ for infinitely many k, hence, $\vartheta(x) \not\mathrel{\mathsf{E}} \vartheta(y)$, a contradiction.

(iii) Fix $r > 0$ with $\mathtt{liminf}_{k\to\infty} \delta(r, X_k) = 0$. For every increasing sequence $n_0 < n_1 < n_2 < \cdots$ we have $\mathsf{D}(\langle X_{n_k}; d_{n_k}\rangle_{k\in\mathbb{N}}) \leq_{\mathsf{B}} \mathsf{D}$. Therefore, it can be assumed that $\mathtt{lim}_k\, \delta(r, X_k) = 0$, and further that $\delta(r, X_k) < \frac{1}{k}$ for all k. (Otherwise choose an appropriate subsequence.) Then every set X_k contains a $\frac{1}{k}$-galaxy $Y_k \subseteq X_k$ such that $\mathtt{diam}\, Y_k \geq r$. As easily $\mathsf{D}(Y_k; d_k) \leq_{\mathsf{B}} \mathsf{D}$, the following lemma suffices to prove (iii).

LEMMA 16.5.2. *Assume that* $r > 0$ *and each* X_k *is a* $\frac{1}{k}$*-galaxy and that* $\mathtt{diam}(X_k) \geq r$. *Then the* $\mathbf{c}_0$*-equality* $\mathsf{D} = \mathsf{D}(\langle X_k; d_k\rangle_{k\in\mathbb{N}})$ *is turbulent and satisfies* $\mathsf{E}_3 \leq_{\mathsf{B}} \mathsf{D}$.

PROOF. We know from the proof of (iii) above that $\mathsf{E}_3 \leq_{\mathsf{B}} \mathsf{D}$. Now prove that the natural action of the Polish group $\mathbb{G}$, defined as in the proof of Lemma 16.2.2, is turbulent under the assumptions of the lemma.

That every D-class is dense in $\mathbb{X} = \prod_k X_k$ (with the product topology on $\mathbb{X}$) is an easy exercise. To see that every D-class $[x]_{\mathsf{D}}$ also is meager in $\mathbb{X}$, note that by the assumptions of the lemma every X_k contains a pair of elements x'_k, x''_k with $d_k(x'_k, x''_k) \geq r$. Let y_k be one of x'_k, x''_k which is d_k-further than $\frac{r}{2}$ from x_k. The set $Z = \{z \in \mathbb{X} : \exists^\infty k\ (z(k) = y_k)\}$ is comeager in $\mathbb{X}$ and disjoint from $[x]_{\mathsf{D}}$.

It remains to prove that local orbits are somewhere dense. Let G be an open nbhd of the neutral element in $\mathbb{G}$ and $\varnothing \neq X \subseteq \mathbb{X}$ be open in $\mathbb{X}$. We can assume that, for some n, G is the $\frac{1}{n}$-ball around the neutral element in $\mathbb{G}$ while $X = \{x \in \mathbb{X} : \forall k < n\ (x(k) = \xi_k)\}$, where elements $\xi_k \in X_k$, $k < n$, are fixed. It is enough to prove that all local orbits, *i.e.*, equivalence classes of $\sim^G_X$, are dense subsets of X. Consider an open set $Y = \{y \in \mathbb{X} : \forall k < m\ (y(k) = \xi_k)\} \subseteq X$, where $m > n$ and elements $\xi_k \in X_k$, $n \leq k < m$, are fixed in addition to the above.

Let $x \in X$. Then $x(k) = \xi_k$ for all $k < n$. Let $n \leq k < m$. The elements ξ_k and $x(k)$ belong to X_k, which is a $\frac{1}{k}$-galaxy. Therefore, there is a chain, of a length $\ell(k)$, of elements of X_k, which connects $x(k)$ to ξ_k so that every step within the chain has d_k-length $< \frac{1}{k}$. Then there is a permutation g_k of X_k such that $g_k^{\ell(k)}(x(k)) = \xi_k$, $g_k(\xi_k) = x(k)$, and $d_k(\xi, g_k(\xi)) < \frac{1}{k}$ for all $\xi \in X_k$.

In addition let g_k be the identity on X_k whenever $k < n$ or $k \geq m$. This defines an element $g \in \mathbb{G}$ which obviously belongs to G. Moreover, the set X is g-invariant and $g^\ell(x) \in U$, where $\ell = \prod_{k=n}^{m-1} \ell(k)$. It follows that $x \sim^G_X g(x)$, as required. □ (Lemma)

□ (Theorem 16.5.1)

16.6. LV-equalities

By definition an LV-*equality* is a $\mathbf{c}_0$-equality $\mathsf{D} = \mathsf{D}(\langle X_k; d_k\rangle_{k\in\mathbb{N}})$ satisfying the following condition:

16.6.1. $\forall m\ \forall \varepsilon > 0\ \forall^\infty k\ \forall x_0, \ldots, x_m \in X_k\ (d_k(x_0, x_m) \leq \varepsilon + \mathtt{max}_{j<m}\, d_k(x_j, x_{j+1}))$.

In other words, the metrics involved are postulated to be *asymptotically close* to ultrametrics. These sorts of $\mathbf{c}_0$-equalities were first considered by LOUVEAU and VELICKOVIC in [**LV94**].

EXERCISE 16.6.2. Put $X_k = \{1, 2, \dots, 2^k\}$ and define the distance functions by $d_k(m,n) = \frac{\log_2(|m-n|+1)}{k}$ for all k and $1 \le m, n \le 2^k$. Prove that $\mathsf{D}(X_k; d_k)$ is an LV-equality and satisfies (co1) of Section 16.5. □

The next theorem of LOUVEAU and VELICKOVIC [**LV94**] is a major application of $\mathbf{c}_0$-equalities. One of its corollaries is that there exist large families of mutually irreducible Borel equivalence relations; see below.

THEOREM 16.6.3. *Let* $\mathsf{D} = \mathsf{D}(\langle X_k; d_k\rangle_{k\in\mathbb{N}})$ *be an* LV*-equality satisfying* (co1) *of Section* 16.5. *Then we can associate, with each infinite set* $A \subseteq \mathbb{N}$, *an* LV*-equality* $\mathsf{D}_A \le_{\mathrm{A}} \mathsf{D}$ *such that for all* $A, B \subseteq \mathbb{N}$ *the following are equivalent*:

(i) $A \subseteq^* B$ (that is, $A \smallsetminus B$ is finite);

(ii) $\mathsf{D}_A \le_{\mathrm{A}} \mathsf{D}_B$ (the additive reducibility);

(iii) $\mathsf{D}_A \le_{\mathrm{B}} \mathsf{D}_B$.

PROOF. Since D is turbulent, the necessary turbulence condition (co1) of Section 16.5 holds. Moreover, as in the proof of Theorem 16.5.1 (part (iii)), we can assume that it takes the following special form for some $r > 0$:

(1) Each X_k is a $(\mathtt{min}\{\frac{r}{2}, \frac{1}{k+1}\})$-galaxy and $\mathtt{diam}(X_k) \ge 4r$.

The intended transformations (reduction to a certain infinite subsequence of spaces $\langle X_k; d_k\rangle$ and then of each X_k to a suitable galaxy $Y_k \subseteq X_k$) preserve the LV-condition 16.6.1, of course. Moreover, we can assume that 16.6.1 holds in the following special form:

(2) $d_k(x_0, x_{\mu_k}) \le \frac{1}{k+1} + \mathtt{max}_{i<\mu_k} d_k(x_i, x_{i+1})$ whenever $x_0, \dots, x_{\mu_k} \in X_k$, where $\mu_k = \prod_{j=0}^{k-1} \mathtt{card}\,(X_j)$ and $\mathtt{card}\, X$ is the number of elements in a finite set X here.

(For if not, then take a suitable subsequence once again.)

We can derive the following important consequence:

(3) For every k there is a set $Y_k \subseteq X_k$ having exactly $\mathtt{card}\,(Y_k) = \mu_k$ elements and such that $d_k(x,y) \ge r$ for all $x \ne y$ in Y_k.

To prove this, note that by (1) there is a set $\{x_0, \dots, x_m\} \subseteq X_k$ such that $d_k(x_0, x_m) \ge 4r$ but $d_k(x_i, x_{i+1}) < r$ for all i. We may assume that m is the least possible length of such a sequence $\{x_i\}$. Define a subsequence $\{y_0, y_1, \dots, y_n\}$ of $\{x_i\}$; the number $n \le m$ will be specified in the course of the construction.

(a) Put $y_0 = x_0$.

(b) If $y_j = x_{i(j)}$ has been defined and there is an index $l > i(j)$, $l \le m$, such that $d_k(y_j, x_l) \ge r$, then let $y_{j+1} = x_l$ for the least such l.

Note that in this case $d_k(y_j, y_{j+1}) < 2r$, for otherwise $d_k(y_j, x_{l-1}) > r$ because $d_k(x_{l-1}, x_l) < r$.

(c) Otherwise put $n = j$ and stop the construction.

By definition $d_k(y_j, y_{j+1}) \geq r$ for all $j < n$, moreover, $d_k(y_{j'}, y_{j+1}) \geq r$ for all $j' < j$ by the minimality of m. Thus $Y_k = \{y_j : j \leq n\}$ satisfies $d_k(x,y) \geq r$ for all $x \neq y$ in Y_k. It remains to prove that $n \geq \mu_k$. Suppose otherwise. Add $y_{n+1} = x_m$ as an extra term. Then $d_k(x_0, x_m) = d_k(y_0, y_{n+1}) \leq 3r$ by (2) because $d_k(y_j, y_{j+1}) < 2r$ (see above). However, we know that $d_k(x_0, x_m) \geq 4r$, a contradiction. This proves (3).

In continuation of the proof of the theorem, define $\mathsf{D}_A = \mathsf{D}(\langle X_k; d_k \rangle_{k \in A})$ for all $A \subseteq \mathbb{N}$. Thus D_A is essentially a $\mathbf{c}_0$-equality on $\prod_{k \in A} X_k$. The direction (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) is routine. Thus it remains to prove (iii) $\Longrightarrow$ (i). In view of Theorem 16.3.2, it is enough to prove the following lemma.

LEMMA 16.6.4. *If $A, B \subseteq \mathbb{N}$ are infinite and disjoint, then $\mathsf{D}_A \leq_{\mathsf{A}} \mathsf{D}_B$ fails.*

PROOF. Suppose toward the contrary that $\mathsf{D}_A \leq_{\mathsf{A}} \mathsf{D}_B$ holds, and let this be witnessed by a reduction Ψ defined (as in Section 5.4) from an increasing sequence $\min B = n_0 < n_1 < n_2 < \cdots$ of numbers $n_k \in B$ and a collection of maps $H_k : X_k \to \prod_{m \in [n_k, n_{k+1}) \cap B} X_m$, $k \in A$. We put

$$f_k(\delta) = \max_{\xi, \eta \in X_k,\, d_k(\xi,\eta) < \delta} \; \max_{m \in [n_k, n_{k+1}) \cap B} \; d_m(H_k(\xi)(m), H_k(\eta)(m)),$$

for $k \in \mathbb{N}$ and $\delta > 0$ (with the understanding that $\max \varnothing = 0$ if applicable). Then $f(\delta) = \sup_{k \in A} f_k(\delta)$ is a non-decreasing map $\mathbb{R}^+ \to [0, \infty)$.

We claim that $\lim_{\delta \to 0} f(\delta) = 0$. Indeed, otherwise there is $\varepsilon > 0$ such that $f(\delta) \geq \varepsilon$ for all δ. Then the numbers

$$s_k = \min_{\xi, \eta \in X_k,\, \xi \neq \eta} d_k(\xi, \eta) \quad \text{(all of them are } > 0)$$

must satisfy $\inf_{k \in A} s_k = 0$. This allows us to define a sequence $k_0 < k_1 < k_2 < \cdots$ of numbers $k_i \in A$, and, for every k_i, a pair of elements $\xi_i, \eta_i \in X_{k_i}$ with $d_{k_i}(\xi_i, \eta_i) \to 0$ and also a number $m_i \in [n_{k_i}, n_{k_i+1}) \cap B$ such that

$$d_{m_i}(H_{k_i}(\xi_i)(m_i), H_{k_i}(\eta_i)(m_i)) \geq \varepsilon.$$

Let $x, y \in \prod_{k \in A} X_k$ satisfy $x(k_i) = \xi_i$ and $y(k_i) = \eta_i$ for all i and $x(k) = y(k)$ for all $k \in A$ not of the form k_i. Then easily $x \;\mathsf{D}_A\; y$ holds but $\Psi(x) \;\mathsf{D}_B\; \Psi(y)$ fails, which is a contradiction. Thus in fact $\lim_{\delta \to 0} f(\delta) = 0$.

Let $k \in A$, and let $Y_k \subseteq X_k$ be as in (3). Then there exist elements $x_k \neq y_k$ in Y_k such that $H_k(x_k) \restriction k = H_k(y_k) \restriction k$. By (1) there is a chain $x_k = \xi_0, \xi_1, \ldots, \xi_n = y_k$ of elements $\xi_i \in X_k$ with $d_k(\xi_i, \xi_{i+1}) \leq \frac{1}{k+1}$ for all $i < n$. Now $H_k(\xi_i) \in \prod_{m \in [n_k, n_{k+1}) \cap B} X_m$ for each $i \leq n$.

Suppose that $m \in [n_k, n_{k+1}) \cap B$, and hence $m \geq n_k \geq k$. The elements $y_i^m = H_k(\xi_i)(m)$, $i \leq n$, satisfy $d_m(y_i^m, y_{i+1}^m) \leq f_k(\frac{1}{k+1})$. Note that $m \neq k$ because $k \in A$ while $m \in B$. Thus we have $m > k$ strictly. It follows that $n \leq \mu_m$. Therefore, by (2), we obtain

$$\text{(4)} \quad d_m(H_k(x_k)(m), H_k(y_k)(m)) \leq f_k(\tfrac{1}{k+1}) + \tfrac{1}{m+1} \leq f(\tfrac{1}{k+1}) + \tfrac{1}{k+1}$$

for all $m \in [n_k, n_{k+1}) \cap B$.

Both $x = \{x_k\}_{k \in A}$ and $y = \{y_k\}_{k \in A}$ are elements of $\prod_{k \in A} X_k$, and $x \;\mathsf{D}_A\; y$ fails because $d_k(x_k, y_k) \geq r$ for all k. On the other hand, we have $\Psi(x) \;\mathsf{D}_B\; \Psi(y)$ by (4), because $\lim_{\delta \to 0} f(\delta) = 0$. This is a contradiction to the assumption that Ψ reduces D_A to D_B. □ (Lemma 16.6.4)

□ (Theorem 16.6.3)

16.7. Non-σ-compact case

For any metric space $\mathbb{X} = \langle X\,; d\rangle$, let $\mathsf{D}(\mathbb{X})$ denote the equivalence relation $\mathsf{D}(X_k; d_k)$ on $X^{\mathbb{N}}$, where $\langle X_k\,; d_k\rangle = \langle \mathbb{X}\,; d\rangle$ for all k. Thus $\mathbf{c}_0$ is equal to $\mathsf{D}(\mathbb{R})$. One may ask, what is the place of equivalence relations of the form $\mathsf{D}(\mathbb{X})$, where $\mathbb{X}$ is a Polish space, in the global $\leq_{\mathrm{B}}$-structure of Borel equivalence relations?

The case of σ-compact Polish spaces here can be reduced to the case of finite spaces, *i.e.*, to $\mathbf{c}_0$-equalities, by Lemma 16.2.3. Thus in this case, we obtain a family of Borel equivalence relations situated $\leq_{\mathrm{B}}$-between the relations E_3 and $\mathbf{c}_0$ by Theorems 16.5.1 and 16.4.1, and this family has a rather rich $\leq_{\mathrm{B}}$-structure by Theorem 16.6.3.

The case of non-σ-compact spaces is much less studied.

EXAMPLE 16.7.1. Let $\mathbb{X} = \mathbb{N}^{\mathbb{N}}$ be the Baire space, with the standard distance $d(a,b) = \frac{1}{m(a,b)+1}$, where $m(a,b)$ (for $a \neq b \in \mathbb{N}^{\mathbb{N}}$) is the largest integer m such that $a \restriction m = b \restriction m$.[3] If $x \in \mathbb{N}^{\mathbb{N}}$ and $n, k \in \mathbb{N}$, then $x(n) \restriction k$ is a finite sequence of k integers. It follows from the fact that $\mathbb{N}^{\mathbb{N}}$ is 0-dimensional that $x\ \mathsf{D}(\mathbb{N}^{\mathbb{N}})\ y$ is equivalent to

$$\forall n\, \exists k_0\, \forall k \geq k_0\ (x(n) \restriction k = y(n) \restriction k)\,.$$

for all $x, y \in \mathbb{N}^{\mathbb{N}}$. □

EXERCISE. Use this to show that $\mathsf{D}(\mathbb{N}^{\mathbb{N}}) \sim_{\mathrm{B}} \mathsf{E}_3$.

QUESTION 16.7.2. Now let $\mathbb{X}$ be the Polish space $C[0,1]$ of all continuous maps $f : [0,1] \to \mathbb{R}$, with the distance $d(f,g) = \mathtt{max}_{0 \leq x \leq 1} |f(x) - g(x)|$. (This space is not σ-compact, of course.) What is the position of $\mathsf{D}(C[0,1])$ in the global $\leq_{\mathrm{B}}$-structure of Borel equivalence relations and what are its $\leq_{\mathrm{B}}$-connections with better-known equivalence relations like E_i, $i = 1,2,3$, E_∞, and ℓ^p, $\mathbf{c}_0$? □

This question (see, *e.g.*, SU GAO [**Gao06**]) remains open. The question is also connected with $\mathbf{c}_0$-equalities, in particular, with $\mathbf{c}_0$ itself from another side. Let us consider the following continual version C_0 of the equivalence relation $\mathbf{c}_0$. If f, g are continuous maps from $[0,+\infty)$ to $\mathbb{R}$, then we define

$$x\ \mathsf{C}_0\ y \quad \text{iff} \quad \lim_{x \to +\infty} |f(x) - g(x)| = 0.$$

It is clear that every continuous map $f : [0,+\infty) \to \mathbb{R}$ can be identified with the sequence of its restrictions to intervals of the form $[n_n, n_{n+1})$, $n \in \mathbb{N}$, that is, with a certain point of the Polish product space $C[0,1]^{\mathbb{N}}$. With such an identification, the domain of C_0 is naturally identified with a certain Borel set in $C[0,1]^{\mathbb{N}}$, while C_0 itself is identified with a Borel equivalence relation equal to $\mathsf{D}(C[0,1])$ on that set. (The domain of $\mathsf{D}(C[0,1])$ is the whole space $C[0,1]^{\mathbb{N}}$.) Question 16.7.2 can also be addressed to C_0.

SU GAO proved in [**Gao06**] that C_0 (there defined as E_K) satisfies $\mathsf{C}_0 \leq_{\mathrm{B}} \mathbf{u}_0^*$, where $\mathbf{u}_0^*$ is an equivalence relation on $\mathbb{R}^{\mathbb{N}\times\mathbb{N}}$ defined as follows:

$$x\ \mathbf{u}_0^*\ y \qquad \text{iff} \qquad \forall \varepsilon > 0\, \exists m_0\, \forall m \geq m_0\, \forall n\ (|x(m,n) - y(m,n)| < \varepsilon)\,.$$

[3] Note that the relation $\mathsf{D}(\mathbb{X})$ depends on the metric rather than topological structure of a space $\mathbb{X}$, and hence it is, generally speaking, essential to specify a concrete distance compatible with the given topology.

In addition, a more complicated Borel equivalence relation $\mathbf{u}_0$ on $\mathbb{R}^{\mathbb{N}\times\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ is defined in **[Gao06]** such that $\mathsf{C}_0 \sim_{\mathrm{B}} \mathbf{u}_0^*$. Investigations of $\mathbf{u}_0$, $\mathbf{u}_0^*$, C_0, $\mathsf{D}(C[0,1])$ remain work in progress.

CHAPTER 17

Pinned equivalence relations

In this chapter we consider a class of equivalence relations E characterized by the property that if E has an equivalence class in a generic extension $\mathbb{V}^+$ of the ground set universe $\mathbb{V}$, definable in $\mathbb{V}^+$ in a certain way in terms of sets in $\mathbb{V}$ as parameters, then this equivalence class contains an element in $\mathbb{V}$. We call them *pinned* equivalence relations.

The main goal will be to prove that certain families of Borel equivalence relations are pinned, while on the other hand the equivalence relation T_2 of the equality of countable sets of the reals is not pinned, and hence not Borel reducible to any pinned equivalence relation. The family of pinned equivalence relations includes, for instance, continuous actions of complete left-invariant groups and some ideals, not necessarily polishable, and is closed under the Fubini product modulo $\mathtt{Fin}$. Reading this chapter is not possible without a substantial knowledge of forcing.

17.1. The definition of pinned equivalence relations

Recall that the equivalence relation T_2 is defined on $(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$ as follows:

$$x \,\mathsf{T}_2\, y \quad \text{iff} \quad \mathtt{ran}\, x = \mathtt{ran}\, y.$$

DEFINITION 17.1.1. $\mathbb{V}$ will denote the ground set universe. In this chapter we consider forcing extensions of $\mathbb{V}$.[1]

Suppose that X is a $\boldsymbol{\Sigma}^1_1$ or $\boldsymbol{\Pi}^1_1$ set in the universe $\mathbb{V}$, and an extension $\mathbb{V}^+$ of $\mathbb{V}$ is considered. In this case, let $X^\#$ denote what results by the definition of X applied in $\mathbb{V}^+$. There is no ambiguity here by SHOENFIELD's absoluteness theorem, and easily $X = X^\# \cap \mathbb{V}$. □

For instance, if, in the universe $\mathbb{V}$, E is a $\boldsymbol{\Sigma}^1_1$ equivalence relation on a fixed Polish space $\mathbb{X}$, then, still by SHOENFIELD's absoluteness theorem $\mathsf{E}^\#$ is a $\boldsymbol{\Sigma}^1_1$ equivalence relation on $\mathbb{X}^\#$. If now $x \in \mathbb{X}$ (hence, $x \in \mathbb{V}$), then the E-class $[x]_{\mathsf{E}} \subseteq \mathbb{X}$ of x (defined in $\mathbb{V}$) is included in a unique $\mathsf{E}^\#$-class $[x]_{\mathsf{E}^\#} \subseteq \mathbb{X}^\#$ (in $\mathbb{V}^+$). Classes of the form $[x]_{\mathsf{E}^\#}$, $x \in \mathbb{X}$, belong to a wider category of $\mathsf{E}^\#$-classes which admit a description from the point of view of the ground universe $\mathbb{V}$.

DEFINITION 17.1.2 (based on an argument of HJORTH [**Hjo99**]). Assume that E is a $\boldsymbol{\Sigma}^1_1$ equivalence relation on a Polish space $\mathbb{X}$ and $\mathbb{P}$ is a notion of forcing in $\mathbb{V}$. A *stable virtual* E*-class* is any $\mathbb{P}$-term $\boldsymbol{\xi}$ such that $\mathbb{P}$ forces $\boldsymbol{\xi} \in \mathbb{X}^\#$ and $\mathbb{P} \times \mathbb{P}$ forces $\boldsymbol{\xi}_{\mathtt{left}} \,\mathsf{E}^\#\, \boldsymbol{\xi}_{\mathtt{right}}$.[2]

[1] Basically, a more rigorous treatment would be either to consider boolean-valued extensions of the universe or to to assume that in fact $\mathbb{V}$ is a countable model in a wider universe.

[2] $\boldsymbol{\xi}_{\mathtt{left}}$ and $\boldsymbol{\xi}_{\mathtt{right}}$ are $\mathbb{P} \times \mathbb{P}$-terms meaning $\boldsymbol{\xi}$ associated with the left and right factors, respectively, $\mathbb{P}$ in the product forcing. Formally, $\boldsymbol{\xi}_{\mathtt{left}}[U \times V] = \boldsymbol{\xi}[U]$ and $\boldsymbol{\xi}_{\mathtt{right}}[U \times V] = \boldsymbol{\xi}[V]$

A stable virtual class is *pinned* if there is, in $\mathbb{V}$, a point $x \in \mathbb{X}$ which pins it, in the sense that $\mathbb{P}$ forces $x \mathsf{E}^{\#} \boldsymbol{\xi}$. Finally, E is *pinned* if, for every forcing notion $\mathbb{P} \in \mathbb{V}$, all stable virtual E-classes are pinned. □

If $\boldsymbol{\xi}$ is a stable virtual E-class, then, in every extension $\mathbb{V}^+$ of $\mathbb{V}$, if U and V are generic subsets of $\mathbb{P}$, then $x = \boldsymbol{\xi}[U]$ and $y = \boldsymbol{\xi}[V]$ belong to $\mathbb{X}^{\#}$ and satisfy $x \mathsf{E}^{\#} y$; hence $\boldsymbol{\xi}$ induces a $\mathsf{E}^{\#}$-class in the extension. If $\boldsymbol{\xi}$ is pinned, then this class contains an element in the ground universe $\mathbb{V}$; in other words, pinned stable virtual classes induce $\mathsf{E}^{\#}$-equivalence classes of the form $[x]_{\mathsf{E}^{\#}}$, $x \in \mathbb{V}$, in the extensions of the universe $\mathbb{V}$.

The following theorem (originally [**KR04**, **KR03**]) is the main result in this chapter. Part (ii) is from [**Hjo99**]. Part (iii) also belongs to HJORTH and is published with his permission.

Recall that a Polish group $\mathbb{G}$ is *complete left invariant* (CLI for brevity) if $\mathbb{G}$ admits a compatible left-invariant complete metric $d_{\mathbb{G}}$. That is, $d_{\mathbb{G}}$ has to be a complete metric compatible with the given Polish topology of $\mathbb{G}$, and in addition $d_{\mathbb{G}}(fg, fh) = d_{\mathbb{G}}(g, h)$ for all $f, g, h \in \mathbb{G}$. This class of groups contains, for instance, all Polish abelian groups, all countable groups (by the trivial reason that the discrete metrics $\delta(x, y) = 1$ whenever $x \neq y$ is invariant), in particular, $\mathbb{Z}$ and $\mathsf{E}_\infty = \mathbb{E}(F_2, 2)$, and many more.

THEOREM 17.1.3. *The class of all pinned* $\boldsymbol{\Sigma}^1_1$ *equivalence relations*:

(i) *is closed under Fubini products modulo* $\mathtt{Fin}$;

and contains the following equivalence relations:

(ii) *all orbit equivalence relations of Borel actions of (Polish)* CLI *groups on Polish spaces*;[3]

(iii) *all Borel equivalence relations, all of whose equivalence classes are* $\boldsymbol{\Sigma}^0_3$, *in particular, all countable Borel equivalence relations*;

(iv) *all equivalence relations of the form* $\mathsf{E}_{\mathscr{I}}$, *where* $\mathscr{I}$ *is an ideal of the form*

$$\mathscr{I} = \mathbf{Exh}_{\{\varphi_i\}} = \{x \subseteq \mathbb{N} : \varphi_\infty(x) = 0\},$$

where φ_i, $i \in 2^{\mathbb{N}}$, *are lower semicontinuous* (LSC) *submeasures on* $\mathbb{N}$ *and* $\varphi_\infty(x) = \mathtt{lim\,sup}_{i \to \infty} \varphi_i(x)$.

On the other hand, T_2 *is not pinned and hence* T_2 *is Borel irreducible to pinned equivalence relations.*

COROLLARY 17.1.4. T_2 *is not Borel reducible to* $\mathbf{c}_0$ *and to* ℓ^∞.

PROOF. ℓ^∞ is an $\mathbf{F}_\sigma$ relation, hence all its equivalence classes are $\mathbf{F}_\sigma$ as well, and we can apply (iii) of the theorem. The equivalence relation $\mathbf{c}_0$ is $\boldsymbol{\Pi}^0_3$, but here (iv) is applicable. Indeed $\mathbf{c}_0 \sim_{\mathrm{B}} \mathsf{Z}_0$ by Lemma 6.2.3. Further Z_0 is $\mathsf{E}_{\mathscr{Z}_0}$, where $\mathscr{Z}_0$ is the density-0 ideal. However, $\mathscr{Z}_0 = \mathbf{Exh}_\varphi$ for a certain LSC submeasure by Lemma 3.3.5. Thus (iv) works. □

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for every $\mathbb{P} \times \mathbb{P}$-generic set $U \times V$, where $\boldsymbol{\xi}[U]$ is the interpretation of a term $\boldsymbol{\xi}$ via a generic set U.

[3] Quite recently, Thompson [**Tho06**] proved that for a Polish group $\mathbb{G}$ to be CLI it is not only necessary (which is by (ii)) but also sufficient that all orbit equivalence relations of Polish actions of $\mathbb{G}$ are pinned.

In fact it is quite clear that each equivalence relation in Figure 1 on page 68 except for T_2 belongs to one of the classes (ii), (iv), (iv), and hence by Theorem 17.1.3, T_2 is not Borel reducible to any reasonable combination of all of them.

17.2. T_2 is not pinned

Here we prove the last claim of Theorem 17.1.3.

CLAIM 17.2.1. *T_2 is not pinned.*

PROOF. To prove that T_2 is not pinned, consider, in $\mathbb{V}$, the collapse forcing notion $\mathbb{P} = \text{COLL}(\mathbb{N}, 2^{\mathbb{N}})$. Thus $\mathbb{P}$ consists of all functions $p : u \to 2^{\mathbb{N}}$, where $u \subseteq \mathbb{N}$ is finite, and $\mathbb{P}$ forces a generic map f from $\mathbb{N}$ onto the set $2^{\mathbb{N}}$ of $\mathbb{V}$. It clearly follows that the $\mathbb{P}$-term $\boldsymbol{\xi}$ for f is a stable virtual T_2-class, but it is not pinned because $2^{\mathbb{N}}$ is uncountable in the ground universe $\mathbb{V}$. □

LEMMA 17.2.2. *If E, F are $\boldsymbol{\Sigma}^1_1$ equivalence relations, $\mathsf{E} \le_{\mathrm{B}} \mathsf{F}$, and F is pinned, then so is E.*

PROOF. Suppose that, in $\mathbb{V}$, $\vartheta : \mathbb{X} \to \mathbb{Y}$ is a Borel reduction of E to F, where $\mathbb{X} = \operatorname{dom} \mathsf{E}$ and $\mathbb{Y} = \operatorname{dom} \mathsf{F}$. We can assume that $\mathbb{X}$ and $\mathbb{Y}$ are just two copies of $2^{\mathbb{N}}$. Let $\mathbb{P}$ be a forcing notion and a $\mathbb{P}$-term $\boldsymbol{\xi}$ be a stable virtual E-class. By SHOENFIELD's absoluteness, $\vartheta^{\#}$ is a reduction of $\mathsf{E}^{\#}$ to $\mathsf{F}^{\#}$ in every extension of $\mathbb{V}$, hence, $\boldsymbol{\sigma}$, a $\mathbb{P}$-term for $\vartheta^{\#}(\boldsymbol{\xi})$, is also a stable virtual F-class. Since F is pinned, there is $y \in \mathbb{Y}$ such that $\mathbb{P}$ forces $y \mathrel{\mathsf{F}^{\#}} \boldsymbol{\sigma}$. Note that it is true in the $\mathbb{P}$-extension that $y \mathrel{\mathsf{F}^{\#}} \vartheta^{\#}(x)$ for some $x \in \mathbb{X}^{\#}$. Hence, by SHOENFIELD's absoluteness theorem [**Sho62**], there is an element $x \in \mathbb{X}$ in the ground universe $\mathbb{V}$ satisfying $y \mathrel{\mathsf{F}} \vartheta(x)$. But then $\mathbb{P}$ forces $x \mathrel{\mathsf{E}^{\#}} \boldsymbol{\xi}$. □

EXERCISE 17.2.3 (S. THOMAS). Using the arguments in the proofs of Claim 17.2.1 and Lemma 17.2.2, prove the following. Suppose that F is a pinned $\boldsymbol{\Sigma}^1_1$ equivalence relation on a Borel set W, and $\vartheta : (2^{\mathbb{N}})^{\mathbb{N}} \to W$ is a homomorphism of T_2 to F. (That is, $x \mathrel{\mathsf{T}_2} y$ implies $\vartheta(x) \mathrel{\mathsf{F}} \vartheta(y)$.) Then there is $w \in W$ such that for all $a \in 2^{\mathbb{N}}$ there exist $x \in (2^{\mathbb{N}})^{\mathbb{N}}$ and $n \in \mathbb{N}$ with $x(n) = a$ and $\vartheta(x) \mathrel{\mathsf{F}} w$. □

It would be interesting to prove Exercise 17.2.3 by a pure topological argument (with no forcing involved) in some particular cases, for instance, in the case $\mathsf{F} = \mathsf{E}_\infty$.

17.3. Fubini product of pinned equivalence relations is pinned

Here we prove part (i) of Theorem 17.1.3. Recall that the Fubini product $\mathsf{E} = \prod_{k \in \mathbb{N}} \mathsf{E}_k \,/\, \mathtt{Fin}$ of equivalence relations E_k on $\mathbb{N}^{\mathbb{N}}$ modulo $\mathtt{Fin}$ is an equivalence relation on $(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$ defined as follows: $x \mathrel{\mathsf{E}} y$ if $x(k) \mathrel{\mathsf{E}_k} y(k)$ for all but finite k.

Suppose that $\boldsymbol{\Sigma}^1_1$ equivalence relations E_k on Polish spaces $\mathbb{X}_k$ are pinned. Prove that the Fubini product $\mathsf{E} = \prod_{k \in \mathbb{N}} \mathsf{E}_k \,/\, \mathtt{Fin}$ is a pinned equivalence relation (on the Polish space $\mathbb{X} = \prod_k \mathbb{X}_k$). Consider a forcing notion $\mathbb{P}$ and a $\mathbb{P}$-term $\boldsymbol{\xi}$. Assume that $\boldsymbol{\xi}$ is a stable virtual E-class. There is a number k_0 and conditions $p, q \in \mathbb{P}$ such that $\langle p, q \rangle$ $(\mathbb{P} \times \mathbb{P})$-forces $\boldsymbol{\xi}_{\mathtt{left}}(k) \mathrel{\mathsf{E}_k^{\#}} \boldsymbol{\xi}_{\mathtt{right}}(k)$ for all $k \ge k_0$. As all E_k are equivalence relations, we conclude that the condition $\langle p, p \rangle$ also forces $\boldsymbol{\xi}_{\mathtt{left}}(k) \mathrel{\mathsf{E}_k^{\#}} \boldsymbol{\xi}_{\mathtt{right}}(k)$ for all $k \ge k_0$. Therefore, since E_k are pinned, there is in $\mathbb{V}$ a sequence of points $x_k \in \mathbb{X}_k$ such that the condition p $\mathbb{P}$-forces $x_k \mathrel{\mathsf{E}_k^{\#}} \boldsymbol{\xi}(k)$ for all $k \ge k_0$. Let $x \in \mathbb{X}$ satisfy $x(k) = x_k$ for all $k \ge k_0$. (The values $x(k) \in \mathbb{X}_k$ for $k < k_0$ can be arbitrary.) Then p obviously $\mathbb{P}$-forces $x \mathrel{\mathsf{E}^{\#}} \boldsymbol{\xi}$.

It remains to show that just every $q \in \mathbb{P}$ also forces $x \,\mathsf{E}^{\#}\, \xi$. Suppose otherwise, that is, some $q \in \mathbb{P}$ forces that $x \,\mathsf{E}^{\#}\, \xi$ *fails.* Consider the pair $\langle p, q\rangle$ as a condition in $\mathbb{P} \times \mathbb{P}$: it forces $x \,\mathsf{E}^{\#}\, \xi_{\mathtt{left}}$ and $\neg\, x \,\mathsf{E}^{\#}\, \xi_{\mathtt{right}}$, as well as $\xi_{\mathtt{left}} \,\mathsf{E}^{\#}\, \xi_{\mathtt{right}}$ by the choice of E and ξ, which is a contradiction.

17.4. Complete left-invariant actions induce pinned relations

Here we prove part (ii) of Theorem 17.1.3. Note that according to Theorem 4.3.3 it is sufficient to consider Polish (= continuous) actions. Thus suppose that $\mathbb{G}$ is a Polish CLI group continuously acting on a Polish space $\mathbb{X}$. By definition $\mathbb{G}$ admits a compatible left-invariant complete metric ρ'. Then easily $\mathbb{G}$ also admits a compatible *right*-invariant complete metric, for instance, $\rho(g,h) = \rho'(g^{-1},h^{-1})$, which will be practically used in the proof.

Let $\mathbb{P}$ be a forcing notion and ξ be a stable virtual E-class. Let $\le$ denote the partial order of $\mathbb{P}$; we assume, as usual, that $p \le q$ means that p is a stronger condition. Let us fix a compatible complete right-invariant metric ρ on $\mathbb{G}$. For every $\varepsilon > 0$, put $G_\varepsilon = \{g \in \mathbb{G} : \rho(g, 1_{\mathbb{G}}) < \varepsilon\}$. Say that $q \in \mathbb{P}$ *is of size* $\le \varepsilon$ if $\langle q, q\rangle$ $\mathbb{P} \times \mathbb{P}$-forces the existence of $g \in {G_\varepsilon}^{\#}$ such that $\xi_{\mathtt{left}} = g \cdot \xi_{\mathtt{right}}$.

LEMMA 17.4.1. *If* $q \in \mathbb{P}$ *and* $\varepsilon > 0$, *then there is a condition* $r \in \mathbb{P}$, $r \le q$, *of size* $\le \varepsilon$.

PROOF. Otherwise for every $r \in \mathbb{P}$, $r \le q$, there is a pair of conditions r', $r'' \in \mathbb{P}$ stronger than r and such that $\langle r', r''\rangle$ $(\mathbb{P} \times \mathbb{P})$-forces that there is no $g \in {G_\varepsilon}^{\#}$ with $\xi_{\mathtt{left}} = g \cdot \xi_{\mathtt{right}}$. Applying an ordinary splitting construction in such a generic extension $\mathbb{V}^+$ of $\mathbb{V}$ where $\mathscr{P}(\mathbb{P}) \cap \mathbb{V}$ is countable, we find an uncountable set $\mathscr{U}$ of generic sets $U \subseteq \mathbb{P}$ with $q \in U$ such that every pair $\langle U, V\rangle$ with $U \ne V$ in $\mathscr{U}$ is $\mathbb{P} \times \mathbb{P}$-generic (over $\mathbb{V}$); hence, there is no $g \in {G_\varepsilon}^{\#}$ with $\xi[U] = g \cdot \xi[V]$.[4] Fix $U_0 \in \mathscr{U}$. We can associate in $\mathbb{V}^+$ with each $U \in \mathscr{U}$, an element $g_U \in G^{\#}$ such that $\xi[U] = g_U \cdot \xi[U_0]$; then $g_U \notin {G_\varepsilon}^{\#}$ by the above. Moreover, we have $g_V g_U^{-1} \cdot \xi[U] = \xi[V]$ for all $U, V \in \mathscr{U}$. Hence $g_V g_U^{-1} \notin {G_\varepsilon}^{\#}$ whenever $U \ne V$, which implies $\rho(g_U, g_V) \ge \varepsilon$ by the right invariance. But this contradicts the separability of G. □ (Lemma)

Coming back to the proof of (iii) of Theorem 17.1.3, suppose toward the contrary that a condition $p \in \mathbb{P}$ forces that there is no $x \in \mathbb{X}$ (in the ground universe $\mathbb{V}$) satisfying $x \,\mathsf{E}^{\#}\, \xi$. According to Lemma 17.4.1, there is, in $\mathbb{V}$, a sequence of conditions $p_n \in \mathbb{P}$ of size $\le 2^{-n}$, and closed sets $X_n \subseteq \mathbb{X}$ with $\mathbb{X}$-diameter $\le 2^{-n}$, such that $p_0 \le p$, $p_{n+1} \le p_n$, $X_{n+1} \subseteq X_n$, and p_n forces $\xi \in {X_n}^{\#}$ for every n. By the completeness of $\mathbb{X}$, let x be the common point of the sets X_n in $\mathbb{V}$.

We assert that p_0 forces $x \,\mathsf{E}^{\#}\, \xi$.

Indeed, otherwise there is condition $q \in \mathbb{P}$, $q \le p_0$, which forces $\neg\, x \,\mathsf{E}^{\#}\, \xi$. Consider an extension $\mathbb{V}^+$ of $\mathbb{V}$ rich enough to contain, for every n, a generic set $U_n \subseteq \mathbb{P}$ with $p_n \in U_n$ such that each pair $\langle U_n, U_{n+1}\rangle$ is $\mathbb{P} \times \mathbb{P}$-generic (over $\mathbb{V}$), and, in addition, $q \in U_0$. Let $x_n = \xi[U_n]$ (an element of $\mathbb{X}^{\#}$), then $\{x_n\} \to x$. Moreover, for every n, both U_n and U_{n+1} contain p_n. Hence, as p_n has size $\le 2^{-n-1}$, there is $g_{n+1} \in {\mathbb{G}_\varepsilon}^{\#}$ with $x_{n+1} = g_{n+1} \cdot x_n$. Thus, $x_n = h_n \cdot x_0$, where $h_n = g_n \cdots g_1$. However $\rho(h_n, h_{n-1}) = \rho(g_n, 1_{\mathbb{G}}) \le 2^{-n+1}$ by the right invariance of the metric, thus, $\{h_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathbb{G}^{\#}$. Let $h = \mathtt{lim}_{n\to\infty} h_n \in \mathbb{G}^{\#}$

[4] $\xi[U]$ is the interpretation of the $\mathbb{P}$-term ξ obtained by taking U as the generic set.

be its limit. As the action considered is continuous, we have $x = \mathtt{lim}_n x_n = h \cdot x_0$. It follows that $x \mathsf{E}^{\#} x_0$ holds in $\mathbb{V}^+$, hence also in $\mathbb{V}[U_0]$. However, $x_0 = \boldsymbol{\xi}[U_0]$ while $q \in U_0$ forces $\neg\, x \mathsf{E}^{\#} \boldsymbol{\xi}$, which is a contradiction.

Thus the condition p_0 $\mathbb{P}$-forces $x \mathsf{E}^{\#} \boldsymbol{\xi}$. Then every conditon $r \in \mathbb{P}$ also forces $x \mathsf{E}^{\#} \boldsymbol{\xi}$. Indeed, if some $r \in \mathbb{P}$ forces $\neg\, x \mathsf{E}^{\#} \boldsymbol{\xi}$, then the pair $\langle p_0, r\rangle$ $\mathbb{P} \times \mathbb{P}$-forces that $x \mathsf{E}^{\#} \boldsymbol{\xi}_{\mathtt{left}}$ and $\neg\, x \mathsf{E}^{\#} \boldsymbol{\xi}_{\mathtt{right}}$, which contradicts the fact that $\mathbb{P} \times \mathbb{P}$ forces $\boldsymbol{\xi}_{\mathtt{left}} \mathsf{E}^{\#} \boldsymbol{\xi}_{\mathtt{right}}$.

17.5. All equivalence relations with Σ^0_3 classes are pinned

Here we prove part (iii) of Theorem 17.1.3. Suppose that E is a Borel equivalence relation on $\mathbb{N}^{\mathbb{N}}$ and all E-equivalence classes are $\boldsymbol{\Sigma}^0_3$, that is, $\mathbf{G}_{\delta\sigma}$. Prove that then E is pinned.

It follows from a theorem of LOUVEAU [**Lou80**] that there is a Borel map γ, defined on $\mathbb{N}^{\mathbb{N}}$, such that $\gamma(x)$ is a $\boldsymbol{\Sigma}^0_3$-code of the equivalence class $[x]_{\mathsf{E}}$ for every $x \in \mathbb{N}^{\mathbb{N}}$, that is, for instance, $\gamma(x) \subseteq \mathbb{N}^2 \times \mathbb{N}^{<\omega}$ and

$$[x]_{\mathsf{E}} = \bigcup_i \bigcap_j \bigcup_{\langle i,j,s\rangle \in \gamma(x)} B_s, \quad \text{where} \quad B_s = \{a \in \mathbb{N}^{\mathbb{N}} : s \subset a\} \text{ for all } s \in \mathbb{N}^{<\omega}.$$

(It is not asserted that $\gamma(x) = \gamma(x')$ whenever $x \mathsf{E} x'$.) Let us fix such a coding map γ.

Now let us consider a forcing notion $\mathbb{P} = \langle \mathbb{P}; \le\rangle$ and a stable virtual E-class $\boldsymbol{\xi}$. By definition $\mathbb{P}\times\mathbb{P}$ forces $\boldsymbol{\xi}_{\mathtt{left}} \mathsf{E}^{\#} \boldsymbol{\xi}_{\mathtt{right}}$; hence there is a number i_0 and a condition $\langle p_0, q_0\rangle \in \mathbb{P} \times \mathbb{P}$ which forces $\boldsymbol{\xi}_{\mathtt{left}} \in \vartheta^{\#}(\boldsymbol{\xi}_{\mathtt{right}})$, where $\vartheta(x) = \bigcap_j \bigcup_{\langle i_0,j,s\rangle \in \gamma(x)} B_s$ for all $x \in \mathbb{N}^{\mathbb{N}}$.

The key idea of the proof is to substitute $\mathbb{P}$ by the Cohen forcing. Let $\mathbb{S}$ denote the set of all $s \in \mathbb{N}^{<\omega}$ such that p_0 does *not* $\mathbb{P}$-force that $s \not\subset \boldsymbol{\xi}$. We consider $\mathbb{S}$ to be a forcing, and $s \subseteq t$ (that is, t is an extension of s) means that t is a stronger condition. And Λ, the empty sequence, is the weakest condition in $\mathbb{S}$. If $s \in \mathbb{S}$, then obviously there is at least one n such that $s^\frown n \in \mathbb{S}$. Therefore, $\mathbb{S}$ forces an element of $\mathbb{N}^{\mathbb{N}}$, whose $\mathbb{S}$-name will be $\mathbf{a}$.

LEMMA 17.5.1. *The pair* $\langle \Lambda, q_0\rangle$ $(\mathbb{S} \times \mathbb{P})$*-forces* $\mathbf{a} \in \vartheta^{\#}(\boldsymbol{\xi})$.

PROOF. Otherwise some condition $\langle s_0, q\rangle \in \mathbb{S} \times \mathbb{P}$ with $q \le q_0$ forces $\mathbf{a} \notin \vartheta^{\#}(\boldsymbol{\xi})$. By the definition of ϑ we can assume that

$$(*) \qquad \langle s_0, q\rangle \quad (\mathbb{S} \times \mathbb{P})\text{-forces} \quad \neg\, \exists s\, (\langle i_0, j_0, s\rangle \in \gamma(\boldsymbol{\xi}) \wedge s \subset \mathbf{a})$$

for some j_0. Since $s_0 \in \mathbb{S}$, there is a condition $p' \in \mathbb{P}$, $p' \le p_0$, which $\mathbb{P}$-forces $s_0 \subset \boldsymbol{\xi}$. By the choice of $\langle p_0, q_0\rangle$ we can assume that

$$\langle p', q'\rangle \quad (\mathbb{P} \times \mathbb{P})\text{-forces} \quad \langle i_0, j_0, s\rangle \in \gamma(\boldsymbol{\xi}_{\mathtt{right}}) \wedge s \subset \boldsymbol{\xi}_{\mathtt{left}}$$

for suitable $s \in \mathbb{S}$ and $q' \in \mathbb{P}$, $q' \le q$. This means that

1) p' $\mathbb{P}$-forces $s \subset \boldsymbol{\xi}$, and

2) q' $\mathbb{P}$-forces $\langle i_0, j_0, s\rangle \in \gamma(\boldsymbol{\xi})$.

In particular, p' forces both $s_0 \subset \boldsymbol{\xi}$ and $s \subset \boldsymbol{\xi}$ by the above. It follows that either $s \subseteq s_0$, then we define $s' = s_0$, or $s_0 \subset s$, then we put $s' = s$. In either case, the condition $\langle s', q'\rangle$ $(\mathbb{S} \times \mathbb{P})$-forces $\langle i_0, j_0, s\rangle \in \gamma(\boldsymbol{\xi})$ and $s \subset \mathbf{a}$, and this is a contradiction to $(*)$. □ (Lemma)

Note that $\mathbb{S}$ is a subforcing of the Cohen forcing $\mathbb{C} = \mathbb{N}^{<\omega}$, therefore, by Lemma 17.5.1, there is a $\mathbb{C}$-term $\boldsymbol{\sigma}$ such that the condition $\langle \Lambda, q_0 \rangle$ $(\mathbb{C} \times \mathbb{P})$-forces $\boldsymbol{\sigma} \in \vartheta^{\#}(\boldsymbol{\xi})$, and hence, forces $\boldsymbol{\sigma} \mathsf{E}^{\#} \boldsymbol{\xi}$. It follows, by consideration of the forcing $\mathbb{C} \times \mathbb{P} \times \mathbb{P}$, that generally $\mathbb{C} \times \mathbb{P}$ forces $\boldsymbol{\sigma} \mathsf{E}^{\#} \boldsymbol{\xi}$. Therefore, by ordinary arguments, first, $\mathbb{C} \times \mathbb{C}$ forces $\boldsymbol{\sigma}_{\mathtt{left}} \mathsf{E}^{\#} \boldsymbol{\sigma}_{\mathtt{right}}$, and second, to prove the theorem it suffices now to find $x \in \mathbb{N}^{\mathbb{N}}$ in $\mathbb{V}$ such that $\mathbb{C}$ forces $x \mathsf{E}^{\#} \boldsymbol{\sigma}$. This is our next goal.

Let $\mathbf{a}$ be a $\mathbb{C}$-name of the Cohen-generic element of $\mathbb{N}^{\mathbb{N}}$. The term $\boldsymbol{\sigma}$ can be of a complicated nature, but we can substitute it by a term of the form $f^{\#}(\mathbf{a})$, where $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is a Borel map in the ground universe $\mathbb{V}$. It follows from the above that $f^{\#}(\mathbf{a}) \mathsf{E}^{\#} f^{\#}(\mathbf{b})$ for every $\mathbb{C} \times \mathbb{C}$-generic, over $\mathbb{V}$, pair $\langle \mathbf{a}, \mathbf{b} \rangle \in \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$. We conclude that $f^{\#}(\mathbf{a}) \mathsf{E}^{\#} f^{\#}(\mathbf{b})$ also holds even for every pair of separately Cohen-generic elements $\mathbf{a}, \mathbf{b} \in \mathbb{N}^{\mathbb{N}}$. Thus, in a generic extension of $\mathbb{V}$, where there are comeager-many Cohen-generic reals, there is a comeager $\mathbf{G}_\delta$ set $X \subseteq \mathbb{N}^{\mathbb{N}}$ such that $f^{\#}(a) \mathsf{E}^{\#} f^{\#}(b)$ for all $a, b \in X$. By SHOENFIELD's absoluteness theorem, the statement of existence of such a set X is true also in $\mathbb{V}$; hence, in $\mathbb{V}$, there is $x \in \mathbb{N}^{\mathbb{N}}$ such that $x \mathsf{E} f(a)$ holds for comeager-many $a \in \mathbb{N}^{\mathbb{N}}$. This is again a SHOENFIELD-absolute property of x,[5] hence, $\mathbb{C}$ forces $x \mathsf{E}^{\#} f^{\#}(\mathbf{a})$, as required.

17.6. Another family of pinned ideals

Here we prove part (iv) of Theorem 17.1.3.

Let us say that a Borel ideal $\mathscr{I}$ is *pinned* if the induced equivalence relation $\mathsf{E}_{\mathscr{I}}$ is such. It follows from Theorem 17.1.3(ii) that every P-ideal is pinned because Borel P-ideals are polishable by Theorem 3.5.1 while all Polish abelian groups are CLI. Yet there are non-P pinned ideals, like $\mathscr{I}_1$.

Suppose that $\{\varphi_i\}_{i \in \mathbb{N}}$ is a sequence of LSC submeasures on $\mathbb{N}$. Define the exhaustive ideal of the sequence,

$$\mathtt{Exh}_{\{\varphi_i\}} = \{x \subseteq \mathbb{N} : \varphi_\infty(x) = 0\}, \quad \text{where} \quad \varphi_\infty(x) = \limsup_{i \to \infty} \varphi_i(x).$$

It follows from Theorem 3.5.1 that for every Borel P-ideal $\mathscr{I}$ there is an LSC submeasure φ such that $\mathscr{I} = \mathtt{Exh}_{\{\varphi_i\}} = \mathtt{Exh}_\varphi$, where $\varphi_i(x) = \varphi(x \cap [i, \infty))$. On the other hand, the non-polishable ideal $\mathscr{I}_1$ also is of the form $\mathtt{Exh}_{\{\varphi_i\}}$, where for $x \subseteq \mathbb{N}^2$ we define an LSC submeasure φ_i by $\varphi_i(x) = 0$ or 1 if $x \subseteq$ or $\not\subseteq \{0, \dots, n-1\} \times \mathbb{N}$, respectively. Therefore, the class of ideals of the form $\mathtt{Exh}_{\{\varphi_i\}}$ includes Borel P-ideals but is strictly larger.

Thus suppose that φ_i is an LSC submeasure on $\mathbb{N}$ for each $i \in \mathbb{N}$. The goal is to prove that the ideal $\mathscr{I} = \mathtt{Exh}_{\{\varphi_i\}}$ is pinned.

We can assume that the submeasures φ_i decrease, that is $\varphi_{i+1}(x) \le \varphi_i(x)$ for all x, for if not, then consider the LSC submeasures $\varphi'_i(x) = \sup_{j \ge i} \varphi_j(x)$.

Suppose toward the contrary that the equivalence relation $\mathsf{E} = \mathsf{E}_{\mathscr{I}}$ is not pinned. Then there is a forcing notion $\mathbb{P}$, a stable virtual E-class $\boldsymbol{\xi}$, and a condition $p \in \mathbb{P}$ which $\mathbb{P}$-forces $\neg\, x \mathsf{E}^{\#} \boldsymbol{\xi}$ for all $x \in \mathscr{P}(\mathbb{N})$ in $\mathbb{V}$. By definition, for every $p' \in \mathbb{P}$ and $n \in \mathbb{N}$ there are $i \ge n$ and conditions $q, r \in \mathbb{P}$ with $q, r \le p'$, such that $\langle q, r \rangle$ $(\mathbb{P} \times \mathbb{P})$-forces the inequality $\varphi_i(\boldsymbol{\xi}_{\mathtt{left}} \,\triangle\, \boldsymbol{\xi}_{\mathtt{right}}) \le 2^{-n-1}$. Hence, $\langle q, q \rangle$ $(\mathbb{P} \times \mathbb{P})$-forces $\varphi_i(\boldsymbol{\xi}_{\mathtt{left}} \,\triangle\, \boldsymbol{\xi}_{\mathtt{right}}) \le 2^{-n}$. It follows that, in $\mathbb{V}$, there is a sequence

[5] Indeed, it is formally written as $\exists x \, \forall^* a \, (x \mathsf{E} f(a))$, where $\forall^* a$ means "for comeager-many a". However it is known that $\forall^* a$ applied to a Borel relation results in a Borel relation, see e.g. [Kec95]. Therefore the formula $\exists x \, \forall^* a \, (x \mathsf{E} f(a))$ can be re-written as a Σ^1_1 formula.

of numbers $i_0 < i_1 < i_2 < \cdots$, and a sequence $p_0 \geq p_1 \geq p_2 \geq \cdots$ of conditions in $\mathbb{P}$, and, for every n, a set $u_n \subseteq [0, n)$, such that $p_0 \leq p$ and

(1) each p_n $\mathbb{P}$-forces $\boldsymbol{\xi} \cap [0, n) = u_n$;

(2) each $\langle p_n, p_n \rangle$ $(\mathbb{P} \times \mathbb{P})$-forces $\varphi_{i_n}(\boldsymbol{\xi}_{\texttt{left}} \bigtriangleup \boldsymbol{\xi}_{\texttt{right}}) \leq 2^{-n}$.

Arguing in the universe $\mathbb{V}$, put $a = \bigcup_n u_n$; then $a \cap [0, n) = u_n$ for all n. We claim that p_0 forces $a \mathsf{E}^{\#} \boldsymbol{\xi}$. This contradicts the assumption above, ending the proof of (iv) of Theorem 17.1.3.

To prove the claim, note that otherwise there is a condition $q_0 \leq p_0$ which forces $\neg\, a \mathsf{E}^{\#} \boldsymbol{\xi}$. Consider a generic extension $\mathbb{V}^+$ of the universe, where there exists a sequence of $\mathbb{P}$-generic sets $U_n \subseteq \mathbb{P}$ such that for every n, the pair $\langle U_n, U_{n+1} \rangle$ is $(\mathbb{P} \times \mathbb{P})$-generic, $p_n \in U_n$, and in addition $q_0 \in U_0$. Then, in $\mathbb{V}^+$, the sets $x_n = \boldsymbol{\xi}[U_n] \in \mathscr{P}(\mathbb{N})$ satisfy $\varphi_{i_n}(x_n \bigtriangleup x_m) \leq 2^{-n}$ by (2), whenever $n \leq m$. It follows that $\varphi_{i_n}(x_n \bigtriangleup a) \leq 2^{-n}$, because $a = \texttt{lim}_m x_m$ by (1). However, we assume that the submeasures φ_j decrease, therefore $\varphi_\infty(x_n \bigtriangleup a) \leq 2^{-n}$. On the other hand, $\varphi_\infty(x_n \bigtriangleup x_0) = 0$ because $\boldsymbol{\xi}$ is a stable virtual E-class. We conclude that $\varphi_\infty(x_0 \bigtriangleup a) \leq 2^{-n}$ for all n. In other words, $\varphi_\infty(x_0 \bigtriangleup a) = 0$, that is, $x_0 \mathsf{E}^{\#} a$, which is a contradiction with the choice of U_0 because $x_0 = \boldsymbol{\xi}[U_0]$ and $q_0 \in U_0$.

□ (Theorem 17.1.3)

One might ask whether all equivalence relations of the form $\mathsf{E}_{\mathscr{I}}$, where $\mathscr{I}$ is a Borel ideal, are pinned. This question is answered in the negative. Indeed it will be proved in the next chapter that for every Borel equivalence relation E there exists a Borel ideal $\mathscr{I} \subseteq \mathscr{P}(\mathbb{N})$ such that $\mathsf{E} \leq_{\mathrm{B}} \mathsf{E}_{\mathscr{I}}$. In particular this is true for the equivalence relation T_2, non-pinned by Theorem 17.1.3. It follows, still by Theorem 17.1.3, that every Borel ideal $\mathscr{I}$ satisfying $\mathsf{T}_2 \leq_{\mathrm{B}} \mathsf{E}_{\mathscr{I}}$ is non-pinned as well.

One more interesting question is motivated by the fact that there seems to be no known mechanics to define non-pinned Borel equivalence relations except for application of T_2 is one or another way.

QUESTION 17.6.1 (KECHRIS). Is it true that T_2 is the $\leq_{\mathrm{B}}$-least non-pinned Borel equivalence relation? □

QUESTION 17.6.2. What is the nature of those Borel ideals $\mathscr{I}$ on $\mathbb{N}$ which satisfy $\mathsf{T}_2 \leq_{\mathrm{B}} \mathsf{E}_{\mathscr{I}}$? □

CHAPTER 18

Reduction of Borel equivalence relations to Borel ideals

The main goal of this chapter is to show that any Borel equivalence relation is Borel reducible to a relation of the form $\mathsf{E}_{\mathscr{I}}$ for some Borel ideal $\mathscr{I}$, and moreover, there is a $\leq_{\mathrm{B}}$-cofinal ω_1-sequence of Borel ideals. The proof of this important result involves a universal analytic equivalence relation generated by an analytic ideal, followed by a well-known construction of upper Borel approximations of $\boldsymbol{\Sigma}^1_1$ sets. In the end we briefly outline the results of subsequent study [**KL06**]: the ideals $\mathscr{I}$ and the corresponding relations $\mathsf{E}_{\mathscr{I}_\xi}$ as above can be explicitly and meaningfully defined on the basis of a certain game.

18.1. Trees

We begin with a review of basic notation related to trees of finite sequences. Recall that for any set X, X^n denotes the set of all sequences, of length n, of elements of X, and $X^{<\omega} = \bigcup_{n\in\mathbb{N}} X^n$—the set of all finite sequences of elements of X. Regarding product sets, note that any $s \in (X_1 \times \cdots \times X_n)^{<\omega}$ is formally a finite sequence of n-tuples $\langle x_1, \ldots, x_n\rangle$, where $x_i \in X_i$, $\forall i$. We identify such a sequence s with the n-tuple $\langle s_1, \ldots, s_n\rangle$, where all $s_i \in {X_i}^{<\omega}$ have the same length as s itself, and $s(i) = \langle s_1(i), \ldots, s_n(i)\rangle$ for all i.

The *length* of a sequence s is $\mathtt{lh}\, s$. Λ, *the empty sequence*, is the only one of length 0. If s is a finite sequence and x any set, then by $s^\frown x$, resp., $x^\frown s$, we denote the result of adjoining x as the new right-most, resp., left-most, term to s. If s, t are sequences, then $s \subseteq t$ means that t is an *extension* of s, that is, $s = t \restriction m$ for some $m \leq \mathtt{lh}\, t$.

A *tree* on a set X is any subset $T \subseteq X^{<\omega}$ closed under restrictions; that is, if $t \in T$, $s \in X^{<\omega}$, and $s \subseteq t$, then $s \in T$. Note that Λ, the empty sequence, belongs to any tree $\varnothing \neq R \subseteq X^{<\omega}$. An *infinite branch* in a tree $T \subseteq X^{<\omega}$ is any infinite sequence $b \in X^{\mathbb{N}}$ such that $b \restriction m \in T$, $\forall m$. A tree T is *well founded* iff it has no infinite branches. Otherwise, T is *ill founded*.

The following transformations of trees on $\mathbb{N}$ preserve in this or another way the properties of well- and ill-foundedness.

Finite union. If S, T are trees, then so is $W = S \cup T$, and clearly $S \cup T$ is ill founded iff so is at least one of S, T.

Contraction. Let $S \subseteq 2^{<\omega}$ be a tree. Fix once and for all a bijection $b : \mathbb{N}^2 \xrightarrow{\text{onto}} \mathbb{N}$. For any sequence $s = \langle k_0, k_1, \ldots, k_n\rangle \in 2^{<\omega}$ with $\mathtt{lh}\, s = n + 1 \geq 2$, define a sequence $s^\downarrow = \langle b(k_0, k_1), k_2, \ldots, k_n\rangle$ of length n. The *contracted tree*

$$S^\downarrow = \{\Lambda\} \cup \{\hat{s} : s \in S \wedge \mathtt{lh}\, s \geq 2\}$$

is ill founded iff S itself is ill founded.

Countable sum. Countable unions do not preserve well-foundedness. Yet there is another useful operation. For any sequence of trees $T_n \subseteq \mathbb{N}^{<\omega}$, we let $\sum_n^* T_n$ denote the tree $T = \{\Lambda\} \cup \{n^\frown t : t \in T_n\}$. Clearly, T is ill founded iff *at least one* of the trees T_n is ill founded.

Countable product. Let $\prod_n^* T_n$ denote the set T of all finite sequences of the form $t = \langle t_0, \dots, t_n \rangle$, where $t_k \in T_k$ and $\mathtt{lh}\, t_k = n$ for all $k \le n$. We put $\langle t_0, \dots, t_n \rangle \preccurlyeq \langle s_0, \dots, s_m \rangle$ iff $n \le m$ and $t_k \subseteq s_k$ (in $\mathbb{N}^{<\omega}$) for all $k \le n$. In addition, let Λ belong to T, with $\Lambda \preccurlyeq t$ for any $t \in T$. Obviously, $\langle T\,; \preccurlyeq \rangle$ is an at most countable tree, order isomorphic to a tree in $\mathbb{N}^{<\omega}$. Moreover, $T = \prod_n^* T_n$ is ill founded iff *every* tree T_n is ill founded.

Component-wise addition. This is a less trivial operation. First of all, if s, $t \in 2^{<\omega}$, then $s \le_{\mathtt{cw}} t$ (the *component-wise* ordering) means that $\mathtt{lh}\, s = \mathtt{lh}\, t$ and $s(i) \le t(i)$ for all $i < \mathtt{lh}\, s$. Similarly, then $s +_{\mathtt{cw}} t$ denotes the component-wise addition of finite sequences s, t of equal length. We now define

$$S +_{\mathtt{cw}} T = \{s +_{\mathtt{cw}} t : s \in S \wedge t \in T \wedge \mathtt{lh}\, s = \mathtt{lh}\, t\}$$

for any trees $S, T \subseteq \mathbb{N}^{<\omega}$. The following lemma shows that the component-wise addition of trees behaves somewhat like the "equal-length" cartesian product

$$S \times T = \{\langle s, t \rangle : s \in S \wedge t \in T \wedge \mathtt{lh}\, s = \mathtt{lh}\, t\}.$$

LEMMA 18.1.1. *Let $S, T \subseteq \mathbb{N}^{<\omega}$ be any trees. The tree $W = S +_{\mathtt{cw}} T$ is ill founded iff both S and T are ill founded.*

PROOF. In the non-trivial direction, suppose that $\gamma \in \mathbb{N}^{\mathbb{N}}$ is an infinite branch in W, *i.e.*, $\gamma \restriction n \in W$ for all n. Then, for each n, there exist $s_n \in S$ and $t_n \in T$ of length n such that $s_n +_{\mathtt{cw}} t_n = \gamma \restriction n$. The sequences s_n, t_n then belong to $\{t \in \mathbb{N}^{<\omega} : t \le_{\mathtt{cw}} \gamma \restriction \mathtt{lh}\, t\}$, a finite-branching tree. Therefore, by KÖNIG's lemma, there exist infinite sequences $\alpha, \beta \in \mathbb{N}^{\mathbb{N}}$ such that

$$\forall m\, \exists n \ge m\, (\alpha \restriction m = s_n \restriction m \wedge \beta \restriction m = t_n \restriction m).$$

Then α, β are infinite branches in S, T, respectively, as required. □

18.2. Louveau–Rosendal transform

Suppose that A is a $\mathbf{\Sigma}_1^1$ subset of $2^{\mathbb{N}} \times 2^{\mathbb{N}}$. It is known from elementary topology of Polish spaces that any $\mathbf{\Sigma}_1^1$ subset of a Polish space S is is equal to the projection of a closed subset of $S \times \mathbb{N}^{\mathbb{N}}$ on S. Thus there exists a closed set $P \subseteq 2^{\mathbb{N}} \times 2^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ satisfying

$$A = \mathsf{dom}\, P = \{\langle x, y \rangle : \exists\, z\, P(x, y, z)\}.$$

Further, there is a tree $R \subseteq (2 \times 2 \times \mathbb{N})^{<\omega}$ (a tree on $2 \times 2 \times \mathbb{N}$) such that

$$P = [R] = \{\langle x, y, \gamma \rangle : \forall n\, R(x \restriction n, y \restriction n, \gamma \restriction n)\},$$

and hence

18.2.1. $\langle x, y \rangle \in A \iff R_{xy} = \{s \in \mathbb{N}^{<\omega} : R(x \restriction \mathtt{lh}\, s, y \restriction \mathtt{lh}\, s, s)\}$ is ill founded.

(Obviously R_{xy} is a tree in $\mathbb{N}^{<\omega}$.) If A is an arbitrary $\mathbf{\Sigma}_1^1$ set, then, perhaps, not much can be established regarding the structure of a tree R which generates A in the sense of 18.2.1. However, assuming that $A = \mathsf{E}$ is an equivalence relation on $2^{\mathbb{N}}$, we can expect a nicer behaviour of R. This is indeed the case.

The following key definition goes back to [**LR05, Ros05**].

DEFINITION 18.2.2. A tree T on a set of the form $X \times \mathbb{N}$ is *normal* if for any $u \in X^{<\omega}$ and $s, t \in \mathbb{N}^{<\omega}$ such that $\mathtt{lh}\, u = \mathtt{lh}\, s = \mathtt{lh}\, t$ and $s \le_{\mathrm{cw}} t$, we have $\langle u, s\rangle \in T \Longrightarrow \langle u, t\rangle \in T$. □

Thus normality means that the tree is $\le_{\mathrm{cw}}$-closed upwards *w.r.t.* the second component. $X = 2 \times 2$ in the next theorem, and the case $X = 2 = \{0, 1\}$ will also be considered. But in all cases, $(X \times \mathbb{N})^{<\omega}$ itself is a normal tree.

THEOREM 18.2.3. *Suppose that* $Q \subseteq (2 \times 2 \times \mathbb{N})^{<\omega}$ *is a tree and the set*

$$(*) \qquad \mathsf{E} = \{\langle x, y\rangle \in 2^{\mathbb{N}} \times 2^{\mathbb{N}} : Q_{xy} \text{ is ill-founded}\}$$

is an equivalence relation on $2^{\mathbb{N}}$. *Then there is a tree* $R \subseteq (2 \times 2 \times \mathbb{N})^{<\omega}$ *satisfying the following requirements.*

(i) *Symmetry*: $R(u, v, s) \Longleftrightarrow R(v, u, s)$, *hence* $R_{xy} = R_{yx}$ *for all* x, y.

(ii) *If* $u \in 2^{\mathbb{N}}$, $s \in \mathbb{N}^{\mathbb{N}}$, $\mathtt{lh}\, s = \mathtt{lh}\, u$, *then* $R(u, u, s)$.

(iii) *Normality*: *if* $R(u, v, s)$, $t \in \mathbb{N}^{\mathbb{N}}$, *and* $s \le_{\mathrm{cw}} t$, *then* $R(u, v, t)$.

(iv) *Transitivity*: *if* $R(u, v, s)$ *and* $R(v, w, t)$, *then* $R(u, w, s +_{\mathrm{cw}} t)$.

(v) *For any* $x, y \in 2^{\mathbb{N}}$, R_{xy} *is ill founded iff so is* Q_{xy}; *hence,* $(*)$ *holds for the tree* R *instead of* Q.

This theorem is equal to Theorem 4 in [**LR05**]. The transformation from Q to R as in the theorem is called here the *Louveau–Rosendal transform*.

PROOF. **Part 1.** We observe that the tree

$$\widehat{Q} = Q \cup \{\langle u, u, s\rangle : u \in 2^{\mathbb{N}} \wedge s \in \mathbb{N}^{\mathbb{N}} \wedge \mathtt{lh}\, s = \mathtt{lh}\, u\} \cup \{\langle u, v, s\rangle : Q(v, u, s)\}$$

satisfies $\widehat{Q}_{xy} = Q_{xy} \cup Q_{yx} \cup D_{xy}$, where $D_{xy} = \mathbb{N}^{<\omega}$ provided $x = y$ and $D_{xy} = \varnothing$, otherwise. It easily follows that $(*)$ still holds for $\widehat{Q}$. In addition, $\widehat{Q}$ obviously satisfies both (i) and (ii). Thus *we can assume, from the beginning, that* Q *satisfies both* (i) *and* (ii).

Part 2. In this assumption to fulfill (iii) we define

$$\widehat{Q} = \{\langle u, v, t\rangle \in (2 \times 2 \times \mathbb{N})^{<\omega} : \exists\, \langle u, v, s\rangle \in Q\ (s \le_{\mathrm{cw}} t)\}.$$

This is still a tree on $2 \times 2 \times \mathbb{N}$, containing Q and satisfying (i), (ii), (iii). In addition, we have $\widehat{Q}_{xy} = Q_{xy} +_{\mathrm{cw}} 2^{<\omega}$ for any $x, y \in 2^{\mathbb{N}}$; therefore, the trees Q_{xy} and $\widehat{Q}_{xy}$ are ill founded simultaneously by Lemma 18.1.1. It follows that $(*)$ still holds for $\widehat{Q}$. Thus, *we can assume that* Q *itself satisfies* (i), (ii), (iii).

Part 3. It is somewhat more difficult to fulfill (iv). A straightforward plan would be to define a new tree R containing all triples of the form $\langle u_0, u_{n+1}, s_0 +_{\mathrm{cw}} \cdots +_{\mathrm{cw}} s_k\rangle$, where $\langle u_i, u_{i+1}, s_i\rangle \in Q$ for all $i = 0, 1, \dots, k$. However, to work properly, such a construction has to be equipped with a kind of counter for the number k of steps in the finite chain. This idea can be realized as follows.

Working in the assumption that Q satisfies (i), (ii), (iii) (see Part 2), we define a tree $R \subseteq (2 \times 2 \times \mathbb{N})^{<\omega}$ as follows. Suppose that $n \in \mathbb{N}$, $u, v \in 2^n$, $s \in \mathbb{N}^n$, $k \in \mathbb{N}$, and $i, j \in 2 = \{0, 1\}$. We put $\langle u^\frown i, v^\frown j, k^\frown s\rangle \in R$ iff

$$(\dagger) \qquad \exists\, u_0, u_1, \dots, u_k \in 2^n\ (u_0 = u \wedge u_k = v \wedge \forall\, \ell < k\ Q(u_\ell, u_{\ell+1}, s)).$$

In addition, we put $\langle \Lambda, \Lambda, \Lambda \rangle \in R$, of course. ($\Lambda$ is the empty sequence.) Note that R is a tree on $2 \times 2 \times \mathbb{N}$ because so is Q.

We claim that, in our assumptions, the tree R satisfies all of (i)–(v).

(i) If $u_0, \dots, u_k$ witness $R(u^\frown i, v^\frown j, k^\frown s)$, then the reversed sequence $u_k, \dots, u_0$ witnesses $R(v^\frown j, u^\frown i, k^\frown s)$ in the sense of (†), because the tree Q satisfies (i).

(iii) Suppose that $\langle u^\frown i, v^\frown j, k^\frown s \rangle \in R$, and let $u_0, \dots, u_k$ witness (†). Let $n = \mathtt{lh}\, u = \mathtt{lh}\, v = \mathtt{lh}\, s = \mathtt{lh}\, u_\ell$, $\forall \ell$. Suppose that $k \le k'$ and $s \le_{\text{cw}} s'$ (still $\mathtt{lh}\, s' = n$). Put $u_\ell = v$ whenever $k < \ell \le k'$. Note that $Q(u_\ell, u_{\ell+1}, s)$ also holds for $k < \ell < k'$ by (ii) for Q. (Indeed, in this case $u_\ell = u_{\ell+1}$.) Thus, $Q(u_\ell, u_{\ell+1}, s')$ holds for all $\ell < k'$ by (iii) for Q. By definition, this witnesses $\langle u^\frown i, v^\frown j, k'^\frown s' \rangle \in R$, as required.

(ii) If $k = 0$ and $u = v$, then Lemma 6.1.1 obviously holds (with the empty list of intermediate sequences $u_1, \dots, u_{k-1}$), and hence $R(u^\frown i, u^\frown j, 0^\frown s)$ holds for all $u \in 2^{\mathbb{N}}$, $s \in \mathbb{N}^{\mathbb{N}}$ of equal length, in particular, $R(u, u, 0^n)$ for all n and $u \in \mathbb{N}^{\mathbb{N}}$ with $\mathtt{lh}\, u = n$. It remains to apply property (iii) just proved.

(iv) Suppose that the triples $\langle u^\frown i, v^\frown j, k^\frown s \rangle$ and $\langle v^\frown j, w^\frown \rho, \kappa^\frown \sigma \rangle$ belong to R, and n is the length of all sequences u, v, s, w, t. Let $R(u^\frown i, v^\frown j, k^\frown s)$ be witnessed, in the sense of (†), by $u_0, \dots, u_k$ and, accordingly, $R(v^\frown j, w^\frown \rho, \kappa^\frown \sigma)$ be witnessed by $v_0, \dots, v_\kappa$. (All u_ℓ and v_ℓ belong to 2^n.) Since Q satisfies (iii), the same sequences also witness $R(u^\frown i, v^\frown j, k^\frown t)$ and $R(v^\frown j, w^\frown \rho, \kappa^\frown t)$, where $t = s +_{\text{cw}} \sigma$ (component-wise). It easily follows that the concatenated complex $u_0, \dots, u_{k-1}, u_k = v_0, v_1, \dots, v_\kappa$ witnesses $R(u^\frown i, w^\frown \rho, (k+\kappa)^\frown t)$, as required.

(v) We observe that, by definition, $Q(u, v, s) \Longrightarrow R(u^\frown i, v^\frown j, 1^\frown s)$ for any i, $j = 0, 1$. It follows that, for any $x, y \in 2^{\mathbb{N}}$, $s \in Q_{xy} \Longrightarrow 1^\frown s \in R_{xy}$, and hence R_{xy} is ill founded provided so is Q_{xy}.

The inverse implication in (v) needs more work. This argument belongs to Louveau and Rosendal [**LR05**]. Assume that R_{xy} is ill founded; that is, there exists an infinite sequence $\delta \in \mathbb{N}^{\mathbb{N}}$ such that $\forall n R(x \restriction n, y \restriction n, \delta \restriction n)$. Let $k = \delta(0)$ and $\gamma(m) = \delta(m+1)$ for all m, so that $\delta = k^\frown \gamma$. By definition, for any n there exist sequences $u_0^n, \dots, u_k^n \in 2^n$ such that $u_0^n = x \restriction n$, $u_k^n = y \restriction n$, and $Q(u_\ell^n, u_{\ell+1}^n, \gamma \restriction n)$ for all $\ell < k$. Each $k+1$-tuple $\langle u_0^n, \dots, u_k^n \rangle \in (2^n)^{k+1}$ can be considered to be an n-tuple in $(2^{k+1})^n$. By König's lemma, there exist infinite sequences $x_0, \dots, x_k \in 2^{\mathbb{N}}$ such that for any m there is a number $n \ge m$ with $x_\ell \restriction m = u_\ell^n \restriction m$ for all $\ell \le k$. It follows that $x_0 = x$, $x_k = y$, and, as Q is a tree, $Q(x_\ell \restriction m, x_{\ell+1} \restriction m, \gamma \restriction m)$ holds for all $\ell < k$ and all m. We conclude that $x_\ell \mathrel{\mathsf{E}} x_{\ell+1}$ for all $\ell < k$ by (∗) for Q. Therefore, $x \mathrel{\mathsf{E}} y$ because E is an equivalence relation. Finally, Q_{xy} is ill founded still by (∗) for Q. □

18.3. Embedding and equivalence of normal trees

Let **NT** denote the set of all non-empty normal trees $T \subseteq (2 \times \mathbb{N})^{<\omega}$. Suppose that $S, T \in \mathbf{NT}$. The set of all finite sequences $f \in \mathbb{N}^{<\omega}$ such that

$$\langle u, s \rangle \in S \Longrightarrow \langle u, s +_{\text{cw}} (f \restriction n) \rangle \in T$$

for all $n \le \mathtt{lh}\, f$ and $u \in 2^n$, $s \in \mathbb{N}^n$, will be denoted by $\textsc{Emb}(S, T)$. Obviously $\textsc{Emb}(S, T)$ is a tree in $\mathbb{N}^{<\omega}$ containing Λ.

We proceed with the following key definition of [**LR05**].

DEFINITION 18.3.1. Suppose that $S, T \in \mathbf{NT}$. Define $S \leq_{\text{NT}} T$ if and only if the tree $\text{EMB}(S, T)$ is ill founded, that is,

$$\exists \gamma \in \mathbb{N}^{\mathbb{N}} \, \forall n \, \forall u \in 2^n \, \forall s \in \mathbb{N}^n \, (\langle u, s \rangle \in S \Longrightarrow \langle u, s +_{\text{cw}} \gamma \restriction n \rangle \in T).$$

Define $S \mathsf{E}_{\text{NT}} T$ if and only if $S \leq_{\text{NT}} T$ and $T \leq_{\text{NT}} S$.[1] □

Thus $S \leq_{\text{NT}} T$ indicates the existence of a certain shift-type embedding of S into T. We assert that the relation $\leq_{\text{NT}}$ is a partial order on the set $\mathbf{NT}$.

Indeed to check that $\leq_{\text{NT}}$ is transitive, suppose that $R \leq_{\text{NT}} S$ and $S \leq_{\text{NT}} T$, where R, S, T are normal trees in $(2 \times \mathbb{N})^{<\omega}$. Then the trees $U = \text{EMB}(R, S)$ and $V = \text{EMB}(S, T)$ (trees in $\mathbb{N}^{<\omega}$) are ill-founded, and hence so is $W = U +_{\text{cw}} V$ by Lemma 18.1.1. On the other hand, easy verification shows that $W \subseteq \text{EMB}(R, T)$. Thus $\text{EMB}(R, T)$ is ill founded, as required. It follows that E_{NT} is an equivalence relation on $\mathbf{NT}$.

Moreover, applying the component-wise addition to the sequences γ that witness $\leq_{\text{NT}}$, one proves that $S \mathsf{E}_{\text{NT}} T$ is equivalent to the existence of $\gamma \in \mathbb{N}^{\mathbb{N}}$ such that for all n and all $u \in 2^n$, $s \in \mathbb{N}^n$, the following holds simultaneously:

$$\langle u, s \rangle \in S \Longrightarrow \langle u, s +_{\text{cw}} \gamma \restriction n \rangle \in T \text{ and } \langle u, s \rangle \in T \Longrightarrow \langle u, s +_{\text{cw}} \gamma \restriction n \rangle \in S.$$

COROLLARY 18.3.2. *If* $S, T \in \mathbf{NT}$, *then* $S \mathsf{E}_{\text{NT}} T$ *iff the tree* $\text{EMB}(S, T) \cap \text{EMB}(T, S)$ *is ill founded.* □

Note that every tree $T \in \mathbf{NT}$ is, by definition, a subset of the countable set $(2 \times \mathbb{N})^{<\omega}$. Therefore $\mathbf{NT}$ is a subset of the Polish space $\mathscr{P}((2 \times \mathbb{N})^{<\omega})$ identified, as usual, with the product space $2^{(2 \times \mathbb{N})^{<\omega}}$. (Elementary estimations show that in fact $\mathbf{NT}$ is a closed set.) It follows that the relations $\leq_{\text{NT}}$ and E_{NT} are, formally, subsets of $\mathscr{P}((2 \times \mathbb{N})^{<\omega}) \times \mathscr{P}((2 \times \mathbb{N})^{<\omega})$.

LEMMA 18.3.3. $\leq_{\text{NT}}$ *and* E_{NT} *are* $\mathbf{\Sigma}^1_1$ *relations.*

PROOF. Straightforward estimations. The principal quantifier expresses the existence of $\gamma \in \mathbb{N}^{\mathbb{N}}$ with certain properties. □

It occurs that E_{NT} belongs to a special type of $\mathbf{\Sigma}^1_1$ equivalence relations.

DEFINITION 18.3.4. An $\mathbf{\Sigma}^1_1$ equivalence relation U is *universal*, or *complete*, if and only if $\mathsf{F} \leq_{\text{B}} \mathsf{U}$ holds for any other $\mathbf{\Sigma}^1_1$ equivalence relation F. □

There is a simple construction that yields a universal $\mathbf{\Sigma}^1_1$ equivalence relation.

EXAMPLE 18.3.5. We begin with a $\mathbf{\Sigma}^1_1$ set $U \subseteq (\mathbb{N}^{\mathbb{N}})^3$, universal in the sense that for any $\mathbf{\Sigma}^1_1$ set $P \subseteq (\mathbb{N}^{\mathbb{N}})^2$ there is an index $x \in \mathbb{N}^{\mathbb{N}}$ such that P is equal to the cross-section $(U)_x = \{\langle y, z \rangle : \langle x, y, z \rangle \in U\}$. (See Section 2.5 on the existence of universal sets.) Define a set $P \subseteq (\mathbb{N}^{\mathbb{N}})^3$ so that every cross-section $(P)_x$ is equal to the *equivalence hull* of $(U)_x$, that is, to the least equivalence relation containing $(U)_x$. Formally, $\langle y, z \rangle \in (P)_x$ iff there is a finite chain $y = y_0, y_1, y_2, \dots y_n, y_{n+1} = z$ such that, for any $k \leq n$, either $\langle y_k, y_{k+1} \rangle$ belongs to U_x, or $\langle y_{k+1}, y_k \rangle$ belongs to $(U)_x$, or just $y_k = y_{k+1}$.

Clearly, P is still a $\mathbf{\Sigma}^1_1$ subset of $(\mathbb{N}^{\mathbb{N}})^3$, with each $(P)_x$ being a $\mathbf{\Sigma}^1_1$ equivalence relation. Moreover, if $(U)_x$ is an equivalence relation, then $(P)_x = (U)_x$. Thus the family of all cross-sections $(P)_x$, $x \in \mathbb{N}^{\mathbb{N}}$, is equal to the family of all $\mathbf{\Sigma}^1_1$ equivalence

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[1] $\leq_{\text{NT}}$ and E_{NT} are denoted in [**Ros05**] by $\leq^*_{\max}$ and $E^*_{\max}$, respectively.

relations on $\mathbb{N}^{\mathbb{N}}$. We claim that the equivalence relation U on $(\mathbb{N}^{\mathbb{N}})^2$, defined so that $\langle x,y\rangle \mathrel{\mathsf{U}} \langle x',y'\rangle$ iff $x = x'$ and $\langle y,y'\rangle \in P_x$, is universal. For instance, take any Σ^1_1 equivalence relation F on $\mathbb{N}^{\mathbb{N}}$. Then $\mathsf{F} = (P)_x$ for some x by the above. Therefore, the map $\vartheta(y) = \langle x,y\rangle$ is a continuous reduction of F to U, as required. □

The next theorem proposes a more meaningful universal equivalence relation.

THEOREM 18.3.6 (Theorem 5 in [**LR05**]). E_{NT} *is a universal* Σ^1_1 *equivalence relation on* **NT**.

PROOF. Consider any Σ^1_1 equivalence relation E on $2^{\mathbb{N}}$. Then E is a Σ^1_1 subset of $2^{\mathbb{N}} \times 2^{\mathbb{N}}$, and hence there is a tree $Q \subseteq (2\times 2\times \mathbb{N})^{<\omega}$ (a tree on $2\times 2\times\mathbb{N}$) such that, for all $x,y \in 2^{\mathbb{N}}$,

$$x \mathrel{\mathsf{E}} y \iff \text{the cross-section tree } Q_{xy} \text{ is ill founded.}$$

It can be assumed, by Theorem 18.2.3, that Q satisfies requirements (i)–(v) of Theorem 18.2.3. We claim that the map

$$x \longmapsto \vartheta(x) = \{\langle u,s\rangle \in (2\times\mathbb{N})^{<\omega} : Q(u, x\restriction \mathtt{lh}\, u, s)\} \qquad (x \in 2^{\mathbb{N}})$$

is a Borel reduction of E to E_{NT}. That ϑ is a Borel, even continuous map, is rather easy. That $\vartheta(x) \in \mathbf{NT}$ immediately follows from (iii). The reduction property follows from the next lemma.

LEMMA 18.3.7. *If a tree* $Q \subseteq (2\times 2\times\mathbb{N})^{<\omega}$ *satisfies requirements* (i)–(iv) *of Theorem* 18.2.3, *and* $x,y\in 2^{\mathbb{N}}$, *then* $\mathrm{EMB}(\vartheta(x),\vartheta(y)) = Q_{xy}$.

PROOF. Suppose that $f \in \mathrm{EMB}(\vartheta(x),\vartheta(y))$, $m = \mathtt{lh}\, f$. Then, by definition,

$$Q(u, x\restriction m, s) \Longrightarrow R(u, y\restriction m, s +_{\mathrm{cw}} f)$$

holds for all $u \in 2^m$ and $s \in \mathbb{N}^m$. Here take $u = x\restriction m$ and $s = 0^m$ (the sequence of m 0s); then $Q(x\restriction m, x\restriction m, 0^n) \Longrightarrow Q(x\restriction m, y\restriction m, f)$. Yet the left-hand side holds by (ii). Therefore, the right-hand side holds, thus $f \in Q_{xy}$.

To prove the converse, let $f \in Q_{xy}$, that is, $Q(x\restriction m, y\restriction m, f)$, where $m = \mathtt{lh}\, f$, and hence $Q(x\restriction n, y\restriction n, f\restriction n)$ for any $n \le m$ as Q is a tree. Assume that $n\le m$ and $u \in 2^n$, $s\in\mathbb{N}^n$. We have to prove

$$Q(u, x\restriction n, s) \Longrightarrow Q(u, y\restriction n, s +_{\mathrm{cw}} (f\restriction n)).$$

So suppose that $Q(u, x\restriction n, s)$. In addition, $Q(x\restriction n, y\restriction n, f\restriction n)$ holds by the above. Then $Q(u, y\restriction n, s +_{\mathrm{cw}} (f\restriction n))$ holds by (iv), as required. □ (Lemma)

To accomplish the proof of Theorem 18.3.6, suppose that $x,y\in 2^{\mathbb{N}}$. Then $x \mathrel{\mathsf{E}} y$ iff the tree R_{xy} is ill founded, iff (by the lemma) $\mathrm{EMB}(\vartheta(y),\vartheta(x))$ is ill founded, iff $\vartheta(x) \mathrel{\mathsf{E}_{\mathrm{NT}}} \vartheta(y)$ (by Definition 18.3.1). □ (Theorem 18.3.6)

18.4. Reduction to Borel ideals: first approach

We present two different proofs of the following theorem, the main result of this chapter.

THEOREM 18.4.1 (ROSENTAL [**Ros05**]). *There is a* $\subseteq$*-decreasing sequence of Borel ideals* $\mathscr{I}_\xi$ $(\xi < \omega_1)$ *on* $\mathbb{N}$, $\le_{\mathrm{B}}$*-cofinal in the sense that every Borel equivalence relation is Borel reducible to one of the relations* $\mathsf{E}_{\mathscr{I}_\xi}$.

Note that this theorem, together with Corollary 13.9.4, accomplishes the proof of Theorem 5.8.1.

The first proof, due to ROSENDAL [**Ros05**], involves the ideal $\mathscr{I}_{\mathrm{NT}}$ on $(2 \times \mathbb{N})^{<\omega}$ finitely generated by all sets of the form $S \triangle T$, where $S, T \subseteq (2 \times \mathbb{N})^{<\omega}$ are normal trees and $S \mathsf{E}_{\mathrm{NT}} T$. Thus $\mathscr{I}_{\mathrm{NT}}$ consists of all subsets of $(2 \times \mathbb{N})^{<\omega}$, covered by unions of finitely many symmetric differences $S \triangle T$ of the type just indicated.

THEOREM 18.4.2. *$\mathscr{I}_{\mathrm{NT}}$ is $\mathbf{\Sigma}^1_1$ as a subset of the Polish space $\mathscr{P}((2 \times \mathbb{N})^{<\omega})$. Furthermore, the equivalence relation E_{NT} is equal to $\mathsf{E}_{\mathscr{I}} \restriction \mathbf{NT}$. This means that for any $S, T \in \mathbf{NT}$, the following holds: $S \mathsf{E}_{\mathrm{NT}} T$ if and only if $S \triangle T \in \mathscr{I}_{\mathrm{NT}}$.*

PROOF. That $\mathscr{I}_{\mathrm{NT}}$ is $\mathbf{\Sigma}^1_1$ is quite clear: the principal quantifier expresses the existence of a finite collection of elements of $\mathbf{NT}$, whose properties are expressible still by a $\mathbf{\Sigma}^1_1$ relation because E_{NT} is $\mathbf{\Sigma}^1_1$.

Suppose that $S \triangle T \in \mathscr{I}_{\mathrm{NT}}$, and prove $S \mathsf{E}_{\mathrm{NT}} T$ (the non-trivial direction). By definition $S \triangle T \subseteq \bigcup_{i=1}^{k}(S_i \triangle T_i)$, where $S_i, T_i \in \mathbf{NT}$ and $S_i \mathsf{E}_{\mathrm{NT}} T_i$. Then the trees $R_i = \mathrm{EMB}(S_i, T_i) \cap \mathrm{EMB}(T_i, S_i)$ are ill founded by Corollary 18.3.2. We have to prove that $\mathrm{EMB}(S, T)$ and $\mathrm{EMB}(T, S)$ are ill-founded trees, too. To check the ill-foundedness of $\mathrm{EMB}(S, T)$, note that the tree $R = R_1 +_{\mathrm{cw}} \cdots +_{\mathrm{cw}} R_k$ is ill founded by Lemma 18.1.1. Thus it remains to prove that $R \subseteq \mathrm{EMB}(S, T)$.

Consider any $r = r_1 +_{\mathrm{cw}} \cdots +_{\mathrm{cw}} r_k \in R$, where all sequences $r_i \in R_i$, $i = 1, \ldots, k$, have one and the same length, say m. Suppose toward the contrary that $r \notin \mathrm{EMB}(S, T)$, i.e., there exists a pair $\langle u, s\rangle \in S$ such that $\langle u, s +_{\mathrm{cw}} (r \restriction n)\rangle \notin T$, where $n = \mathtt{lh}\, u = \mathtt{lh}\, s \le m$. Then

$$(\ddagger) \qquad \langle u, s +_{\mathrm{cw}} r'\rangle \notin T \quad \text{whenever} \quad r' \in 2^n,\ r' \le_{\mathrm{cw}} r \restriction n.$$

In particular, $\langle u, s\rangle \notin T$ by normality, and hence $\langle u, s\rangle \in S \triangle T$, thus $\langle u, s\rangle \in S_{i_1} \triangle T_{i_1}$ for some $1 \le i_1 \le k$. This implies $\langle u, s_1\rangle \in S_{i_1} \cap T_{i_1}$, where $s_1 = s +_{\mathrm{cw}} (r_{i_1} \restriction n)$. (Indeed we have $\langle u, s\rangle \in S_{i_1} \cup T_{i_1}$ by the choice of i_1. If say $\langle u, s\rangle \in S_{i_1}$, then $\langle u, s_1\rangle \in T_{i_1}$ because $r_{i_1} \in R_{i_1} \subseteq \mathrm{EMB}(S_{i_1}, T_{i_1})$. In addition $\langle u, s_1\rangle \in S_{i_1}$ by the normality of S_{i_1}.)

Once again, we have $\langle u, s_1\rangle \in S \smallsetminus T$ by ($\ddagger$). It follows that $\langle u, s_1\rangle \in S_{i_2} \triangle T_{i_2}$ for some $1 \le i_2 \le k$ by the same argument. This implies $\langle u, s_2\rangle \in S_{i_2} \cap T_{i_2}$, where $s_2 = s_1 +_{\mathrm{cw}} (r_{i_2} \restriction n)$, because r_{i_2} belongs to R_{i_2}. Note that $i_2 \ne i_1$ as $\langle u, s_1\rangle \in S_{i_1} \cap T_{i_1}$, and still $\langle u, s_2\rangle \in S_{i_1} \cap T_{i_1}$ since S_i and T_i are normal trees.

After k steps of this construction, all indices $1 \le i \le k$ will be considered, and the final sequence $s_k = s +_{\mathrm{cw}} (r \restriction n)$ will satisfy $\langle u, s_k\rangle \in S_i \cap T_i$ for all $i = 1, \ldots, k$. It follows that $\langle u, s_k\rangle \notin S \triangle T$. However, $\langle u, s_k\rangle \in S$ since $\langle u, s\rangle \in S$ and S is a normal tree. Thus, $\langle u, s_k\rangle$ belongs to T, contrary to the above. □

Theorems 18.4.2 and 18.3.6 imply

COROLLARY 18.4.3. *$\mathsf{E}_{\mathscr{I}_{\mathrm{NT}}}$ is a universal $\mathbf{\Sigma}^1_1$ equivalence relation.* □

Let us show now that these properties of $\mathscr{I}_{\mathrm{NT}}$ suffice to prove Theorem 18.4.1.

We begin with a very general fact of basic descriptive set theory. As any $\mathbf{\Sigma}^1_1$ set, $\mathscr{I}_{\mathrm{NT}}$ can be presented in the form $\mathscr{I}_{\mathrm{NT}} = \bigcap_{\xi<\omega_1} \mathscr{I}^{\xi}_{\mathrm{NT}}$, where $\mathscr{I}^{\xi}_{\mathrm{NT}}$ are Borel subsets of $\mathscr{P}((2 \times \mathbb{N})^{<\omega})$, $\xi < \eta \Longrightarrow \mathscr{I}^{\eta}_{\mathrm{NT}} \subseteq \mathscr{I}^{\xi}_{\mathrm{NT}}$, and for any $\mathbf{\Pi}^1_1$ set X in the same Polish space containing $\mathscr{I}_{\mathrm{NT}}$, there is an ordinal $\xi < \omega_1$ such that $\mathscr{I}^{\xi}_{\mathrm{NT}} \subseteq X$.

(This *index restriction* property was first established by LUSIN and SIERPIŃSKI [**LS18**], essentially in the dual form saying that the canonical representation of any

$\mathbf{\Pi}^1_1$ set C, in the form of a union $C = \bigcup_{\xi<\omega_1} C_\xi$ of $\subseteq$-increasing Borel approximations, has the property that for any $\mathbf{\Sigma}^1_1$ set $X \subseteq C$ there is an index $\xi < \omega_1$ with $X \subseteq C_\xi$. The shortest proof consists of observation that otherwise the relation $x \preccurlyeq y$ iff x appears in sets C_ξ not later than y on X is a $\mathbf{\Sigma}^1_1$ prewellordering of uncountable length, contrary to the KUNEN–MARTIN prewellordering theorem, see, *e.g.*, [**Mos80**, 2G.2].)

The sets $\mathscr{I}^\xi_{\rm NT}$ are called *(upper) Borel approximations* of $\mathscr{I}_{\rm NT}$.

The next lemma is the key fact.

LEMMA 18.4.4. *For any $\xi < \omega_1$ there exists an ordinal ν, $\xi < \nu < \omega_1$, such that the Borel approximation $\mathscr{I}^\nu_{\rm NT}$ is still an ideal.*

PROOF. **Step 1.** We claim that for any $\xi < \omega_1$, there is an ordinal $\eta = \eta(\xi)$, $\xi < \eta < \omega_1$, such that $y \subseteq x \in \mathscr{I}^\eta_{\rm NT} \Longrightarrow y \in \mathscr{I}^\xi_{\rm NT}$. Indeed the set

$$P = \{x \in \mathscr{I}^\xi_{\rm NT} : \forall\, y \subseteq x\ (y \in \mathscr{I}^\xi_{\rm NT})\}$$

is a $\mathbf{\Pi}^1_1$ superset of $\mathscr{I}_{\rm NT}$ (since $\mathscr{I}_{\rm NT}$ is an ideal). It follows that there is an ordinal $\eta > \xi$ with $\mathscr{I}^\eta_{\rm NT} \subseteq P$.

Step 2. We claim that for any $\xi < \omega_1$, there is an ordinal $\zeta = \zeta(\xi)$, $\xi < \zeta < \omega_1$, such that $x, y \in \mathscr{I}^\zeta_{\rm NT} \Longrightarrow x \cup y \in \mathscr{I}^\xi_{\rm NT}$. The argument contains two substeps. First, the set $X = \{x \in \mathscr{I}^\xi_{\rm NT} : \forall\, y \in \mathscr{I}_{\rm NT}\ (x \cup y \in \mathscr{I}^\xi_{\rm NT})\}$ is a $\mathbf{\Pi}^1_1$ superset of $\mathscr{I}_{\rm NT}$ since $\mathscr{I}_{\rm NT}$ is an ideal. Thus there is an ordinal $\alpha > \xi$ with $\mathscr{I}^\alpha_{\rm NT} \subseteq X$. Then we have $x \cup y \in \mathscr{I}^\xi_{\rm NT}$ whenever $x \in \mathscr{I}^\alpha_{\rm NT}$ and $y \in \mathscr{I}_{\rm NT}$. It follows that the $\mathbf{\Pi}^1_1$ set

$$Y = \{y \in \mathscr{I}^\alpha_{\rm NT} : \forall\, x \in \mathscr{I}^\alpha_{\rm NT}\ (x \cup y \in \mathscr{I}^\xi_{\rm NT})\}$$

is a superset of $\mathscr{I}_{\rm NT}$, and hence there is an an ordinal $\eta > \alpha$ such that $\mathscr{I}^\eta_{\rm NT} \subseteq Y$. Obviously η is as required.

Final argument. Put $\xi_0 = \xi$ and $\xi_{n+1} = \eta(\zeta(\xi_n))$ for all n. The ordinal $\nu = \sup_n \xi_n$ is as required. □

It follows that the set $\Xi = \{\xi < \omega_1 : \mathscr{I}^\xi_{\rm NT}$ is an ideal$\}$ is unbounded in ω_1. We also note that $\mathsf{E}_{\mathscr{I}^\xi_{\rm NT}}$ is a Borel equivalence relation on $\mathscr{P}((2\times\mathbb{N})^{<\omega})$ for any $\xi \in \Xi$, and the sequence of these equivalence relations is $\subseteq$-decreasing and satisfies $\mathsf{E}_{\mathscr{I}_{\rm NT}} = \bigcap_{\xi\in\Xi} \mathsf{E}_{\mathscr{I}^\xi_{\rm NT}}$. The proof of Theorem 18.4.1, our main result here, is accomplished with the following lemma.

LEMMA 18.4.5. *If E is a Borel equivalence relation on a Polish space X, then there is an ordinal $\xi \in \Xi$ such that $\mathsf{E} \le_{\rm B} \mathsf{E}_{\mathscr{I}^\xi_{\rm NT}}$.*

PROOF. It follows from Corollary 18.4.3 that $\mathsf{E} \le_{\rm B} \mathsf{E}_{\mathscr{I}_{\rm NT}}$, that is, there exists a Borel map $\vartheta : X \to \mathscr{P}((2\times\mathbb{N})^{<\omega})$ such that $x \mathrel{\mathsf{E}} y \Longleftrightarrow \vartheta(x) \mathbin{\triangle} \vartheta(y) \in \mathscr{I}_{\rm NT}$. Thus the full ϑ-image $\vartheta[P]$ of the set $P = (X\times X) \smallsetminus \mathsf{E}$ is a $\mathbf{\Sigma}^1_1$ set disjoint from $\mathscr{I}_{\rm NT}$. Then by Lemma 18.4.4 there is an ordinal $\xi \in \Xi$ such that $\vartheta[P]$ does not intersect $\mathscr{I}^\xi_{\rm NT}$, too. Thus ϑ reduces E not only to $\mathsf{E}_{\mathscr{I}_{\rm NT}}$ but also to the approximating Borel equivalence relation $\mathsf{E}_{\mathscr{I}^\xi_{\rm NT}}$. □

□ (Theorem 18.4.1, first proof)

18.5. Reduction to Borel ideals: second approach

Is there any method to prove Theorem 18.4.1 by a sequence of more "effective" and mathematically meaningful upper Borel approximations of a $\le_{\rm B}$-maximal analytic ideal? A suitable definition is given in [**KL06**].

First of all recall that any tree $T \subseteq X^{<\omega}$ admits the *rank function*, a unique map $\mathtt{rnk}_R : R \to \mathtt{Ord} \cup \{\infty\}$, where ∞ denotes a formal element larger than any ordinal, satisfying the following requirements:

(a) $\mathtt{rnk}_R(r) = -1$ whenever $r \notin R$;

(b) $\mathtt{rnk}_R(r) = \sup_{r^\frown n \in R} \mathtt{rnk}_R(r^\frown n)$ for any $r \in R$.[2] In particular, $\mathtt{rnk}_R(r) = 0$ if and only if $r \in R$ is a $\subseteq$-maximal element of R;

(c) $\mathtt{rnk}_R(r) = \infty$ if and only if R has an infinite branch containing r, i.e., there exists $\gamma \in X^{\mathbb{N}}$ such that $\gamma \restriction n \in R$ for all n, and $\gamma \restriction \mathtt{lh}\, r = r$.

In addition, put $\mathtt{rnk}(\varnothing) = -1$ for the empty tree $\varnothing$, and $\mathtt{rnk}(R) = \mathtt{rnk}_R(\Lambda)$ for any non-empty tree R. (Λ, the empty sequence, belongs to any tree $\varnothing \neq R \subseteq X^{<\omega}$.) Obviously, any tree R is well founded iff $\mathtt{rnk}(R) < \infty$.

DEFINITION 18.5.1. Suppose that $S, T \in \mathbf{NT}$ and $\xi < \omega_1$.
Define $S \leq^{\xi}_{\text{NT}} T$ iff the tree $\text{EMB}(S,T)$ satisfies $\mathtt{rnk}(\text{EMB}(S,T)) \geq \xi$.[3]
Define $S \mathsf{E}^{\xi}_{\text{NT}} T$ iff both $S \leq^{\xi}_{\text{NT}} T$ and $T \leq^{\xi}_{\text{NT}} S$. □

It is demonstrated in [**KL06**] by simple and rather straightforward arguments that all relations $\mathsf{E}^{\xi}_{\text{NT}}$ are Borel equivalence relations on $\mathbf{NT}$ of certain explicitly defined Borel ranks. A notable part of this result is the proof of transitivity of $\leq^{\xi}_{\text{NT}}$ and $\mathsf{E}^{\xi}_{\text{NT}}$, based on the following generalization of Lemma 18.1.1.

LEMMA 18.5.2 (Lemma 4 in [**KL06**]). $\mathtt{rnk}(S +_{\text{cw}} T) = \mathtt{min}\{\mathtt{rnk}(S), \mathtt{rnk}(T)\}$ *for any trees* $S, T \subseteq \mathbb{N}^{<\omega}$*, well or ill founded independently of each other.* □

In addition, $\mathsf{E}_{\text{NT}} = \bigcap_{\xi<\omega_1} \mathsf{E}^{\xi}_{\text{NT}}$, and this intersection has the same restriction property as above: if P is a $\mathbf{\Pi}^1_1$ subset of $\mathbf{NT} \times \mathbf{NT}$ containing E_{NT}, then there is an ordinal $\xi < \omega_1$ such that $\mathsf{E}^{\xi}_{\text{NT}} \subseteq P$.

It follows, essentially by the same arguments as above, that the sequence of Borel relations $\mathsf{E}^{\xi}_{\text{NT}}$ is $\leq_{\text{B}}$-cofinal among all Borel equivalence relations.

The following construction of Borel ideals that generate the equivalence relations $\mathsf{E}^{\xi}_{\text{NT}}$ is a modification of a construction in [**KL06**].

Consider a set $X \subseteq (2 \times \mathbb{N})^{<\omega}$. Suppose that $f \in \mathbb{N}^{<\omega}$, $u \in 2^{<\omega}$, $n = \mathtt{lh}\, u \leq \mathtt{lh}\, f$. Let $\mathbf{G}^u_f(X)$ be the game in which player I plays $s_1, s_2 \dots \in \mathbb{N}^n$, player II plays $t_1, t_2 \dots \in \mathbb{N}^n$ so that $t_1 +_{\text{cw}} \dots +_{\text{cw}} t_m \leq_{\text{cw}} f \restriction n$ for all m, and player I wins if and only if $\langle u, \widehat{s}_k \rangle \in X$ for all k, where $\widehat{s}_k = s_1 +_{\text{cw}} t_1 +_{\text{cw}} \dots +_{\text{cw}} s_{k-1} +_{\text{cw}} t_{k-1} +_{\text{cw}} s_k$.

Define $\text{WID}(X)$ to be the tree of all $f \in \mathbb{N}^{<\omega}$ such that for any $n \leq \mathtt{lh}\, f$ and $u \in 2^n$, player II has a winning strategy in $\mathbf{G}^u_f(X)$. Thus, informally, $f \in \text{WID}(X)$ can be seen as a statement of the possibility of leaving X for good in finitely many steps, the $+_{\text{cw}}$-total length of which is at most f. Let $\mathscr{I}_{\text{NT}}$ be the collection of all sets $X \subseteq (2 \times \mathbb{N})^{<\omega}$ such that $\text{WID}(X)$ is ill founded. For $\xi < \omega_1$, let $\mathscr{I}^{\xi}_{\text{NT}}$ be the collection of all sets $X \subseteq (2 \times \mathbb{N})^{<\omega}$ with $\mathtt{rnk}(\text{WID}(X)) \geq \xi$.

LEMMA 18.5.3. $\mathscr{I}_{\text{NT}}$ *and all sets* $\mathscr{I}^{\xi}_{\text{NT}}$ *are ideals on* $(2 \times \mathbb{N})^{<\omega}$.

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[2] We define $\sup \Omega$, for $\Omega \subseteq \mathtt{Ord}$, to be the least ordinal strictly larger than all ordinals in Ω. We also define $\sup \Omega = \infty$ provided Ω contains ∞.

[3] The inequality $\mathtt{rnk}(\text{EMB}(S,T)) \geq \xi$ means that either $\text{EMB}(S,T)$ (a tree in $\mathbb{N}^{<\omega}$) is ill founded (then $\mathtt{rnk}(\text{EMB}(S,T)) = \infty$) or it is well founded and its rank is an ordinal $\geq \xi$.

PROOF. Suppose that sets $X, Y \subseteq (2 \times \mathbb{N})^{<\omega}$ belong to $\mathscr{I}_{\rm NT}$, and hence the trees $F = \text{WID}(X)$ and $G = \text{WID}(Y)$ are ill founded. Then the tree $F +_{\rm cw} G$ is ill founded by Lemma 18.1.1 (to be replaced by Lemma 18.5.2 for the ideals $\mathscr{I}^{\xi}_{\rm NT}$), and hence it suffices to prove that $F +_{\rm cw} G \subseteq \text{WID}(X \cup Y)$.

Take any $f \in F$ and $g \in G$ with $\mathtt{lh}\, f = \mathtt{lh}\, g$. To prove that $h = f +_{\rm cw} g$ belongs to $\text{WID}(X \cup Y)$, fix any $u \in 2^n$, $n \le \mathtt{lh}\, f$, and a pair of winning strategies ξ, η for player II in games $\mathbf{G}^u_f(X)$ and $\mathbf{G}^u_g(Y)$, respectively. To describe a winning strategy for player II in $\mathbf{G}^u_h(X \cup Y)$, let $s_1, t_1, s_2, t_2, \ldots$ be a full sequence of moves. Put

$$K = \{k : \widehat{s}_k \in X\} \quad \text{and} \quad K' = \{k : \widehat{s}_k \in Y \smallsetminus X\}$$

and let $K = \{k_1, k_2, \ldots\}$ and $K' = \{k'_1, k'_2, \ldots\}$, in the increasing order.

For every k, if $k = k_j \in K$, then player II plays $t_k = \xi(\sigma_1, \tau_1, \ldots, \sigma_{j-1}, \tau_{j-1}, \sigma_j)$, where $\tau_i = t_{k_i}$ and, for all $1 \le i \le j$,

$$\sigma_i = s_{k_{i-1}+1} +_{\rm cw} t_{k_{i-1}+1} +_{\rm cw} s_{k_{i-1}+2} +_{\rm cw} t_{k_{i-1}+2} +_{\rm cw} \cdots +_{\rm cw} s_{k_i-1} +_{\rm cw} t_{k_i-1} +_{\rm cw} s_{k_i}.$$

Accordingly, if $k = k'_j \in K'$, then $t_k = \eta(\sigma'_1, \tau'_1, \ldots, \sigma'_{j-1}, \tau'_{j-1}, \sigma'_j)$, where

$$\sigma'_i = s_{k'_{i-1}+1} +_{\rm cw} t_{k'_{i-1}+1} +_{\rm cw} s_{k'_{i-1}+2} +_{\rm cw} t_{k'_{i-1}+2} +_{\rm cw} \cdots +_{\rm cw} s_{k'_i-1} +_{\rm cw} t_{k'_i-1} +_{\rm cw} s_{k'_i}$$

and $\tau'_i = t'_{k_i}$ for any $1 \le i \le j$. If to the contrary, player I wins, then $K \cup K' = \mathbb{N}$. Let, say, $K = \{k_1, k_2, \ldots\}$ be infinite. Then player II must win the auxiliary play $\sigma_1, \tau_1, \sigma_2, \tau_2, \ldots$ in $\mathbf{G}^u_f(X)$. Hence one of the finite sums

$$\widehat{\sigma}_j = \sigma_1 +_{\rm cw} \tau_1 +_{\rm cw} \cdots +_{\rm cw} \sigma_{j-1} +_{\rm cw} \tau_{j-1} +_{\rm cw} \sigma_j$$

satisfies $\widehat{\sigma}_j \notin X$. But obviously $\widehat{\sigma}_j = \widehat{s}_{k_j}$, which is a contradiction with $k_j \in K$. $\square$

Thus $\mathscr{I}_{\rm NT}$ is a $\mathbf{\Sigma}^1_1$ ideal while each $\mathscr{I}^{\xi}_{\rm NT}$ is a Borel ideal.

THEOREM 18.5.4. *The equivalence relation* $\mathsf{E}_{\rm NT}$ *is equal to* $\mathsf{E}_{\mathscr{I}_{\rm NT}} \restriction \mathbf{NT}$, *while for any* ξ, $\mathsf{E}^{\xi}_{\rm NT}$ *is equal to* $\mathsf{E}_{\mathscr{I}^{\xi}_{\rm NT}} \restriction \mathbf{NT}$.

PROOF. Consider any $S, T \in \mathbf{NT}$. Assume that $S \mathrel{\mathsf{E}_{\rm NT}} T$. Then the trees $F = \text{EMB}(S,T)$ and $G = \text{EMB}(T,S)$ are ill founded, and hence $H = F +_{\rm cw} G$ is also by Lemma 18.1.1. (Lemma 18.5.2 is used in the case of $\mathscr{I}^{\xi}_{\rm NT}$.) Note that $H \subseteq G \cap F$ since both S and T are $\le_{\rm cw}$-transitive to the right. Thus it suffices to prove that $G \cap F \subseteq \text{WID}(S \vartriangle T)$. Consider any $f \in G \cap F$. By definition, for any $\langle u, s\rangle \in S \cup T$, $\mathtt{lh}\, u = \mathtt{lh}\, s = n \le \mathtt{lh}\, f$, we have $\langle u, s +_{\rm cw} (f \restriction n)\rangle \in S \cap T$. In particular, $\langle u, s\rangle \in S \vartriangle T \Longrightarrow \langle u, s +_{\rm cw} (f \restriction n)\rangle \notin S \vartriangle T$, and easily $f \in \text{WID}(S \vartriangle T)$.

To prove the converse, suppose that $S \vartriangle T \in \mathscr{I}_{\rm NT}$, thus $\text{WID}(S \vartriangle T)$ is ill founded. It suffices to prove that $\text{WID}(S \vartriangle T) \subseteq \text{EMB}(S,T)$. Suppose toward the contrary that $f \in \text{WID}(S \vartriangle T)$ but $f \notin \text{EMB}(S,T)$. The latter means that there exists a pair $\langle u, s\rangle \in S$, $\mathtt{lh}\, u = \mathtt{lh}\, s = n \le \mathtt{lh}\, f$, such that $\langle u, s +_{\rm cw} (f \restriction n)\rangle \notin T$. Then also $\langle u, s\rangle \notin T$, and hence both $\langle u, s\rangle$ and $\langle u, s +_{\rm cw} (f \restriction n)\rangle$ belong to $S \smallsetminus T$. We conclude that

18.5.5. $\langle u, s + g\rangle \in S \smallsetminus T$ *holds for all* $g \in \mathbb{N}^n$ *with* $g \le_{\rm cw} (f \restriction n)$.

Now consider a play in $\mathbf{G}^u_f(S \vartriangle T)$ in which player II follows its winning strategy (which exists because $f \in \text{WID}(S \vartriangle T)$) while player I plays $s_1 = s$ and $s_k = 0^n$ (the sequence of n zeros) on every round $k \ge 2$. Let $t_1, t_2, \ldots$ be the sequence of player II's moves. Then $t_1 +_{\rm cw} \cdots +_{\rm cw} t_k \le_{\rm cw} (f \restriction n)$ for all k, and hence, by 18.5.5, the sum $\widehat{s}_k = s +_{\text{cw}} t_1 +_{\text{cw}} \cdots +_{\text{cw}} t_k$ satisfies $\langle u, \widehat{s}_k \rangle \in S \bigtriangleup T$, which contradicts the choice of the strategy. □

□ (Theorem 18.4.1, second proof)

18.6. Some questions

It can be reasonably conjectured that $\mathsf{E}_{\text{NT}}^{\eta} <_{\text{B}} \mathsf{E}_{\text{NT}}^{\omega\nu} <_{\text{B}} \mathsf{E}_{\text{NT}}^{\omega\nu+n}$ whenever $\eta < \omega\nu$ and $n \ge 1$. The background idea here is that there is no $\le_{\text{B}}$-largest Borel equivalence relation (noted in [**HKL98**]). Therefore, the sequence of equivalence relations $\mathsf{E}_{\text{NT}}^{\xi}$ has uncountably many indices of $<_{\text{B}}$-increase (in strict sense). On the other hand, it seems plausible that $\mathsf{E}_{\text{NT}}^{\omega\nu+n} \sim_{\text{B}} \mathsf{E}_{\text{NT}}^{\omega\nu+n+1}$ provided $n \ge 1$.

A couple of more interesting questions.

Which Borel classes contain complete equivalence relations?

A related problem can be discussed here. It was once considered a viable conjecture (see, *e.g.*, [**KR03**]) that the equivalence relation T_2 is not Borel reducible to any equivalence relation $\mathsf{E}_{\mathscr{I}}$ induced by a Borel ideal $\mathscr{I} \subseteq \mathscr{P}(\mathbb{N})$. It follows from Theorem 18.4.1 that this is not the case, in fact there is an ordinal $\xi < \omega_1$ such that $\mathsf{T}_2 \le_{\text{B}} \mathsf{E}_{\text{NT}}^{\xi}$. What is the least ordinal ξ satisfying this statement, and what is the nature of the corresponding ideal?

Finally, it should be stressed that all evaluations of the Borel class of equivalence relations in this paper were related to the actual Borel class in Cantor's discontinuum-like spaces. A somewhat deeper approach of "potential" Borel classes of equivalence relations in [**HKL98**] may require suitable adjustment of arguments.

APPENDIX A

On Cohen and Gandy–Harrington forcing over countable models

Forcing was invented as a tool to prove independence results, and it has been used extensively in this role in set theoretic investigations; see *e.g.*, [**Kun83**]. But here we are mainly interested in applications of forcing, especially of Cohen forcing and the Gandy–Harrington forcing in proofs of usual theorems in descriptive set theory. Cohen forcing, connected with the Baire property in Polish spaces, facilitates arguments with Cohen-generic points of Polish spaces. Gandy–Harrington forcing makes much more transparent several difficult arguments in the theory of Borel reducibility.

This appendix is by no means a manual on forcing. And reading it requires a certain minimal knowledge of this common set theoretic tool and related topics like models, formulas, theories, *etc.*

A.1. Models of a fragment of ZFC

Basically, forcing is a method of extension of models of set theory to bigger models by adding *generic* objects. As long as we deal with independence problems, the models considered are normally models of **ZFC** or even stronger theories. But working in **ZFC** we cannot use models of the full **ZFC** as a default prerequisite. Fortunately, models of different fragments of **ZFC** are sufficient substitutions in many cases.

Definition A.1.1. $\mathbf{ZFC}^-$ includes all **ZFC** axioms, minus the Power Set axiom but plus the axiom that says, "for every set X, the countable power set $\mathscr{P}_{\mathtt{ctbl}}(X) = \{y \subseteq X : \mathtt{card}\, y \le \aleph_0\}$ exists." □

This theory is strong enough to prove the existence of such sets as $\mathbb{N}$, $\mathbb{R}$, $2^{\mathbb{N}}$, $\mathbb{N}^{\mathbb{N}}$, ω_1, the set HC of all *hereditarily countable* sets, and many more of the same types, as well as typical properties of them. For instance, $\mathbf{ZFC}^-$ proves the existence of the cartesian power set X^Y for all sets X and all at most countable sets Y.

The following definition introduces a natural model of $\mathbf{ZFC}^-$.

Definition A.1.2. $\mathrm{H}_{\mathfrak{c}^+}$ is the collection of all sets x such that the transitive closure $\mathrm{TC}(x)$ has cardinality $\mathtt{card}\,\mathrm{TC}(x) \le \mathfrak{c}$. □

Recall that $\mathfrak{c} = 2^{\aleph_0} = \mathtt{card}\, 2^{\mathbb{N}}$ is the cardinality of the continuum, and $\mathrm{TC}(x)$ is the least transitive set y such that $x \subseteq y$.

A set y is *transitive* iff $a \in b \in y \Longrightarrow a \in y$.

Note that $\mathrm{H}_{\mathfrak{c}^+}$ is a rather large set. For instance, it obviously contains sets such as $\mathbb{N}$, $\mathbb{R}$, $2^{\mathbb{N}}$, $\mathbb{N}^{\mathbb{N}}$, ω_1, and $\mathrm{H}_{\mathfrak{c}^+}$ satisfies $\mathbf{ZFC}^-$ (in the presence of the axiom of choice), simply because $\mathfrak{c}^{\aleph_0} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \times \aleph_0} = 2^{\aleph_0} = \mathfrak{c}$.

REMARK A.1.3 (Submodels). It is known from model theory that if $X \subseteq \mathrm{H}_{\mathfrak{c}^+}$ is at most countable, then there exists a countable elementary submodel $M \subseteq \mathrm{H}_{\mathfrak{c}^+}$ such that $X \subseteq M$. Note that M contains all natural numbers, the set $\mathbb{N}$, sets like $\mathbb{R}$, $2^{\mathbb{N}}$, $\mathbb{N}^{\mathbb{N}}$, ω_1, and generally all sets $x \in \mathrm{H}_{\mathfrak{c}^+}$ $\in$-definable in $\mathrm{H}_{\mathfrak{c}^+}$.

The set M is not necessarily transitive, but it can be converted to a transitive set by means of the *Mostowski collapse*.

Namely, define a function ϕ on M such that $\phi(x) = \{\phi(y) : y \in x \cap M\}$ for all $x \in M$. This is a definition by $\in$-induction, in particular $\phi(\varnothing) = \varnothing$. The range $\mathfrak{M} = \{\phi(x) : x \in M\}$ of ϕ is a transitive set, and clearly ϕ is an $\in$-isomorphism of M onto $\mathfrak{M}$. Therefore, $\mathfrak{M}$ is a model of $\mathbf{ZFC}^-$ as well because M is also. □

THEOREM A.1.4. *Suppose that* $M, \mathfrak{M}, \phi$ *are as in Remark* A.1.3. *Then*

(i) $\phi(n) = n$ *for all* $n \in \mathbb{N}$, $\phi(\mathbb{N}) = \mathbb{N}$, $\phi(x) = x$ *for all* $x \in M \cap \mathbb{N}^{\mathbb{N}}$, $\phi(X) = X \cap \mathfrak{M}$ *for all* $X \in M$, $X \subseteq \mathbb{N}^{\mathbb{N}}$;

(ii) $\mathfrak{M}$ *is an elementary submodel of* $\mathrm{H}_{\mathfrak{c}^+}$ *w.r.t. all* $\in$*-formulas that contain only elements of* $(\mathbb{N}^{\mathbb{N}} \cup \mathbb{N} \cup \mathbb{R}) \cap \mathfrak{M}$ *as parameters*;

(iii) *therefore,* $\mathfrak{M}$ *is an elementary submodel of the universe of all sets w.r.t. all analytic formulas (in the sense of Section* 1.4*) that contain only elements of* $\mathbb{N}^{\mathbb{N}} \cap \mathfrak{M}$ *as parameters*;

(iv) *if* $X \in M$ *is a countable set, then* X *is countable in* M, *that is, there exists a map* $h \in M$, $h : \mathbb{N} \xrightarrow{\text{onto}} X$, *and hence* $\phi(X)$ *is countable in* $\mathfrak{M}$;

(v) *if* $X \in M$ *is countable and* $X \subseteq \mathbb{N}^{\mathbb{N}} \cup \mathbb{R}$, *then* $\phi(X) = X$.

PROOF. We leave the proof as an easy exercise. □

COROLLARY A.1.5. *If* X *is a hereditarily countable set (that is, the transitive closure* $\mathrm{TC}(X)$ *is at most countable), then there is a countable transitive model* $\mathfrak{M}$ *of* $\mathbf{ZFC}^-$ *such that* $X \in \mathfrak{M}$ *and* $\mathfrak{M}$ *is an elementary submodel of the universe w.r.t. all analytic formulas with elements of* $\mathfrak{M}$ *as parameters.*

PROOF. Let M be any countable elementary submodel of $\mathrm{H}_{\mathfrak{c}^+}$ satisfying $X \in M$ and $\mathrm{TC}(X) \subseteq M$. Define ϕ and $\mathfrak{M}$ as in Remark A.1.3. Use Theorem A.1.4 to prove that $\phi(x) = x$ for every $x \in \mathrm{TC}(X)$, and therefore $\phi(X) = X \in \mathfrak{M}$. In addition, we have $\phi(x) = x$ for all $x \in \mathfrak{M} \cap \mathbb{N}^{\mathbb{N}}$. It follows that $\mathfrak{M}$, M, $\mathrm{H}_{\mathfrak{c}^+}$ are elementarily equivalent *w.r.t.* all analytic formulas. And finally note that $\mathrm{H}_{\mathfrak{c}^+}$ contains all elements of $\mathbb{N}^{\mathbb{N}}$, so that $\mathrm{H}_{\mathfrak{c}^+}$ is an elementary submodel of the universe *w.r.t.* all analytic formulas. □

A few words about absoluteness. A formula φ is said to be *absolute w.r.t.* a model $\mathfrak{M}$ if φ is either true in both $\mathfrak{M}$ and the universe of all sets or false in both $\mathfrak{M}$ and the universe of all sets. Here it is assumed that φ is a formula with sets in $\mathfrak{M}$ as parameters. See [**Sho62**] on the next well-known theorem.

THEOREM A.1.6. *Suppose that* $\mathfrak{M}$ *is a transitive model of* $\mathbf{ZFC}^-$. *Then every* Σ^1_1 *formula with parameters in* $\mathfrak{M}$ *is absolute for* $\mathfrak{M}$ (MOSTOWSKI).

If in addition $\omega_1 \subseteq \mathfrak{M}$, *then every* Σ^1_2 *formula with parameters in* $\mathfrak{M}$ *is absolute for* $\mathfrak{M}$ (SHOENFIELD). □

A.2. Coding uncountable sets in countable models

Suppose that $\mathfrak{M}$ is a countable transitive model of $\mathbf{ZFC}^-$, for instance a model defined as in Remark A.1.3. By definition it cannot contain uncountable sets. However, many uncountable sets, especially sets in Polish spaces, can be explicitly coded in such a model.

(i) By a code of a Polish space $\mathbb{X}$, we understand a pair that consists of a countable dense subset $D_{\mathbb{X}} \subseteq \mathbb{X}$ and the distance function restricted to $D_{\mathbb{X}}$, that is, $d_{\mathbb{X}} \upharpoonright D_{\mathbb{X}}$. We consider here only those Polish spaces $\mathbb{X}$ that satisfy $\mathbb{X} \subseteq \mathbb{N}^{\mathbb{N}}$ (but the distance and the topology may have nothing in common with those of $\mathbb{N}^{\mathbb{N}}$), but obviously every Polish space is isometric to one of this type. In this case the sets $\mathbb{X}$, $d_{\mathbb{X}}$, $D_{\mathbb{X}}$, and $d_{\mathbb{X}} \upharpoonright D_{\mathbb{X}}$ belong to $\mathrm{H}_{\mathfrak{c}^+}$, and if M, $\mathfrak{M}$, ϕ are as in Remark A.1.3 and the sets D and $d_{\mathbb{X}} \upharpoonright D_{\mathbb{X}}$ belong to M, then $\phi(D_{\mathbb{X}}) = D_{\mathbb{X}}$ and $\phi(d_{\mathbb{X}} \upharpoonright D_{\mathbb{X}}) = d_{\mathbb{X}} \upharpoonright D_{\mathbb{X}}$. It follows by Lemma A.1.4 that the sets D and $d_{\mathbb{X}} \upharpoonright D_{\mathbb{X}}$ belong to $\mathfrak{M}$ as well and D is countable in $\mathfrak{M}$.

Say that $\mathbb{X}$ *is coded in* $\mathfrak{M}$, if the sets $D_{\mathbb{X}}$ and $d_{\mathbb{X}} \upharpoonright D_{\mathbb{X}}$ belong to $\mathfrak{M}$ and $D_{\mathbb{X}}$ is countable in $\mathfrak{M}$ in the sense that there is a map $f \in \mathfrak{M}$, $f : \mathbb{N} \overset{\text{onto}}{\longrightarrow} D_{\mathbb{X}}$.

(ii) Say that an open set $U \subseteq \mathbb{X}$ *is coded in* $\mathfrak{M}$, if there exists a set $c \in \mathfrak{M}$, $c \subseteq D_{\mathbb{X}} \times \mathbb{Q}^+$ such that $U = \bigcup_{\langle x,r\rangle \in c} U_r^{\mathbb{X}}(x)$, where $U_r^{\mathbb{X}}(x) = \{y \in \mathbb{X} : d_{\mathbb{X}}(x,y) < r\}$ (a basic open ball in $\mathbb{X}$). Such a set c is called a *code* of U.

Say that a $\mathbf{G}_\delta$-set $W \subseteq \mathbb{X}$ *is coded in* $\mathfrak{M}$, if it can be presented in the form $W = \bigcap_n U_n$, where all $U_n \subseteq \mathbb{X}$ are open sets, and there exists a sequence $c = \{c_n\}_{n\in\mathbb{N}} \in \mathfrak{M}$ such that each c_n is a code of U_n (as an open set). Such a sequence c is called a *code* of W.

(iii) Now suppose that Polish spaces $\mathbb{X}$ and $\mathbb{Y}$ are coded in $\mathfrak{M}$ as in (i), and $\vartheta : \mathbb{X} \to \mathbb{Y}$ is a Borel map that we would like to code as well. The coding of arbitrary Borel sets as in Section 2.8 can be employed, yet it is too complicated. In addition, our goal is somewhat simpler: we need to code not the function ϑ as a whole but rather the restriction $\vartheta \upharpoonright X$, where $X \subseteq \mathbb{X}$ is a $\mathbf{G}_\delta$ set already coded in $\mathfrak{M}$ as in (ii), and such that $\vartheta \upharpoonright X$ is continuous.

Note that the set $X_{yr} = \{x \in X : d_{\mathbb{Y}}(y, \vartheta(x)) < r\}$ is relatively open in X for every $y \in D_{\mathbb{Y}}$ and $r \in \mathbb{Q}^+$. Therefore, there exists an open (in $\mathbb{X}$) set $U_{yr} \subseteq \mathbb{X}$ such that $X_{yr} = X \cap U_{yr}$. By a code of $\vartheta \upharpoonright X$ we understand an arbitrary set $c = \{c_{yr}\}_{y\in D_{\mathbb{Y}},\, r\in\mathbb{Q}^+}$ such that every element c_{yr} is a code of the open set U_{yr} as in (ii). And we say that $\vartheta \upharpoonright X$ is coded in $\mathfrak{M}$ if this model contains a code of $\vartheta \upharpoonright X$ and also contains a code of X itself (as a $\mathbf{G}_\delta$ set).

A.3. Forcing over countable models

Suppose that $\mathfrak{M}$ is a countable transitive model of $\mathbf{ZFC}^-$, for instance, a model chosen as in Remark A.1.3. Let $P = \langle P\,; \leq\rangle \in \mathfrak{M}$ be a partially ordered set. It is called a *notion of forcing* in $\mathfrak{M}$. Elements $p \in P$ are called (forcing) *conditions*, and $p \leq q$ means that p is *stronger* than q. A set $D \subseteq P$ is

dense: if for every condition $q \in P$ there is a stronger condition $p \in D$;

open dense: if in addition for every $p \in D$, all conditions $q \in P$ stronger than p also belong to D.

An easy argument shows that *w.l.o.g.* dense sets can be replaced by open dense ones in the following definition of genericity. A set $G \subseteq P$ is *P-generic over* $\mathfrak{M}$[1] if it satisfies the following three conditions:

1) if $p, q \in G$, then there is $r \in G$ such that $r \le p$ and $r \le q$;

2) if $p \in G$, $q \in P$, $p \le q$, then $q \in G$;

3) if $D \in \mathfrak{M}$, $D \subseteq P$ is dense, then $G \cap D \neq \varnothing$.

If D is such, then there is a certain unique countable model $\mathfrak{M}[G]$ that contains both G and all sets in $\mathfrak{M}$ and is the least model satisfying these requirements. This model is called a *P-generic extension* of $\mathfrak{M}$ by adjoining G. It can be defined explicitly by $\mathfrak{M}[G] = \{x[G] : x \in \mathfrak{M}\}$, where $x[G]$ is defined by induction on the set theoretic rank of x so that $\varnothing[G] = \varnothing$ and, for all $x \in \mathfrak{M}$, $x \neq \varnothing$,

$$x[G] = \{y[G] : \exists p \in G\,(\langle p, y\rangle \in x)\}.$$

To see that every $x \in \mathfrak{M}$ belongs to $\mathfrak{M}[G]$, define $\mathring{x} \in \mathfrak{M}$ by induction on the set theoretic rank of x so that $\mathring{\varnothing} = \varnothing$ and $\mathring{x} = \{\mathring{y} : y \in x\}$ for $x \neq \varnothing$. Then obviously $\mathring{x}[G] = x$. To see that $G \in \mathfrak{M}[a]$, define $\underline{G} \in \mathfrak{M}$ to be the set of all pairs of the form $\langle p, p\rangle$, where $p \in P$. Then $\underline{G}[G] = G$ for all G.

Sometimes they identify $\mathring{x}$ with x itself and also $\underline{G}$ with G. Regarding the latter identification, one has to keep in mind that by definition $\underline{G}$ is a fixed element of $\mathfrak{M}$ that does not depend on any choice of a factual generic set G.

LEMMA A.3.1. *If $\mathfrak{M}$ is a countable transitive model of* $\mathbf{ZFC}^-$, *P is countable in $\mathfrak{M}$ and $G \subseteq P$ is P-generic over $\mathfrak{M}$, then $\mathfrak{M}[a]$ still is a model of* $\mathbf{ZFC}^-$.

PROOF. Emulate the proof that generic extensions of models of **ZFC** are still models of **ZFC**. □

The next "forcing product lemma" is helpful in some arguments.

THEOREM A.3.2. *Let $P_1, P_2 \in \mathfrak{M}$ be partially ordered sets. A set $G \subseteq P_1 \times P_2$ is $P_1 \times P_2$-generic over $\mathfrak{M}$ iff there exist a P_1-generic set $G_1 \subseteq P_1$ over $\mathfrak{M}$ and a P_2-generic set $G_2 \subseteq P_2$ over $\mathfrak{M}[G_1]$ such that $G = G_1 \times G_2$.* □

Finally, if $\varphi(y_1, \ldots, y_n)$ is an $\in$-formula with sets $y_1, \ldots, y_n \in \mathfrak{M}$ as parameters, and $p \in P$, then $p \Vdash \varphi(y_1, \ldots, y_n)$ (the *P-forcing relation*) means that if $G \subseteq P$ is P-generic over $\mathfrak{M}$, then the sentence $\varphi(y_1[G], \ldots, y_n[G])$ (with sets $x_1[G], \ldots, x_n[G] \in \mathfrak{M}[G]$ as parameters) is true in $\mathfrak{M}[G]$.

They mostly use this relation in the case when the given sets y_i have the form $\mathring{x}$ or $\underline{G}$. For instance, if $\varphi(x_1, \ldots, x_n, x)$ is an $\in$-formula, $x_1, \ldots, x_n \in \mathfrak{M}$, and $p \in P$, then $p \Vdash \varphi(\mathring{x}_1, \ldots, \mathring{x}_n, \underline{G})$ iff $\varphi(x_1, \ldots, x_n, G)$ is true in $\mathfrak{M}[G]$ for every set $G \subseteq P$ P-generic over $\mathfrak{M}$.

The following is the main forcing theorem.

THEOREM A.3.3. *If $\varphi(x_1, \ldots, x_n, x)$ is an $\in$-formula, $x_1, \ldots, x_n \in \mathfrak{M}$, a set $G \subseteq P$ is P-generic over $\mathfrak{M}$, and the sentence $\varphi(x_1, \ldots, x_n, G)$ is true in $\mathfrak{M}[G]$, then there is a condition $p \in G$ such that $p \Vdash \varphi(\mathring{x}_1, \ldots, \mathring{x}_n, \underline{G})$.* □

See [**Sho71**], Chapter 4 in [**Bar78**], or [**Kun83**] (a standard reference) for more on forcing.

[1] Sometimes a set is $\mathfrak{M}$-*generic* if P is fixed in the context.

A.4. Cohen forcing

Cohen forcing belongs to a wide class of forcing notions that "add a real".

Let us fix a countable transitive model $\mathfrak{M}$ of **ZFC**$^-$ and a Polish space $\mathbb{X}$ coded in $\mathfrak{M}$ so that a certain countable dense set $D_{\mathbb{X}} \subseteq \mathbb{X}$ belongs to $\mathfrak{M}$ along with the restricted distance function $d_X \restriction D_X$.

Definition A.4.1. The *Cohen forcing* $\mathbf{C}_{\mathbb{X}}$ consists of all open balls in $\mathbb{X}$ of the form $U_r(x) = \{y \in \mathbb{X} : d_{\mathbb{X}}(x,y) < r\}$, where $r \in \mathbb{Q}^+$ and $x \in D_{\mathbb{X}}$.

We order $\mathbf{C}_{\mathbb{X}}$ by inclusion, so that smaller balls are stronger conditions. Therefore, following the general definition in Section A.3, a set $D \subseteq \mathbf{C}_{\mathbb{X}}$ is:

dense: if for every condition $X \in \mathbf{C}_{\mathbb{X}}$ there is a condition $Y \in D$, $Y \subseteq X$;

open dense: if in addition $X \in D \Longrightarrow Y \in D$ whenever $X, Y \in \mathbf{C}_{\mathbb{X}}$, $Y \subseteq X$. □

Sets of the form $U_r(x)$ are, generally speaking, uncountable, and hence they cannot belong to $\mathfrak{M}$. However, the set $\mathbf{C}'_{\mathbb{X}} = \mathbb{Q}^+ \times D_{\mathbb{X}}$ with the order

$$\langle r, x\rangle \le \langle r', x'\rangle \quad \text{iff} \quad U_r(x) \subseteq U_{r'}(x')$$

does belong to $\mathfrak{M}$ under the assumptions of Definition A.4.1, and is similar (order isomorphic) to $\mathbf{C}_{\mathbb{X}}$. In this sense $\mathbf{C}_{\mathbb{X}}$ can be adequately considered as a forcing in $\mathfrak{M}$. Accordingly, a set $G \subseteq \mathbf{C}_{\mathbb{X}}$ is $\mathbf{C}_{\mathbb{X}}$*-generic*, or *Cohen-generic*, over $\mathfrak{M}$ iff it satisfies the following modified forms of 1), 2), and 3) of Section A.3:

1) if $X, Y \in G$, then there is $Z \in G$ such that $Z \subseteq X \cap Y$;

2) if $X \in G$, $Y \in \mathbf{C}_{\mathbb{X}}$, $Y \subseteq X$, then $Y \in G$;

3) if $D \subseteq \mathbf{C}_{\mathbb{X}}$ is dense and *coded in* $\mathfrak{M}$ in the sense that the corresponding set $D' = \{\langle r, x\rangle \in \mathbf{C}'_{\mathbb{X}} : U_r(x) \in D\}$ belongs to $\mathfrak{M}$, then $G \cap D \ne \varnothing$.

Theorem A.4.2. *If $\mathfrak{M}$, $\mathbb{X}$ are as above, then $\mathbf{C}_{\mathbb{X}}$ adds a real in the sense that if a set $G \subseteq \mathbf{C}_{\mathbb{X}}$ is Cohen-generic over $\mathfrak{M}$, then the intersection $\bigcap G$ of all sets $X \in G$ is a singleton, which will be denoted by a_G.*

Proof. The intersection A_G of all closed balls

$$\overline{U}_r(x) = \{y \in \mathbb{X} : d_{\mathbb{X}}(x,y) \le r\}, \quad \text{where} \quad U_r(x) \in G,$$

is a singleton, say, a_G, because the space is complete. Fix an arbitrary $X = U_{r'}(x') \in G$ and prove that $a_G \in X$. It is quite clear that the set

$$\mathscr{D}(X) = \{Y \in \mathbf{C}_{\mathbb{X}} : Y \cap X = \varnothing \vee \overline{Y} \subseteq X\}$$

is a dense subset of $\mathbf{C}_{\mathbb{X}}$ coded in $\mathfrak{M}$. Therefore, there is $Y \in G \cap \mathscr{D}(X)$. However $Y \cap X = \varnothing$ is impossible because $X \in G$, too. Therefore, $\overline{Y} \subseteq X$. But $a_G \in \overline{Y}$ by definition, as required. □

Elements of $\mathbb{X}$ of the form a_G, where $G \subseteq \mathbf{C}_{\mathbb{X}}$ is a Cohen-generic set over $\mathfrak{M}$, are called *Cohen-generic over* $\mathfrak{M}$, too.

The following characterization of Cohen generic points is well known:

Proposition A.4.3. *Suppose that $a \in \mathbb{X}$. Either of the two following conditions is necessary and sufficient for a to be Cohen-generic over $\mathfrak{M}$:*

(i) *the set $G(a) = \{X \in \mathbf{C}_{\mathbb{X}} : a \in X\}$ is Cohen-generic over $\mathfrak{M}$;*

(ii) *a belongs to every dense open set $B \subseteq \mathbb{X}$ coded in $\mathfrak{M}$.*

In addition if a set $G \subseteq \mathbf{C}_{\mathbb{X}}$ *is Cohen-generic over* $\mathfrak{M}$, *then* $\mathfrak{M}[G] = \mathfrak{M}[a_G]$ *and* $G = G(a_G)$. □

COROLLARY A.4.4. *The set of all points* $x \in \mathbb{X}$ *Cohen-generic over* $\mathfrak{M}$ *is a dense* $\mathbf{G}_\delta$ *subset of* $\mathbb{X}$ (since $\mathfrak{M}$ is countable). □

REMARK A.4.5. Some forcing theorems and definitions admit convenient modifications in terms of generic points in the case of Cohen forcing. In particular

> *the condition* $X \in \mathbf{C}_{\mathbb{X}}$ *forces* $\varphi(\mathring{x}_1, \dots, \mathring{x}_n, \underline{G})$ *if and only if* $\varphi(x_1, \dots, x_n, G(a))$ *is true in* $\mathfrak{M}[a]$ *for every Cohen-generic point* $a \in X$.

Theorem A.3.3 can be reformulated as follows:

> *If* $a \in \mathbb{X}$ *is Cohen-generic over* $\mathfrak{M}$ *and* $\varphi(x_1, \dots, x_n, G(a))$ *is true in* $\mathfrak{M}[a]$, *then there exists a condition* $X \in \mathbf{C}_{\mathbb{X}}$ *such that* $a \in X$ *and* $X \Vdash \varphi(\mathring{x}_1, \dots, \mathring{x}_n, \underline{G})$.

Theorem A.3.2 now says that a pair $\langle x, y\rangle \in \mathbb{X} \times \mathbb{Y}$ is Cohen-generic over $\mathfrak{M}$ iff x is Cohen-generic over $\mathfrak{M}$ and y is Cohen-generic over $\mathfrak{M}[x]$. □

REMARK A.4.6. Some Polish spaces $\mathbb{X}$ admit special, more convenient definitions of the Cohen forcing notion. For instance if $\mathbb{X} = \mathbb{N}^{\mathbb{N}}$, then $\mathbf{C}_{\mathbb{X}}$ can be identified with the set $\mathbb{N}^{<\omega}$ of all finite sequences of natural numbers, longer sequences being stronger conditions. And every sequence $s \in \mathbb{N}^{<\omega}$ produces the clopen ball $U_s = \{a \in \mathbb{N}^{\mathbb{N}} : s \subset a\}$. A similar definition works for $2^{\mathbb{N}}$.

As for the space $\mathscr{P}(\mathbb{N})$ (essentially the same as $2^{\mathbb{N}}$, see Section 3.1), we can define $\mathbf{C}_{\mathscr{P}(\mathbb{N})}$ to be equal to the set $\mathbb{C}$ of all pairs $\langle n, u\rangle$, where $n \in \mathbb{N}$ and $u \subseteq [0, n)$, with the order $\langle n, u\rangle \leq \langle m, v\rangle$ iff $m \leq n$ and $v = u \cap [0, m)$. The set $\mathbb{C}$ obviously belongs to every countable transitive model of $\mathbf{ZFC}^-$. To see that this is essentially the same as $\mathbf{C}_{\mathscr{P}(\mathbb{N})}$ in the sense of the general definition (Definition A.4.1) identify each condition $p = \langle n, u\rangle \in \mathbb{C}$ with the clopen set $\{a \in \mathscr{P}(\mathbb{N}) : a \cap [0, n) = u\}$.

And it is quite easy to prove that a set $a \in \mathscr{P}(\mathbb{N})$ is Cohen-generic over a transitive model $\mathfrak{M}$ of $\mathbf{ZFC}^-$ iff the set $G(a) = \{p = \langle n, u\rangle \in \mathbb{C} : a \cap [0, n) = u\}$ is $\mathbb{C}$-generic over $\mathfrak{M}$. □

A.5. Gandy-Harrington forcing

Here we discuss some special issues related to Gandy–Harrington forcing.

DEFINITION A.5.1. Fix a countable elementary submodel $M \subseteq \mathrm{H}_{\mathfrak{c}^+}$ and use the Mostowski collapse to get a countable transitive model $\mathfrak{M} \subseteq \mathrm{H}_{\mathfrak{c}^+}$ as in Remark A.1.3. Then $\mathfrak{M}$ is a model of $\mathbf{ZFC}^-$ and an elementary submodel of the universe of all sets *w.r.t.* all analytic formulas with parameters in $\mathfrak{M}$ by Theorem A.1.4. □

DEFINITION A.5.2. The set $\mathbb{P} = \mathbb{P}[\mathbb{N}^{\mathbb{N}}] = \{\varnothing \neq X \subseteq \mathbb{N}^{\mathbb{N}} : X \text{ is } \Sigma^1_1\}$ is the *Gandy–Harrington forcing* for the space $\mathbb{N}^{\mathbb{N}}$. Smaller (in the sense of $\subseteq$) sets in $\mathbb{P}$ are stronger conditions. □

Forcing notions $\mathbb{P}[2^{\mathbb{N}}]$, $\mathbb{P}[(2^{\mathbb{N}})^{\mathbb{N}}]$, *etc.*, can be defined similarly, and since all product spaces are Δ^1_1-isomophic those will be isomorphic forcing notions.

REMARK A.5.3. Obviously $\mathbb{P} \notin$ and $\not\subseteq \mathfrak{M}$, of course, but $\mathbb{P}$ can be adequately coded in $\mathfrak{M}$. For instance, if $X \in \mathbb{P}$, then $X \cap \mathfrak{M}$ belongs to $\mathfrak{M}$ and is a Σ^1_1 set in $\mathfrak{M}$. In fact, as $\mathfrak{M}$ is an elementary submodel of the universe *w.r.t.* all analytic formulas, the set $X \cap \mathfrak{M}$ can be defined in $\mathfrak{M}$ by the same Σ^1_1 formula as X itself is defined in the universe. Therefore, the set $\mathbb{P}' = \{X \cap \mathfrak{M} : X \in \mathbb{P}\}$ belongs to $\mathfrak{M}$. And by the same reasoning we have $X \subseteq Y$ iff $X \cap \mathfrak{M} \subseteq Y \cap \mathfrak{M}$ for all $X, Y \in \mathbb{P}$, and hence the $\subseteq$-ordering of $\mathbb{P}$ is similar to the $\subseteq$-ordering of $\mathbb{P}'$. □

Thus, $\mathbb{P}$ can be adequately treated as a forcing notion in $\mathfrak{M}$. Similar to Section A.4, a set $G \subseteq \mathbb{P}$ is *$\mathbb{P}$-generic over $\mathfrak{M}$* iff

1) for every pair of $X, Y \in G$ there exists $Z \in G$ such that $Z \subseteq X \cap Y$;
2) if $X \in G$ and $Y \in \mathbb{P}$, $X \subseteq Y$, then $Y \in G$; and
3) for every set $D \in \mathfrak{M}$, $D \subseteq \mathbb{P}$ dense in $\mathbb{P}$, and *coded in* $\mathfrak{M}$ in the sense that the corresponding set $D' = \{X \cap \mathfrak{M} : X \in D\}$ (then dense in $\mathbb{P}'$) belongs to $\mathfrak{M}$, the intersection $D \cap G$ is non-empty.

A set $D \subseteq \mathbb{P}$ is *dense* if we have $X \in D$ whenever $X \in \mathbb{P}$ and $X \subseteq Y \in D$.

Similar to Cohen forcing, the Gandy–Harrington forcing adds a real.

THEOREM A.5.4. *If $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over $\mathfrak{M}$, then $\bigcap G$ contains a single element of $\mathbb{N}^{\mathbb{N}}$, denoted by x_G. Every $A \in \mathbb{P}$ forces that $x_{\underline{G}}$ belongs to A.*

Elements of the form x_G, G as in the theorem, are called $\mathbb{P}$-*generic*, or *Gandy–Harrington generic* (over $\mathfrak{M}$).

PROOF. Let us return to the proof of Theorem 2.10.4. We defined there a collection $\mathscr{D}(P, s, t)$ of non-empty Σ^1_1 sets $X \subseteq \mathbb{N}^{\mathbb{N}}$ for each triple P, s, t that consists of a Π^0_1 set $P \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ and $s, t \in \mathbb{N}^{<\omega}$. We proved that all sets $\mathscr{D}(P, s, t)$ are dense subsets of $\mathbb{P} = \mathscr{F}$, and the (obviously countable) family of all collections $\mathscr{D}(P, s, t)$ witnesses the Polish-likeness of $\mathbb{P}$. In addition, the sets $\mathscr{D}(P, s, t)$ are obviously coded in $\mathfrak{M}$. Therefore, $G \cap \mathscr{D}(P, s, t) \neq \varnothing$ for all P, s, t as indicated.

An elementary forcing technique allows us now to obtain a $\subseteq$-decreasing sequence of Gandy–Harrington conditions $X_n \in G$ which

1) has a term X_n in common with every family of the form $\mathscr{D}(P, s, t)$ (n depends on P, s, t), and in addition
2) for every $X \in \mathbb{P}$ there exists n such that $X_n \subseteq X$ or $X_n \cap X = \varnothing$.

Then by the Polish-likeness and 1) $\bigcap_n X_n \neq \varnothing$, and hence $\bigcap G \neq \varnothing$. That the intersection cannot contain more than one element easily follows from 1), for take as X all sets of the form $X_s = \{a \in \mathbb{N}^{\mathbb{N}} : s \subset a\}$, where $s \in \mathbb{N}^{<\omega}$. □

Bibliography

[Add59a] J. W. Addison, *Separation principles in the hierarchies of classical and effective descriptive set theory*, Fund. Math. **46** (1959), 123–135 (English)

[Add59b] ———, *Some consequences of the axiom of constructibility*, Fund. Math. **46** (1959), 337–357 (English).

[AK00] Scot Adams and Alexander S. Kechris, *Linear algebraic groups and countable Borel equivalence relations*, J. Amer. Math. Soc. **13** (2000), no. 4, 909–943 (electronic). MR1775739 (2001g:03086)

[Bae37] Reinhold Baer, *Abelian groups without elements of finite order*, Duke Math. J. **3** (1937), no. 1, 68–122. MR1545974

[Bar78] Jon Barwise (ed.), *Handbook of mathematical logic. With the cooperation of H. J. Keisler, K. Kunen, Y. N. Moschovakis, A. S. Troelstra. 2nd printing*, Studies in Logic and the Foundations of Mathematics, Vol. 90, Amsterdam, New York, Oxford: North-Holland Publishing Company. XI, 1165 p., 1978 (English).

[BK96] Howard Becker and Alexander S. Kechris, *The descriptive set theory of Polish group actions*, Cambridge University Press, Cambridge, 1996. MR98d:54068

[CFW81] A. Connes, J. Feldman, and B. Weiss, *An amenable equivalence relation is generated by a single transformation*, Ergodic Theory Dynam. Systems **1** (1981), no. 4, 431–450 (1982). MR662736 (84h:46090)

[DH99] Randall Dougherty and Greg Hjorth, *Reducibility and nonreducibility between l^p equivalence relations*, Trans. Amer. Math. Soc. **351** (1999), no. 5, 1835–1844. MR2000d:03103

[DJK94] R. Dougherty, S. Jackson, and A. S. Kechris, *The structure of hyperfinite Borel equivalence relations*, Trans. Amer. Math. Soc. **341** (1994), no. 1, 193–225. MR94c:03066

[Dye63] H. A. Dye, *On groups of measure preserving transformations. I. II.*, Amer. J. Math. **81** and **85** (1959 and 1963), 119–159 and 551–576 (English).

[Far99] Ilijas Farah, *Ideals induced by Tsirelson submeasures*, Fund. Math. **159** (1999), no. 3, 243–258. MR2001b:03047

[Far00] ———, *Analytic quotients: theory of liftings for quotients over analytic ideals on the integers*, Mem. Amer. Math. Soc. **148** (2000), no. 702, xvi+177. MR2001c:03076

[Far01a] ———, *Basis problem for turbulent actions. I. Tsirelson submeasures*, Ann. Pure Appl. Logic **108** (2001), no. 1–3, 189–203. MR1819053 (2002c:03074)

[Far01b] ———, *Basis problem for turbulent actions. II. c_0-equalities*, Proc. London Math. Soc. (3) **82** (2001), no. 1, 1–30. MR1 794 255

[FM77] Jacob Feldman and Calvin C. Moore, *Ergodic equivalence relations, cohomology, and von Neumann algebras. I*, Trans. Amer. Math. Soc. **234** (1977), no. 2, 289–324. MR0578656 (58 #28261a)

[FR85] J. Feldman and A. Ramsay, *Countable sections for free actions of groups*, Adv. Math. **55** (1985), 224–227 (English).

[Fri00] Harvey M. Friedman, *Borel and Baire reducibility*, Fund. Math. **164** (2000), no. 1, 61–69. MR1 784 655

[Gab00] Damien Gaboriau, *Coût des relations d'équivalence et des groupes*, Invent. Math. **139** (2000), no. 1, 41–98. MR1728876 (2001f:28030)

[Gao98] Su Gao, *The isomorphism relation between countable models*, Ph.D. thesis, University of California, Los Angeles, 1998.

[Gao06] Su Gao, *Equivalence relations and classical Banach spaces*, Mathematical Logic in Asia. Proceedings of the 9th Asian Logic Conference. Novosibirsk, Russia, 16–19 August 2005, World Scientific, 2006, pp. 70–89 (English).

[GJ07] Su Gao and Steve Jackson, *Countable abelian group actions and hyperfinite equivalence relations*, Preprint, May, 2007.

[Hal74] Paul R. Halmos, *Measure theory. 2nd printing*, Graduate Texts in Mathematics. 18. New York - Heidelberg - Berlin: Springer-Verlag. XI, 304 p., 1974 (English).

[Hjo99] Greg Hjorth, *Orbit cardinals: on the effective cardinalities arising as quotient spaces of the form X/G where G acts on a Polish space X*, Israel J. Math. **111** (1999), 221–261. MR2000h:03091

[Hjo00a] G. Hjorth, *Actions by the classical Banach spaces*, J. Symbolic Logic **65** (2000), no. 1, 392–420. MR1 782 128

[Hjo00b] Greg Hjorth, *Classification and orbit equivalence relations*, Math. Surveys Monogr., 75, American Mathematical Society, Providence, RI, 2000. MR2000k:03097

[Hjo05] ———, *A converse to Dye's theorem*, Trans. Amer. Math. Soc. **357** (2005), no. 8, 3083–3103 (electronic). MR2135736 (2005m:03093)

[HK96] Greg Hjorth and Alexander S. Kechris, *Borel equivalence relations and classifications of countable models*, Ann. Pure Appl. Logic **82** (1996), no. 3, 221–272. MR99m:03073

[HK97] ———, *New dichotomies for Borel equivalence relations*, Bull. Symbolic Logic **3** (1997), no. 3, 329–346. MR98m:03101

[HK01] ———, *Recent developments in the theory of Borel reducibility*, Fund. Math. **170** (2001), no. 1–2, 21–52. MR1881047 (2002k:03076)

[HK05] ———, *Rigidity theorems for actions of product groups and countable Borel equivalence relations*, Mem. Amer. Math. Soc. **177** (2005), no. 833, viii+109. MR2155451 (2006f:03078)

[HKL90] L. A. Harrington, A. S. Kechris, and A. Louveau, *A Glimm-Effros dichotomy for Borel equivalence relations*, J. Amer. Math. Soc. **3** (1990), no. 4, 903–928. MR91h:28023

[HKL98] Greg Hjorth, Alexander S. Kechris, and Alain Louveau, *Borel equivalence relations induced by actions of the symmetric group*, Ann. Pure Appl. Logic **92** (1998), no. 1, 63–112. MR2000a:03081

[Jec71] Thomas J. Jech, *Lectures in set theory with particular emphasis on the method of forcing*, Lecture Notes in Mathematics. 217. Berlin-Heidelberg-New York: Springer-Verlag. 137 p., 1971 (English).

[JK84] Winfried Just and Adam Krawczyk, *On certain Boolean algebras $P(\omega)/I$*, Trans. Amer. Math. Soc. **285** (1984), no. 1, 411–429. MR86f:04003

[JKL02] S. Jackson, A. S. Kechris, and A. Louveau, *Countable Borel equivalence relations*, J. Math. Log. **2** (2002), no. 1, 1–80. MR1900547 (2003f:03066)

[JN76] S.-A. Jalali-Naini, *The monotone subsets of Cantor space, filters, and descriptive set theory*, Ph.D. thesis, Oxford, 1976.

[Jus90a] Winfried Just, *More mutually irreducible ideals*, Preprint dated 20.07.1990, 1990.

[Jus90b] ———, *Mutually irreducible ideals*, Preprint dated 27.04.1990, 1990.

[Kan96] Vladimir Kanovei, *Topologies generated by effectively Suslin sets and their applications in descriptive set theory*, Uspekhi Mat. Nauk **51** (1996), no. 3(309), 17–52, (English transl. in *Russian Math. Surveys* 51 (1996), no. 3, 385–417). MR97g:54051

[Kan98] ———, *Ulm classification of analytic equivalence relations in generic universes*, MLQ Math. Log. Q. **44** (1998), no. 3, 287–303. MR2000a:03082

[Kan99] ———, *On non-wellfounded iterations of the perfect set forcing*, J. Symbolic Logic **64** (1999), no. 2, 551–574. MR1 777 770

[Kec91] Alexander S. Kechris, *Amenable equivalence relations and Turing degrees*, J. Symbolic Logic **56** (1991), no. 1, 182–194. MR1131739 (93c:03061)

[Kec92] ———, *Countable sections for locally compact group actions*, Ergodic Theory Dyn. Syst. **12** (1992), no. 2, 283–295 (English).

[Kec93] ———, *Amenable versus hyperfinite Borel equivalence relations*, J. Symbolic Logic **58** (1993), no. 3, 894–907. MR1242044 (95f:03081)

[Kec95] ———, *Classical descriptive set theory*, Graduate Texts in Mathematics, 156, Springer-Verlag, New York, 1995. MR96e:03057

[Kec98] ———, *Rigidity properties of Borel ideals on the integers*, Topology Appl. **85** (1998), no. 1–3, 195–205, 8th Prague Topological Symposium on General Topology and Its Relations to Modern Analysis and Algebra (1996). MR99k:03040

[Kec99] ———, *New directions in descriptive set theory*, Bull. Symbolic Logic **5** (1999), no. 2, 161–174. MR1 791 302

[Kec02] ———, *Actions of Polish groups and classification problems*, Analysis and Logic (Mons, 1997), London Math. Soc. Lecture Note Ser., vol. 262, Cambridge Univ. Press, Cambridge, 2002, pp. 115–187. MR1967835 (2004b:03070)

[Kec07] ———, *Set theory and dynamical systems*, Proceedings of the 13th International Congress of Logic, Methodology and Philosophy of Science, Beijing 2007, to appear (English).

[KL97] Alexander S. Kechris and Alain Louveau, *The classification of hypersmooth Borel equivalence relations*, J. Amer. Math. Soc. **10** (1997), no. 1, 215–242. MR97e:03067

[KL04] V. G. Kanoveĭ and V. A. Lyubetskiĭ, *On the set of constructive real numbers*, Tr. Mat. Inst. Steklova **247** (2004), no. Geom. Topol. i Teor. Mnozh., 95–128. MR2168165 (2006h:03047)

[KL06] ———, *A cofinal family of equivalence relations and the Borel ideals generating them*, Tr. Mat. Inst. Steklova **252** (2006), no. Geom. Topol., Diskret. Geom. i Teor. Mnozh., 94–113. MR2255972 (2007f:37004)

[KM04] Alexander S. Kechris and Benjamin D. Miller, *Topics in orbit equivalence*, Lecture Notes in Mathematics, vol. 1852, Springer-Verlag, Berlin, 2004. MR2095154

[KR03] Vladimir Kanovei and Michael Reeken, *Some new results on the Borel irreducibility of equivalence relations*, Izv. Ross. Akad. Nauk Ser. Mat. **67** (2003), no. 1, 59–82, (English translation in *Izv. Math.* 67 (2003), no. 1, 55–76.). MR1957916 (2004c:03060)

[KR04] ———, *Borel irreducibility between two large families of Borel equivalence relations*, Logic Colloquium '99, Lect. Notes Log., vol. 17, Assoc. Symbol. Logic, Urbana, IL, 2004, pp. 100–110. MR2088768

[KS05] Jakob Kellner and Saharon Shelah, *Saccharinity*, 2005.

[Kun83] Kenneth Kunen, *Set theory. An introduction to independence proofs. 2nd print*, Studies in Logic and the Foundations of Mathematics, 102. Amsterdam-New York-Oxford, North-Holland. XVI, 313 p., 1983 (English).

[Kur66] K. Kuratowski, *Topology. Vol. I*, New edition, revised and augmented. Translated from the French by J. Jaworowski, Academic Press, New York, 1966. MR0217751 (36 #840)

[Leb05] Henry Lebesgue, *Sur les fonctions représentable analytiquement*, Journ. de Math. (Sér. 6) **1** (1905), 139–216 (French).

[Lou80] Alain Louveau, *A separation theorem for Σ^1_1 sets*, Trans. Amer. Math. Soc. **260** (1980), 363–378 (English).

[Lou94] ———, *On the reducibility order between Borel equivalence relations*, Logic, Methodology and Philosophy of Science, IX (Uppsala, 1991), Stud. Logic Found. Math., vol. 134, North-Holland, Amsterdam, 1994, pp. 151–155. MR1327979 (96g:03081)

[LR05] Alain Louveau and Christian Rosendal, *Complete analytic equivalence relations*, Trans. Amer. Math. Soc. **357** (2005), no. 12, 4839–4866 (English).

[LS18] N. Lusin and W. Sierpinski, *Sur quelques proprietes des ensembles mesurables (A)*, Krak. Anz. **1918** (1918), 35–48 (French).

[Lus72] Nicolas Lusin, *Lecons sur les ensembles analytiques et leurs applications. 2nd corrected ed.*, New York, N.Y.: Chelsea Publishing Company. XIV, 326 p., 1972 (French).

[LV94] Alain Louveau and Boban Veličković, *A note on Borel equivalence relations*, Proc. Amer. Math. Soc. **120** (1994), no. 1, 255–259. MR94f:54076

[Mat72] A. R. D. Mathias, *Solution of problems of Choquet and Puritz*, Conference in Mathematical Logic—London '70 (Bedford Coll., London, 1970), Springer, Berlin, 1972, pp. 204–210. Lecture Notes in Math., Vol. 255. MR51 #166

[Mat75] ———, *A remark on rare filters*, Infinite and Finite Sets (Colloq., Keszthely, 1973; dedicated to P. Erdös on his 60th birthday), Vol. III, North-Holland, Amsterdam, 1975, pp. 1095–1097. Colloq. Math. Soc. János Bolyai, Vol. 10. MR51 #10098

[Maz91] Krzysztof Mazur, *F_σ-ideals and $\omega_1\omega_1^*$-gaps in the Boolean algebras $P(\omega)/I$*, Fund. Math. **138** (1991), no. 2, 103–111. MR92g:06019

[Mil95] Arnold W. Miller, *Descriptive set theory and forcing*, Springer-Verlag, Berlin, 1995, How to Prove Theorems about Borel Sets the Hard Way. MR98g:03119

[Mos80] Yiannis N. Moschovakis, *Descriptive set theory*, Studies in Logic and the Foundations of Mathematics, Vol. 100. Amsterdam, New York, Oxford: North-Holland Publishing Company. XII, 637 p., 1980 (English).

[MW85] Richard Mansfield and Galen Weitkamp, *Recursive aspects of descriptive set theory*, The Clarendon Press, Oxford University Press, New York, 1985, With a chapter by Stephen Simpson. MR86g:03003

[Oli03] Michael R. Oliver, *An inquiry into the number of isomorphism classes of Boolean algebras and the Borel cardinality of certain Borel equivalence relations*, Ph.D. thesis, University of California, Los Angeles, 2003.

[Ros05] Christian Rosendal, *Cofinal families of Borel equivalence relations and quasiorders*, J. Symbolic Logic **70** (2005), no. 4, 1325–1340.

[Sco64] Dana Scott, *Invariant Borel sets*, Fund. Math. **56** (1964), 117–128. MR0172809 (30 #3027)

[Sho62] J. R. Shoenfield, *The problem of predicativity*, Essays Found. Math., dedicat. to A. A. Fraenkel on his 70th Anniv. (1962), 132–139.

[Sho71] ———, *Unramified forcing*, Axiomatic Set Theory, Proc. Sympos. Pure Math. 13, Part I, (1971), 357–381.

[Sho01] Joseph R. Shoenfield, *Mathematical logic. Reprint of the 1967 original*, Natick, MA: Association for Symbolic Logic. vii, 344 p., 2001 (English).

[Sie18] Waclaw Sierpiński, *L'axiome de M. Zermelo et son rôle dans la théorie des ensembles et l'analyse*, Bull. Acad. Sci. Cracovie (1918), 97–152.

[Sil80] Jack H. Silver, *Counting the number of equivalence classes of Borel and coanalytic equivalence relations*, Ann. Math. Logic **18** (1980), no. 1, 1–28. MR81d:03051

[Sol70] R. M. Solovay, *A model of set-theory in which every set of reals is Lebesgue measurable*, Ann. Math. (2) **92** (1970), 1–56 (English).

[Sol96] Sławomir Solecki, *Analytic ideals*, Bull. Symbolic Logic **2** (1996), no. 3, 339–348. MR97i:04002

[Sol99] ———, *Analytic ideals and their applications*, Ann. Pure Appl. Logic **99** (1999), no. 1–3, 51–72. MR2000g:03112

[Sol00] ———, *Actions of non-compact and non-locally compact Polish groups*, J. Symbolic Logic **65** (2000), no. 4, 1881–1894 (English).

[Sri81] S. M. Srivastava, *Selection and representation theorems for σ-compact valued multifunctions*, Proc. Amer. Math. Soc. **83** (1981), no. 4, 775–780. MR82m:04002

[SS88] T. Slaman and J. Steel, *Definable functions on degrees*, Cabal Seminar 81–85, Springer, Berlin, 1988, pp. 37–55. Lecture Notes in Math., Vol. 1333.

[Tal80] Michel Talagrand, *Compacts de fonctions mesurables et filtres non mesurables*, Studia Math. **67** (1980), no. 1, 13–43. MR82e:28009

[Tho02] Simon Thomas, *Some applications of superrigidity to Borel equivalence relations*, Set theory (Piscataway, NJ, 1999), DIMACS Ser. Discrete Math. Theoret. Comput. Sci., vol. 58, Amer. Math. Soc., Providence, RI, 2002, pp. 129–134. MR1903855 (2003j:03061)

[Tho03a] ———, *The classification problem for torsion-free abelian groups of finite rank*, J. Amer. Math. Soc. **16** (2003), no. 1, 233–258 (electronic). MR1937205 (2004b:20081)

[Tho03b] ———, *Superrigidity and countable Borel equivalence relations*, Ann. Pure Appl. Logic **120** (2003), no. 1–3, 237–262. MR1949709 (2004e:03081)

[Tho06] Alex Thompson, *A metamathematical condition equivalent to the existence of a complete left invariant metric for a Polish group*, J. Symbolic Logic **71** (2006), no. 4, 1108–1124 (English).

[Tho07] Simon Thomas, *Popa superrigidity and countable Borel equivalence relations*, to appear in Ann. Pure Appl. Logic.

[Vel99] Boban Veličković, *A note on Tsirelson type ideals*, Fund. Math. **159** (1999), no. 3, 259–268. MR2000f:03142

[Wag93] Stan Wagon, *The Banach-Tarski paradox*, Cambridge University Press, Cambridge, 1993, With a foreword by Jan Mycielski, Corrected reprint of the 1985 original. MR1251963 (94g:04005)

[Zap04] Jindrich Zapletal, *Descriptive set theory and definable forcing*, Mem. Amer. Math. Soc. **167** (2004), no. 793, viii + 141 pp. MR2023448 (2005e:03107)

[Zim84] Robert J. Zimmer, *Ergodic theory and semisimple groups*, Monogr. Math., vol. 81, Birkhäuser Verlag, Basel, 1984. MR776417 (86j:22014)

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