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Higher order homogenization for random non-autonomous parabolic operators

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Abstract

We consider Cauchy problem for a divergence form second order parabolic operator with rapidly oscillating coefficients that are periodic in spatial variables and random stationary ergodic in time. As was proved in Zhikov et al. (Mat Obshch 45:182–236, 1982) and Kleptsyna and Piatnitski (Homogenization and applications to material sciences. GAKUTO Internat Ser Math Sci Appl vol 9, pp 241–255. Gakkōtosho, Tokyo, 1995) in this case the homogenized operator is deterministic. The paper focuses on the diffusion approximation of solutions in the case of non-diffusive scaling, when the oscillation in spatial variables is faster than that in temporal variable. Our goal is to study the asymptotic behaviour of the normalized difference between solutions of the original and the homogenized problems.

Keywords Homogenization · Diffusion approximation · Random operator · Asymptotic expansion

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1 Introduction

In this work we consider the asymptotic behaviour of solutions to the following Cauchy problem

$$\begin{aligned} \frac{\partial}{\partial t} u^\varepsilon &= \operatorname{div}\Big[a\Big(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha}\Big)\nabla u^\varepsilon\Big] \qquad \text{in } \mathbb{R}^d \times (0, T] \\ u^\varepsilon(x, 0) &= \varphi(x). \end{aligned} \tag{1}$$

Here ε is a small positive parameter that tends to zero, $\alpha \in (0, 2)$, $a(z, s)$ is a positive definite matrix whose entries are periodic in z variable and random stationary ergodic in s.

It was shown in [29] and [17] that this problem admits homogenization and that the homogenized operator is deterministic and has constant coefficients both in space and in time. The homogenized Cauchy problem takes the form

$$\begin{aligned} \frac{\partial}{\partial t} u^0 &= \operatorname{div}(a^{\text{eff}} \nabla u^0) \\ u^0(x, 0) &= \varphi(x). \end{aligned} \tag{2}$$

The formula for the effective matrix a^{eff} is given in (5) in Sect. 2 (see also [17]).

The goal of this paper is to study the limit behaviour of the difference $u^\varepsilon - u^0$, as $\varepsilon \to 0$.

In applications problem (1) describes for instance various processes in porous media in the presence of gas bubble formations. The gas bubble formations might essentially affect the characteristics of the media. The artificial porous materials used in physical experiments and in the industry often have a periodic microstructure, while the gas bubble formations and migration are the processes which are random in time, see [24] for further details.

Also, when studying the diffusion in porous media whose characteristics might depend on the atmosphere pressure, humidity, etc. we often face the model problem in (1).

The first rigorous homogenization results for second order elliptic operators with random coefficients were obtained in pioneer works [21] and [25]. The approach developed in [21] and [25] for random divergence form elliptic operators also applies to the corresponding parabolic problems and yields the semigroup convergence.

In the existing literature there is a number of works devoted to homogenization of random parabolic equations with time dependent coefficients.

In the presence of large lower order terms the limit dynamics might remain random and show diffusive or even more complicated behaviour. The papers [6, 20, 26] focus on time dependent parabolic operators with large lower order terms in the case of periodic in spatial variables and random in time coefficients. Under the natural assumptions the limit (effective) dynamics is described by a SPDE with a multiplicative noise. The fully random case has been studied in [4, 5, 14, 27]. Here the structure of the limit operator might depend on the dimension.

One of the important aspects of homogenization theory is estimating the rate of convergence. For random operators the first estimates have been obtained in [16]. An important progress in this direction was achieved in the works [12, 13]. Further references and a complete presentation of the theory that allows to obtain sharp estimates in stochastic homogenization can be found in the recent book [2]. The pathwise structure of fluctuations in stochastic homogenization has been studied in [8].

The paper [1] deals with stochastic homogenization of divergence form parabolic equations with time dependent coefficients. The authors consider the diffusive scaling and obtain optimal estimates for the rate of convergence.

Problem (1) in the case of diffusive scaling $\alpha = 2$ was studied in our previous work [18]. It was shown that, under proper mixing conditions, the difference $u^\varepsilon - u^0$ is of order ε, and that the normalized difference $\varepsilon^{-1}(u^\varepsilon - u^0)$ after subtracting an appropriate corrector, converges in law to a solution of some limit SPDE.

In the present paper we consider the case $0 < \alpha < 2$. In other words, bearing in mind the diffusive scaling, we assume that the oscillation in spatial variables is faster than that in time.

In this case, due to the disagreement with the diffusive scaling, the principal part of the asymptotic expansion of $u^\varepsilon - u^0$ consists of a finite number of correctors, the oscillating part of each of them being a solution of an elliptic PDE with periodic in spatial variable coefficients. The number of correctors increases as α approaches 2. After subtracting these correctors, the resulting expression divided by $\varepsilon^{\alpha/2}$ converges in law to a solution of the limit SPDE. See Sect. 2.4 for further details.

Previously, higher order correctors were used in stochastic homogenization of elliptic operators for rather different purposes, see works [10] and [9]. In [10] the higher order correctors are introduced in the context of the large-scale regularity. In [9] the authors construct higher-order two-scale expansion and obtain improved large-scale estimates using annealed Calderon–Zygmund estimates for elliptic operators with random coefficients.

In contrast with the diffusive scaling, for $\alpha < 2$ the interplay between the scalings in spatial variables and time and the necessity to construct higher order correctors results in additional regularity assumptions on the coefficients. Indeed, each corrector is introduced as a solution of some elliptic equation in which time is a parameter, thus this corrector has the same regularity in time as the coefficients of the equation. When we construct the higher-order terms of the expansion, at each step the corrector obtained at the previous step is differentiated in time, which reduces the regularity. The result mentioned in the previous paragraph holds if the coefficients $a^{ij}(z, s)$ in (1) are sufficiently smooth in temporal variable.

We also consider in the paper the special case of diffusive dependence on time. Namely, we assume in this case that $a(z, s) = \mathrm{a}(z, \xi_s)$, where $\xi_\cdot$ is a stationary diffusion process in $\mathbb{R}^n$ and $\mathrm{a}(z, y)$ is a smooth deterministic function that is periodic in z. It should be emphasized that in the said diffusive case Theorem 1 does not apply because the coefficients a^{ij} do not possess the required regularity in time. This lack of regularity leads to additional difficulties in treating the diffusive case. As was shown in our previous work [19], the statement of Theorem 1 remains valid if $\alpha < 1$. Also, for $1 \leq \alpha < 2$ in dimension one the issues can be addressed using the equation satisfied by the potential of the discrepancy. This technique fails to work in dimension higher than 1. Here we treat the case $\alpha = 1$ in any dimension.

The paper is organized as follows.

- In Sect. 2 we introduce the problem and provide all the assumptions. Then we formulate the main result (Theorem 1) of the paper concerning the smooth case for $\alpha < 2$.

  Sect. 2.4 focuses on the proof of Theorem 1. At the beginning we consider a number of auxiliary problems and define the higher order terms of the asymptotics of solution.
- In Sect. 3 we consider the special case of diffusive dependence on time for $\alpha \le 1$. We extend to the dimension d the result of [19] in Theorem 2.

2 The smooth case

In this section we provide all the assumptions on the data of problem (1), introduce some notations and formulate the main result.

For the Cauchy problem (1), where ε is a small positive parameter, we assume that the following conditions hold true:

a1. the matrix $a(z,s) = \{a^{ij}(z,s)\}_{i,j=1}^d$ is symmetric and satisfies uniform ellipticity condition: there exists $\lambda > 0$ such that for any $(z,s) \in \mathbb{R}^d \times \mathbb{R}$ and any $\zeta \in \mathbb{R}^d$:

$$\lambda|\zeta|^2 \le a(z,s)\zeta\cdot\zeta \le \lambda^{-1}|\zeta|^2 .$$

a2. $\varphi \in C_0^\infty(\mathbb{R}^d)$. In fact, this condition can be essentially relaxed, see Remark 3.

In the first setting it is assumed that the coefficients of matrix a are smooth functions that have good mixing properties in time variable. The smoothness is important because our approach relies on auxiliary elliptic equations that depend on time as a parameter, and we have to differentiate these equations w.r.t. time.

In the case of smooth coefficients our assumptions read:

h1. The coefficients $a^{ij}(z,s)$ are periodic in z with the period $[0,1]^d$ and random stationary ergodic in s. Given a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with an ergodic measure-preserving dynamical system τ_s, we assume that $a^{ij}(z,s,\omega) = \mathrm{a}^{ij}(z,\tau_s\omega)$, where $\{\mathrm{a}^{ij}(z,\omega)\}_{i,j=1}^d$ is a collection of random periodic in z functions that satisfy the above uniform ellipticity condition.

h2. Realizations $a^{ij}(z,s)$ are almost surely elements of $C^\infty\big([0,+\infty); C^{1,\beta}(\mathbb{T}^d)\big)$ for some $\beta > 0$, and for any $N \geqslant 0$ and $k \geqslant 1$ there exist $C_{k,N}$ such that

$$\mathbf{E}\,\|\partial_s^N a^{ij}\|^k_{C^0([0,\infty);C^{1,\beta}(\mathbb{T}^d))} \le C_{k,N}$$

(see Remark 3).

Here and in what follows we identify periodic functions with functions on the torus $\mathbb{T}^d$, $\mathbf{E}$ stands for the expectation.

Our last condition concerns the decay of the strong mixing coefficient of $a(\cdot)$. We recall here the definition of the strong mixing coefficient. Let $\mathcal{F}_{\le s}$ and $\mathcal{F}_{\ge s}$ be the σ-algebras generated by $\{a(z,t)\,:\, z\in\mathbb{T}^d, t\le s\}$ and $\{a(z,t)\,:\, z\in\mathbb{T}^d, t\ge s\}$, respectively. We set

$$\gamma(r)=\sup\left|\mathbf{P}(A\cap B)-\mathbf{P}(A)\mathbf{P}(B)\right|,$$

where the supremum is taken over all $A\in\mathcal{F}_{\le 0}$ and $B\in\mathcal{F}_{\ge r}$. See among others [15, Chapter 8.3.101] or [23, Chapter 5.8] for more details or different mixing conditions.

h3. Mixing condition. The strong mixing coefficient $\gamma(r)$ of $a(z,\cdot)$ satisfies the inequality

$$\int_0^\infty (\gamma(r))^{1/2}dr<\infty.$$

We say that Condition **(H)** holds if **a1**, **a2** and **h1** – **h3** are fulfilled.

Remark 1 In the proof of Theorem 1 formulated below we deal with stationary ergodic processes taking on values in finite dimensional spaces. For such processes the law of large number holds, however, the functional central limit theorem (invariance principle) need not hold without additional assumptions. One of the typical conditions that ensure the applicability of the CLT is condition **h3.** For an ergodic process the σ-algebras generated by the process evaluated at times t and $t+s$ are getting less and less dependent as s is growing. The strong mixing coefficient is one of the characteristics that quantify this dependence, and condition **h3.** implies that this coefficient decay fast enough so that the invariance principle holds.

2.1 Homogenized problem and first corrector

According to [17], under **(H)**, the sequence u^ε converges in probability, as $\varepsilon\to 0$, to a solution u^0 of problem (2). For the reader convenience we provide here the definition of the effective matrix $a^{\rm eff}$. We solve the following auxiliary problem

$$\operatorname{div}\big(a(z,s,\omega)\nabla\chi^0(z,s,\omega)\big)=-\operatorname{div}a(z,s,\omega),\quad z\in\mathbb{T}^d; \tag{3}$$

here s and ω are parameters, and χ^0 is an unknown vector function: $\chi^0=(\chi^{0,1},\dots,\chi^{0,d})$. In what follows *we usually do not indicate explicitly the dependence of* ω. Due to ellipticity of the matrix a Eq. (3) has a unique, up to an additive constant vector, periodic solution, $\chi^0\in\big(L^\infty(\mathbb{T}^d)\cap H^1(\mathbb{T}^d)\big)^d$. This constant vector is chosen in such a way that

$$\int_{\mathbb{T}^d}\chi^0(z,s)\,dz=0\qquad\text{for all } s \text{ and } \omega. \tag{4}$$

Then we define the effective matrix a^{eff} by

$$a^{\text{eff}} = \mathbf{E} \int_{\mathbb{T}^d} a(z,s)\big(\mathbf{I} + \nabla \chi^0(z,s)\big)\, dz, \tag{5}$$

where $\mathbf{I}$ stands for the unit matrix, and $\{\nabla \chi^0(z,s)\}^{ij} = \frac{\partial}{\partial z_i} \chi^{0,j}$.

It is known that the matrix a^{eff} is positive definite (see, for instance, [17]). Therefore, problem (2) is well posed, and function u^0 is uniquely defined. Under assumption **a2** the function u^0 is smooth and satisfies the estimates

$$\left| (1+|x|)^N \frac{\partial^{\mathbf{k}} u^0(x,t)}{\partial t^{k_0} \partial x_1^{k_1} \ldots \partial x_d^{k_d}} \right| \le C_{N,\mathbf{k}} \tag{6}$$

for all $N > 0$ and all multi index $\mathbf{k} = (k_0, k_1, \ldots, k_d)$, $k_i \ge 0$.

2.2 Main result for smooth coefficients with good mixing properties

Here we assume that condition **(H)** holds. In order to formulate the main results we need a number of auxiliary functions and quantities. For $j = 1, 2, \ldots, J^0$ with $J^0 = \lfloor \frac{\alpha}{2(2-\alpha)} \rfloor + 1$, the higher order correctors are introduced as periodic solutions to the equations

$$\operatorname{div}\big(a(z,s)\nabla \chi^j(z,s)\big) = \partial_s \chi^{j-1}(z,s), \tag{7}$$

where $\lfloor \cdot \rfloor$ stands for the integer part. Due to (4) for $j = 1$ this equation is solvable in the space of periodic in z functions. A solution χ^1 is uniquely defined up to an additive constant vector. Choosing the constant vector in a proper way yields

$$\int_{\mathbb{T}^d} \chi^1(z,s)\, dz = 0 \qquad \text{for all } s \text{ and } \omega$$

and thus the solvability of the equation for χ^2. Iterating this procedure, we define all χ^j, $j = 1, 2, \ldots, J^0$.

Next, we introduce the functions $u^j = u^j(x,t)$, $j = 1, \ldots, J^0$. They solve the following problems:

$$\frac{\partial}{\partial t} u^j = \operatorname{div}(a^{\text{eff}} \nabla u^j) + \sum_{k=1}^{j} \operatorname{div}(a^{k,\text{eff}} \nabla u^{j-k}), \quad u^j(x,0) = 0 \tag{8}$$

with

$$a^{k,\text{eff}} = \mathbf{E} \int_{\mathbb{T}^d} a(z,s) \nabla \chi^k(z,s)\, dz. \tag{9}$$

To characterize the diffusive term in the limit equation we introduce the matrix

$$\varXi(s) = \int_{\mathbb{T}^d} \Big[\big(a(z,s) + a(z,s)\nabla\chi^0(z,s) \big) - \mathbf{E}\big\{ a(z,s) + a(z,s)\nabla\chi^0(z,s) \big\} \Big] dz \tag{10}$$

By construction the matrix function $\varXi$ is stationary and its entries satisfy condition **h3** (mixing condition). Denote

$$\varLambda = \int_0^\infty \mathbf{E}\Big(\varXi(s) \otimes \varXi(0) + \varXi(0) \otimes \varXi(s) \Big) ds, \qquad \varLambda = \{\varLambda^{ijkl}\}, \tag{11}$$

where $(\varXi(s) \otimes \varXi(0))^{ijkl} = \varXi^{ij}(s)\varXi^{kl}(0)$ (see [15, Theorem VIII.5.56] or [23, Chapter 9, Theorem 2]). Under condition **h3** the integral on the right-hand side converges. Indeed we can use [15, Lemma VIII.3.102] to obtain that coordinate by coordinate

$$\begin{aligned} \Big| \mathbf{E}\Big(\varXi(s) \otimes \varXi(0) \Big) \Big| &\le \Big[\mathbf{E}\Big(\varXi(0)^2 \Big) \Big]^{1/2} \Big[\mathbf{E}\Big(\mathbf{E}\big[\varXi(s) \big| \mathcal{F}_0 \big] \Big)^2 \Big]^{1/2} \\ &\le 4 \Big[\mathbf{E}\Big(\varXi(0)^2 \Big) \Big]^{1/2} \gamma(r)^{1/2} \| a(\mathbf{I} + \nabla\chi^0) \|_{\mathbb{L}^\infty(\mathbb{T}^d \times [0,+\infty))}. \end{aligned}$$

Since $\gamma^{1/2}$ is integrable, the claim follows.

The first main result of this paper is

Theorem 1 *Let Condition* (**H**) *be fulfilled. Then the functions*

$$U^\varepsilon = \varepsilon^{-\alpha/2} \Big(u^\varepsilon - u^0 - \sum_{j=1}^{J_0} \varepsilon^{j(2-\alpha)} u^j \Big)$$

converge in law, as $\varepsilon \to 0$, in $L^2(\mathbb{R}^d \times (0,T))$ to the unique solution of the following SPDE

$$\begin{aligned} dv^0 &= \operatorname{div}(a^{\text{eff}} \nabla v^0)\, dt + (\varLambda^{1/2})^{ijkl} \frac{\partial^2}{\partial x_i \partial x_j} u^0\, dW_t^{kl} \\ v^0(x,0) &= 0; \end{aligned} \tag{12}$$

where $W_\cdot = \{W_\cdot^{ij}\}$ is the standard d^2-dimensional Brownian motion.

Remark 2 According to [7, Theorem 5.4] Eq. (12) is well-posed and has exactly one weak solution. The definition of a solution to problem (12) can also be found in [7].

Remark 3 The regularity assumption on φ given in condition **a2** can be weakened. Namely, the statement of Theorem 1 holds if φ is $J^0 + 1$ times continuously differentiable and the corresponding partial derivatives decay at infinity sufficiently fast. The regularity assumptions imposed on $a(\cdot)$ in condition **h2** can also be relaxed. Our result holds if a is $J^0 + 1$ times continuously differentiable w.r.t. the time variable as a function from $(-\infty, +\infty)$ to $C^{1,\beta}(\mathbb{T}^d)$.

The scheme of the proof is the following. We write down the following ansatz

$$V^\varepsilon(x,t) = \varepsilon^{-\frac{\alpha}{2}} \Bigg\{ u^\varepsilon(x,t) - \sum_{k=0}^{J^0} \varepsilon^{k\delta} \Big[u^k(x,t) + \sum_{j=0}^{J^0-k} \varepsilon^{(j\delta+1)} \chi^j\Big(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha}\Big) \cdot \nabla u^k(x,t) \Big] \Bigg\},$$

here and in what follows the symbol $\delta = \delta_\alpha$ stands for $2-\alpha$. Then we substitute V^ε for u^ε in (1) and we obtain for V^ε a PDE with random coefficients when **(H)** is in force. We prove that V^ε converges in law in the suitable functional space to the solution of (12). We combine the definition of correctors, formula (8) and the Cental Limit Theorem for stationary mixing processes. After some manipulations this yields the desired convergence (see Sect. 2.4). The rest of this section concerns the proof of this result.

2.3 Heuristic scheme

As $\varepsilon \to 0$, the *random* solution u^ε converges to the *deterministic* limit u^0. Since we study the asymptotic behaviour of the difference $u^\varepsilon - u^0$, we need to construct higher order terms of the asymptotic expansion of u^ε. If we try to follow the same scheme as in the case $\alpha = 2$ (see [18]), then, letting $t = \frac{s}{\varepsilon^2}$, $z = \frac{x}{\varepsilon}$, for the first corrector $\chi(z,s)$ we obtain the equation

$$\partial_s \chi(z,s) - \operatorname{div}\big(a(z,\varepsilon^\delta s)\nabla\chi(\xi,s)\big) = -\operatorname{div} a(z,\varepsilon^\delta s), \quad (z,s) \in \mathbb{T}^d \times (-\infty,+\infty),$$

which, in contrast with the case $\alpha = 2$, depends on ε. It was shown in [18] that, for each $\varepsilon > 0$, this equation has a unique up to an additive constant stationary solution. Moreover, it is not difficult to see that, at least for continuous in time coefficients, the solution of this equation is close to that of the elliptic equation

$$\operatorname{div}\big(a(z,\varepsilon^\delta s)\nabla\chi^0(\xi,\varepsilon^\delta s)\big) = -\operatorname{div} a(z,\varepsilon^\delta s), \quad z \in \mathbb{T}^d,$$

in the latter equation s is a parameter. Indeed, for time independent coefficients the solution of parabolic equation stabilizes at exponential rate to the solution of the corresponding elliptic equation, and the desired closeness follows from the perturbation theory arguments. In order to obtain a more precise asymptotics of χ we represent it as $\chi(z,s) = \chi^0(z,\varepsilon^\delta s) + \widetilde{\chi}(z,s)$. Substituting this expression in the above parabolic equation we have

$$\partial_s \widetilde{\chi}(z,s) - \operatorname{div}\big(a(z,\varepsilon^\delta s)\nabla\widetilde{\chi}(\xi,s)\big) = -\partial_s\chi^0(z,\varepsilon^\delta s), \quad (z,s) \in \mathbb{T}^d \times (-\infty,+\infty).$$

Since $\partial_s\chi^0(z,\varepsilon^\delta s) = \varepsilon^\delta \partial_\tau \chi^0(z,\tau)\big|_{\tau=\varepsilon^\delta s}$, this suggests that the next term of the expansion should be of order ε^δ. Repeating the above arguments we conclude that $\widetilde{\chi}(z,s)$

is close to the function $\varepsilon^\delta \chi^1(z, \varepsilon^\delta s)$ with $\chi^1(z, \tau)$ being a solution of the equation

$$\operatorname{div}\big(a(z,\tau)\nabla\chi^1(\xi,\tau)\big) = -\partial_s\chi^0(z,\tau), \quad z \in \mathbb{T}^d,$$

Iterating this step we arrive at Eqs. (7). Observe that the right hand side in the latter equation and in the higher order equations in (7) is well defined only if $a(z, s)$ is sufficiently regular in the temporal variable.

Since the corrector χ is represented as a sum of several terms, the regular part of the asymptotics also takes the form $u^0(x,t) + \varepsilon^\delta u^1(x,t) + \ldots$, see (8). Indeed, collecting the terms of order ε^0 in the asymptotic expansion and taking the average of these terms with respect to the fast variables we obtain the homogenized equation, u^0 being a solution to this equation. The terms of order ε^δ also have a non-trivial average w.r.t. the fast variables. Taking this average results in the equation for u^1. Iterating this procedure we obtain $\{u^j\}$ for all j such that $\delta j \leqslant \frac{\alpha}{2}$

Considering the difference $u^\varepsilon - u^0$ and dividing it by ε^δ, one can pass to the limit; if the limit is deterministic we consider the difference $u^\varepsilon - u^0 - \varepsilon^\delta u^1$, divide it by $\varepsilon^{2\delta}$ and pass to the limit again. We iterate this procedure until at some stage we reach a *random* limit; it happens as soon as $j\delta \geqslant \frac{\alpha}{2}$, where j is the number of iterations. Returning to u^ε, we obtain its expansion being a sum of terms of increasing order of ε, and the random term coming with the scaling factor $\varepsilon^{\alpha/2}$.

It should be noted that the stochastic term in the expansion is of order $\varepsilon^{\alpha/2}$ even for the coefficients having a finite range of dependence. This is just the normalization of the central limit theorem.

2.4 Proof of Theorem 1

Auxiliary problems.

We begin by considering problem (3). This equation has a unique up to an additive constant vector periodic solution such that

$$\|\chi^0\|_{L^\infty(\mathbb{T}^d\times[0,\infty))} \leq C. \tag{13}$$

Indeed, multiplying Eq. (3) by χ^0, using the Schwartz and Poincaré inequalities and considering (4), we conclude that $\|\chi^0(\cdot,s)\|_{H^1(\mathbb{T}^d)} \leq C$ for all $s \in \mathbb{R}$. Estimate (13) then follows from [11, Theorem 8.15]. And from [11, Theorem 8.24], $\chi^0(\cdot,s)$ is Hölder continuous in z. If $a(\cdot,s) \in C^{1,\beta}$ for some $\beta > 0$, $-\operatorname{div} a(\cdot,s,\omega) \in C^\beta$, and from the Schauder estimates [11, Theorem 6.2], $\chi^0(\cdot,s) \in C^{2,\beta}$. Now we can deduce that $\partial_s\chi^0$, solution of

$$\begin{aligned}\operatorname{div}\big(a(z,s,\omega)\nabla(\partial_s\chi^0)(z,s,\omega)\big) &= -\operatorname{div}\partial_s a(z,s,\omega)\\ &\quad - \operatorname{div}\big(\partial_s a(z,s,\omega)\nabla\chi^0(z,s,\omega)\big), \quad z \in \mathbb{T}^d,\end{aligned}$$

inherits the same regularity $C^{2,\beta}$.

Since $\chi^0(\cdot, s)$ only depends on $a(\cdot, s)$, the solution with zero average is stationary and the strong mixing coefficient of the pair $(a(\cdot, s), \chi^0(\cdot, s))$ coincides with that for $a(\cdot, s)$. The same statement is valid for any finite collection $(a(\cdot, s), \chi^0(\cdot, s), \chi^1(\cdot, s), \ldots)$.

By the similar arguments, the solutions χ^j of Eq. (7) are stationary, satisfy strong mixing condition with the same coefficient $\gamma(r)$. Indeed from Schauder estimate [11, Theorem 6.2], χ^1 belongs to $C^{2,\beta}$ and with the same arguments as before, $\partial_s \chi^1 \in C^{2,\beta}$. By recursion the same property holds for any j: for any $k \geq 0$ and $N = 1, \ldots, J^0$

$$\mathbf{E}\|\chi^j\|^k_{C^0([0,\infty);C^{2,\beta}(\mathbb{T}^d))} + \mathbf{E}\|\partial_s^N \chi^j\|^k_{C^0([0,\infty);C^{2,\beta}(\mathbb{T}^d))} \leq C_{k,N}. \tag{14}$$

Solutions u^0 and u^j, $j = 1, 2, \ldots, J_0$, of problems (2) and (8) are smooth functions. Moreover, for any $\mathbf{k} = (k_0, k_1, \ldots, k_d)$ and $N > 0$ there exists a constant $C_{\mathbf{k},N}$ such that

$$|D^{\mathbf{k}} u^j| \leq C_{\mathbf{k},N}(1 + |x|)^{-N}, \tag{15}$$

where $D^{\mathbf{k}} f(x,t) = \dfrac{\partial^{k_0}}{\partial t^{k_0}} \dfrac{\partial^{k_1}}{\partial x_1^{k_1}} \ldots \dfrac{\partial^{k_d}}{\partial x_d^{k_d}} f(x,t)$.

The proof of Theorem 1.

For the sake of brevity we use the following notational conventions

$$\partial_{z_j} = \frac{\partial}{\partial z_j}, \quad \partial_t = \frac{\partial}{\partial t},$$
$$(\partial_{x_j} f)\Big(\frac{x}{\varepsilon}\Big) = \partial_{z_j} f(z)\big|_{z=x/\varepsilon}, \quad (\partial_t f)\Big(\frac{t}{\varepsilon^\alpha}\Big) = \partial_s f(s)\big|_{s=t/\varepsilon^\alpha}. \tag{16}$$

Denote

$$\widehat{a}^{0,ij}(z,s) = a^{ij}(z,s) + \sum_{m=1}^{d} \Big[a^{im}(z,s)\partial_{z_m}\chi^{0,j}(z,s) + \partial_{z_m}\big(a^{mi}(z,s)\chi^{0,j}(z,s)\big)\Big],$$

and for $k = 1, 2, \ldots$

$$\widehat{a}^{k,ij}(z,s) = \sum_{m=1}^{d} \Big[a^{im}(z,s)\partial_{z_m}\chi^{k,j}(z,s) + \partial_{z_m}\big(a^{mi}(z,s)\chi^{k,j}(z,s)\big)\Big],$$

and (see (9))

$$a^{k,\mathrm{eff}} = \mathbf{E}\int_{\mathbb{T}^d} \big[\widehat{a}^k(z,s)\big]dz.$$

Substituting V^ε for u^ε in (1) yields

$$\begin{aligned}
&\partial_t V^\varepsilon - \operatorname{div}\big[a\big(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha}\big)\nabla V^\varepsilon\big] \\
&= -\varepsilon^{-\frac{\alpha}{2}} \sum_{k=0}^{J^0} \varepsilon^{k\delta}\Big[\partial_t u^k \\
&\quad + \sum_{j=0}^{J^0-k} \varepsilon^{(j\delta+1-\alpha)}\big(\partial_t \chi^j\big)\big(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha}\big)\nabla u^k + \sum_{j=0}^{J^0-k} \varepsilon^{(j\delta+1)}\chi^j\big(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha}\big)\partial_t \nabla u^k\Big] \\
&\quad + \varepsilon^{-\frac{\alpha}{2}} \sum_{k=0}^{J^0} \varepsilon^{k\delta-1}\Big[(\operatorname{div} a)\big(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha}\big) + \sum_{j=0}^{J^0-k} \varepsilon^{j\delta}\big(\operatorname{div}(a\nabla\chi^j)\big)\big(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha}\big)\Big]\nabla u^k \\
&\quad + \varepsilon^{-\frac{\alpha}{2}} \sum_{k=0}^{J^0}\sum_{j=0}^{J^0-k} \varepsilon^{(k+j)\delta} \sum_{1\le \ell,m\le d}\Big[\widehat{a}^{j,\ell m}\big(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha}\big)\frac{\partial^2}{\partial x_\ell \partial x_m}u^k\Big] \\
&\quad + \varepsilon^{-\frac{\alpha}{2}} \sum_{k=0}^{J^0}\sum_{j=0}^{J^0-k} \varepsilon^{(k+j)\delta+1} \sum_{1\le i,\ell,m\le d}\Big[(a^{im}\chi^{j,\ell})\big(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha}\big)\frac{\partial^3}{\partial x_i \partial x_m \partial x_\ell}u^k\Big],
\end{aligned} \tag{17}$$

with

$$V^\varepsilon(x,0) = \sum_{j=0}^{J^0} \varepsilon^{(j\delta+1)}\chi^j\Big(\frac{x}{\varepsilon}, 0\Big)\nabla u^0(x,0).$$

Due to (3) and (7) and with $\delta = 2-\alpha$:

$$\begin{aligned}
&-\sum_{k=0}^{J^0}\varepsilon^{k\delta}\sum_{j=0}^{J^0-k} \varepsilon^{(j\delta+1-\alpha)}\big(\partial_t \chi^j\big)\big(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha}\big)\nabla u^k \\
&\quad + \sum_{k=0}^{J^0} \varepsilon^{k\delta-1}\Big[(\operatorname{div} a)\big(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha}\big) + \sum_{j=0}^{J^0-k} \varepsilon^{j\delta}\big(\operatorname{div}(a\nabla\chi^j)\big)\big(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha}\big)\Big]\nabla u^k \\
&= -\varepsilon^{(J^0+1)\delta-1}\sum_{k=0}^{J^0}\big(\partial_t \chi^{J^0-k}\big)\big(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha}\big)\nabla u^k.
\end{aligned}$$

Considering our choice of J^0 we have: $(J^0+1)\delta - 1 > \alpha/2$. Therefore, with the help of (2) and (8) the first relation in (17) can be rearranged as follows

$$\partial_t V^\varepsilon - \operatorname{div}\big[a\big(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha}\big)\nabla V^\varepsilon\big] = -\varepsilon^{-\frac{\alpha}{2}}\sum_{k=0}^{J^0}\varepsilon^{k\delta}\partial_t u^k$$

$$+\varepsilon^{-\frac{\alpha}{2}}\sum_{k=0}^{J^0}\sum_{j=0}^{J^0-k}\varepsilon^{(k+j)\delta}\operatorname{Tr}\left[\widehat{a}^j\left(\frac{x}{\varepsilon},\frac{t}{\varepsilon^\alpha}\right)\nabla^2u^k\right]+\mathcal{R}^\varepsilon(x,t)$$

$$=\varepsilon^{-\frac{\alpha}{2}}\sum_{j=0}^{J^0}\sum_{k=0}^{J^0-j}\varepsilon^{(k+j)\delta}\operatorname{Tr}\left[\left(\widehat{a}^j\left(\frac{x}{\varepsilon},\frac{t}{\varepsilon^\alpha}\right)-a^{j,\mathrm{eff}}\right)\nabla^2u^k\right]+\mathcal{R}^\varepsilon(x,t),\quad (18)$$

where we identify $a^{0,\mathrm{eff}}$ with a^{eff}, Tr is the trace, ∇^2 the Hessian matrix w.r.t. x, and $\mathcal{R}^\varepsilon$ is the sum of all the terms on the right-hand side in (17) that are multiplied by a positive power of ε. One can easily check that

$$\mathcal{R}^\varepsilon(x,t)=\varepsilon^{-\alpha/2}\sum_{j=0}^{J_0}\varepsilon^{1+j\delta}\theta^j\left(\frac{x}{\varepsilon},\frac{t}{\varepsilon^\alpha}\right)\Phi^j(x,t),\quad (19)$$

where $\theta^j(z,s)$ are periodic in z, stationary in s and satisfy the estimates

$$\mathbf{E}\left(\|\theta^j\|^k_{C(\mathbb{T}^d\times[0,\infty))}\right)\le C_k;\quad (20)$$

Φ^j are smooth functions such that

$$|D^{\mathbf{k}}\Phi^j|\le C_{\mathbf{k},N}(1+|x|)^{-N};\quad (21)$$

we do not specify these quantities explicitly because we do not need them. We represent V^ε as the sum $V^\varepsilon=V_1^\varepsilon+V_2^\varepsilon$, where V_1^ε and V_2^ε solve the following problems:

$$\begin{cases}\partial_tV_1^\varepsilon-\operatorname{div}\left[a\left(\frac{x}{\varepsilon},\frac{t}{\varepsilon^\alpha}\right)\nabla V_1^\varepsilon\right]\\ \qquad=\varepsilon^{-\alpha/2}\sum_{k=0}^{J^0}\sum_{j=0}^{J^0-k}\varepsilon^{(k+j)\delta}\operatorname{Tr}\left[\left(\widehat{a}^j\left(\frac{x}{\varepsilon},\frac{t}{\varepsilon^\alpha}\right)-a^{j,\mathrm{eff}}\right)\nabla^2u^k\right],\\ V_1^\varepsilon(x,0)=0,\end{cases}\quad (22)$$

and

$$\begin{cases}\partial_tV_2^\varepsilon-\operatorname{div}\left[a\left(\frac{x}{\varepsilon},\frac{t}{\varepsilon^\alpha}\right)\nabla V_2^\varepsilon\right]=\mathcal{R}^\varepsilon(x,t),\\ V_2^\varepsilon(x,0)=V^\varepsilon(x,0).\end{cases}\quad (23)$$

From (13) and (14) it follows that the initial condition in the latter problem satisfies for any $k>0$ the estimate $\mathbf{E}\|V^\varepsilon(\cdot,0)\|^k_{C(\mathbb{R}^d)}\le C_k\varepsilon^{k\delta/2}$. If we multiply Eq. (23) by V_2^ε and integrate the resulting relation over $\mathbb{R}^d\times(0,T)$, then integrating by parts and combining estimates (19), (20) and the estimates for Φ^j, we obtain

$$\mathbf{E}\|V_2^\varepsilon\|^2_{L^2(\mathbb{R}^d\times(0,T))}\le C\varepsilon^\delta.\quad (24)$$

Denote

$$\langle a\rangle^0(s)=\int_{\mathbb{T}^d}\widehat{a}^0(z,s)dz$$
$$\langle a\rangle^k(s)=\int_{\mathbb{T}^d}\widehat{a}^k(z,s)dz,\quad k=1,\,2,\ldots$$

It follows from the definition of $\widehat{a}^k$ that for any $\ell>0$ there is a constant C_ℓ such that $\mathbf{E}\|(\widehat{a}^k-\langle a\rangle^k)\|^\ell_{C^1(\mathbb{T}^d\times[0,\infty))}\le C_\ell$. Since for each $s\in\mathbb{R}$ the mean value of $(\widehat{a}^k(\cdot,s)-\langle a\rangle^k(s))$ is equal to zero, the problem

$$\Delta_z\zeta^{k,im}(z,s)=(\widehat{a}^k(z,s)-\langle a\rangle^k(s))^{im}$$

has for each i and m a unique up to an additive constant periodic solution. Letting $\Theta^{k,im}(z,s)=\nabla\zeta^{k,im}(z,s)$, we obtain a stationary in s vector functions $\Theta^{k,im}$ such that

$$\operatorname{div}\Theta^{k,im}(z,s)=(\widehat{a}^k(z,s)-\langle a\rangle^k(s))^{im}, \tag{25}$$

with

$$\mathbf{E}\|\Theta^{k,im}\|^\ell_{C^1(\mathbb{T}^d\times[0,\infty))}\le C_\ell.$$

It is then straightforward to check that for the functions

$$\begin{aligned}F^\varepsilon(x,t)&=\varepsilon^{-\alpha/2}\sum_{k=0}^{J^0}\sum_{j=0}^{J^0-k}\varepsilon^{(k+j)\delta}\sum_{1\le\ell,m\le d}\Big[\widehat{a}^j\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^\alpha}\Big)-\langle a\rangle^j\Big(\frac{t}{\varepsilon^\alpha}\Big)\Big]^{\ell m}\frac{\partial^2}{\partial x_\ell\partial x_m}u^k(x,t)\\&=\varepsilon^{1-\frac{\alpha}{2}}\sum_{k=0}^{J^0}\sum_{j=0}^{J^0-k}\varepsilon^{(k+j)\delta}\sum_{1\le\ell,m\le d}\Big\{\operatorname{div}\Big[\Theta^{j,\ell m}\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^\alpha}\Big)\frac{\partial^2}{\partial x_\ell\partial x_m}u^k(x,t)\Big]\\&\quad-\Theta^{j,\ell m}\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^\alpha}\Big)\sum_i\Big(\frac{\partial^3}{\partial x_i\partial x_\ell\partial x_m}u^k(x,t)\Big)\Big\}\end{aligned}$$

the following estimate is fulfilled:

$$\mathbf{E}\|F^\varepsilon\|^2_{L^2(0,T;H^{-1}(\mathbb{R}^d))}\le C\varepsilon^\delta. \tag{26}$$

Therefore, a solution to the problem

$$\begin{cases}\partial_t V^\varepsilon_{1,2}-\operatorname{div}\Big[a\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^\alpha}\Big)\nabla V^\varepsilon_{1,2}\Big]=F^\varepsilon(x,t),\\ V^\varepsilon_{1,2}(x,0)=0,\end{cases} \tag{27}$$

admits the estimate

$$\mathbf{E}\|V_{1,2}^{\varepsilon}\|^2_{L^2(0,T;H^1(\mathbb{R}^d))} \le C\varepsilon^{\delta}. \tag{28}$$

Splitting $V_1^{\varepsilon} = V_{1,1}^{\varepsilon} + V_{1,2}^{\varepsilon}$, we conclude that $V_{1,1}^{\varepsilon}$ solves the following problem

$$\begin{cases} \partial_t V_{1,1}^{\varepsilon} - \operatorname{div}\Big[a\Big(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^{\alpha}}\Big)\nabla V_{1,1}^{\varepsilon}\Big] \\ \qquad = \varepsilon^{-\alpha/2}\displaystyle\sum_{k=0}^{J^0}\sum_{j=0}^{J^0-k}\varepsilon^{(k+j)\delta}\mathrm{Tr}\left[\left(\langle a\rangle^j\Big(\frac{t}{\varepsilon^{\alpha}}\Big) - a^{j,\mathrm{eff}}\right)\nabla^2 u^k\right], \\ V_{1,1}^{\varepsilon}(x,0) = 0. \end{cases} \tag{29}$$

By construction the strong mixing coefficient of $\widehat{a}^k$ remains unchanged and is equal to $\gamma(\cdot)$. Denote by $V_{1,1}^{0,\varepsilon}$ the solution of the following problem:

$$\begin{cases} \partial_t V_{1,1}^{0,\varepsilon} - \operatorname{div}\Big[a\Big(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^{\alpha}}\Big)\nabla V_{1,1}^{0,\varepsilon}\Big] = \varepsilon^{-\frac{\alpha}{2}}\mathrm{Tr}\left[\left(\langle a\rangle^0\Big(\frac{t}{\varepsilon^{\alpha}}\Big) - a^{\mathrm{eff}}\right)\nabla^2 u^0\right], \\ V_{1,1}^{0,\varepsilon}(x,0) = 0. \end{cases} \tag{30}$$

Lemma 1 *The solution of problem (30) converges in law, as* $\varepsilon \to 0$, *in* $L^2(\mathbb{R}^d \times (0,T))$ *equipped with strong topology, to the solution of (12).*

Proof We consider an auxiliary problem

$$\begin{cases} \partial_t V_{\mathrm{aux}}^{\varepsilon} - \operatorname{div}\Big[a^{\mathrm{eff}}\nabla V_{\mathrm{aux}}^{\varepsilon}\Big] = \varepsilon^{-\alpha/2}\mathrm{Tr}\left[\left(\langle a\rangle^0\Big(\frac{t}{\varepsilon^{\alpha}}\Big) - a^{\mathrm{eff}}\right)\nabla^2 u^0\right] \\ V_{\mathrm{aux}}^{\varepsilon}(x,0) = 0, \end{cases} \tag{31}$$

and notice that this problem admits an explicit solution

$$V_{\mathrm{aux}}^{\varepsilon}(x,t) = \mathrm{Tr}\left[\varepsilon^{\alpha/2}\zeta\left(\frac{t}{\varepsilon^{\alpha}}\right)\nabla^2 u^0(x,t)\right] \text{ with } \zeta(s) = \int_0^s \left[\langle a\rangle^0(r) - a^{\mathrm{eff}}\right]dr. \tag{32}$$

Since the matrix Ξ defined by (10) is equal to $\langle a\rangle^0\Big(\frac{\cdot}{\varepsilon^{\alpha}}\Big) - a^{\mathrm{eff}}$, then, due to [15, Lemma VIII.3.102], there exists a constant C that depends only on the $\mathbb{L}^{\infty}$-norm of the functions a, χ^0 and of their first order spatial derivatives, such that for any $1 \le \ell, m \le d$ we have

$$\mathbf{E}\left(\mathbf{E}\left[(\langle a\rangle^0(r) - a^{\mathrm{eff}})^{\ell m}\Big|\mathcal{F}_0\right]\right)^2 \le C\gamma(r).$$

Hence with assumption **h3**,

$$\int_0^\infty \left[\mathbf{E}\left(\mathbf{E}\left[(\langle a\rangle^0(r)-a^{\mathrm{eff}})^{\ell m}\Big|\mathcal{F}_0\right]\right)^2\right]^{1/2} dr \le C\int_0^\infty \gamma(r)^{1/2}dr < +\infty.$$

Moreover, since $\left(\langle a\rangle^0(0)-a^{\mathrm{eff}}\right)^{\ell m}$ are bounded, we have

$$\mathbf{E}\left[\left(\langle a\rangle^0(0)-a^{\mathrm{eff}}\right)^{\ell m}\right]^2 < +\infty.$$

From [15, Theorem VIII.3.97] (with $p=q=2$ and $r=\infty$), the invariance principle holds for the process $\varepsilon^{\alpha/2}\zeta\left(\frac{t}{\varepsilon^\alpha}\right)$, that is $\varepsilon^{\alpha/2}\zeta\left(\frac{t}{\varepsilon^\alpha}\right)$, converges in law, as $\varepsilon\to 0$, in $C([0,T])^{d^2}$ to a d^2-dimensional Brownian motion with the covariance matrix Λ given by (11) (see [28, Theorem 3]). Since u^0 satisfies estimates (6), the last convergence implies that $V^\varepsilon_{\mathrm{aux}}$ converges in law in $C((0,T);L^2(\mathbb{R}^d))$ to the solution of problem (12). Note that $\nabla V^\varepsilon_{\mathrm{aux}}$ can be expressed in terms of the third derivative of u^0, it is sufficient to differentiate (32) in x. Thus $V^\varepsilon_{\mathrm{aux}}$ also converges in law in $C((0,T);H^1(\mathbb{R}^d))$.

Next, we represent $V^{0,\varepsilon}_{1,1}$ as $V^{0,\varepsilon}_{1,1}(x,t)=\mathcal{Z}^\varepsilon(x,t)+V^\varepsilon_{\mathrm{aux}}(x,t)$. Then $\mathcal{Z}^\varepsilon$ solves the problem

$$\begin{cases}\partial_t\mathcal{Z}^\varepsilon-\mathrm{div}\Big[a\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^\alpha}\Big)\nabla\mathcal{Z}^\varepsilon\Big]=\mathrm{div}\Big(\Big[a\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^\alpha}\Big)-a^{\mathrm{eff}}\Big]\nabla V^\varepsilon_{\mathrm{aux}}(x,t)\Big)\\ \mathcal{Z}^\varepsilon(x,0)=0,\end{cases} \tag{33}$$

and our goal is to show that $\mathcal{Z}^\varepsilon$ goes to zero in probability in $L^2((0,T)\times\mathbb{R}^d)$, as $\varepsilon\to 0$. To this end we consider one more auxiliary problem that reads

$$\begin{cases}\partial_t\mathcal{Y}^\varepsilon-\mathrm{div}\Big[a\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^\alpha}\Big)\nabla\mathcal{Y}^\varepsilon\Big]=\mathrm{div}\Big(\Big[a\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^\alpha}\Big)-a^{\mathrm{eff}}\Big]\Upsilon(x,t)\Big)\\ \mathcal{Y}^\varepsilon(x,0)=0.\end{cases} \tag{34}$$

We now prove that $\mathcal{Y}^\varepsilon$ converges to zero in probability in $L^2((0,T)\times\mathbb{R}^d)$. The arguments are basically the same as in [18, Lemma 5.1]. If the vector function $\Upsilon\in L^2((0,T)\times\mathbb{R}^d)$, then problem (34) has a unique solution, and, by the standard energy estimate,

$$\|\mathcal{Y}^\varepsilon\|_{L^2(0,T;H^1(\mathbb{R}^d))}+\|\partial_t\mathcal{Y}^\varepsilon\|_{L^2(0,T;H^{-1}(\mathbb{R}^d))}\le C\|\Upsilon\|_{L^2((0,T)\times\mathbb{R}^d)}.$$

According to [22, Lemma 1.5.2] the family $\{\mathcal{Y}^\varepsilon\}$ is locally compact in $L^2(\mathbb{R}^d\times(0,T))$. Combining this with Aronson's estimate (see [3]) we conclude that the family $\{\mathcal{Y}^\varepsilon\}$ is compact in $L^2((0,T)\times\mathbb{R}^d)$.

Assume for a while that Υ is smooth and satisfies estimates (6). Multiplying Eq. (34) by a test function of the form $\varphi(x,t)+\varepsilon\chi^0\left(\frac{x}{\varepsilon},\frac{t}{\varepsilon^\alpha}\right)\nabla\varphi(x,t)$ with $\varphi\in C_0^\infty((0,T)\times\mathbb{R}^d)$

and integrating the resulting relation yields

$$-\int_0^T\int_{\mathbb{R}^d}\mathcal{Y}^\varepsilon\Big(\partial_t\varphi+\varepsilon^{1-\alpha}(\partial_t\chi^0)\big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^\alpha}\big)\nabla\varphi+\varepsilon\chi^0\big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^\alpha}\big)\partial_t\nabla\varphi(x,t)\Big)dxdt$$
$$+\int_0^T\int_{\mathbb{R}^d}\sum_{1\le i,j,m\le d}\partial_{x_m}\mathcal{Y}^\varepsilon a^{im}\big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^\alpha}\big)\Big[\partial_{x_i}\varphi+\big(\partial_{x_i}\chi^{0,j}\big)\big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^\alpha}\big)\partial_{x_j}\varphi$$
$$+\varepsilon\chi^{0,j}\big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^\alpha}\big)\frac{\partial^2\varphi}{\partial x_i\partial x_j}\Big]dxdt$$
$$=\int_0^T\int_{\mathbb{R}^d}\sum_{1\le i,j,m\le d}\big[a\big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^\alpha}\big)-a^{\mathrm{eff}}\big]^{im}\Upsilon^m\big[\partial_{x_i}\varphi+\big(\partial_{x_i}\chi^{0,j}\big)\big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^\alpha}\big)\partial_{x_j}\varphi$$
$$+\varepsilon\chi^{0,j}\big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^\alpha}\big)\frac{\partial^2\varphi}{\partial x_i\partial x_j}\big]dxdt;$$

here we have used the relation $\varphi(\cdot,0)=\varphi(\cdot,T)=0$. Since $\int_{\mathbb{T}^d}\chi^0(z,s)dz=0$, we have $\|(\partial_t\chi^0)(x/\varepsilon,t/\varepsilon^\alpha)\nabla\varphi\|_{L^2(0,T;H^{-1}(\mathbb{R}^d))}\le C\varepsilon$. Therefore

$$\int_0^T\int_{\mathbb{R}^d}\mathcal{Y}^\varepsilon\varepsilon^{1-\alpha}(\partial_t\chi^0)\big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^\alpha}\big)\nabla\varphi dxdt$$

tends to zero, as $\varepsilon\to0$. Considering (3) and (5) we obtain

$$\int_0^T\int_{\mathbb{R}^d}\sum_{1\le i,j,m\le d}\partial_{x_m}\mathcal{Y}^\varepsilon a^{im}\big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^\alpha}\big)\Big[\partial_{x_i}\varphi+\big(\partial_{x_i}\chi^{0,j}\big)\big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^\alpha}\big)\partial_{x_j}\varphi\Big]dxdt$$
$$=-\int_0^T\int_{\mathbb{R}^d}\mathcal{Y}^\varepsilon\sum_{1\le i,j\le d}\big\{a\big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^\alpha}\big)[\mathbf{I}+(\nabla\chi^0)\big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^\alpha}\big)]\big\}^{ij}\frac{\partial^2\varphi}{\partial x_i\partial x_j}dxdt$$

and

$$\int_0^T\int_{\mathbb{R}^d}\sum_{1\le i,j,m\le d}\big[a\big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^\alpha}\big)-\mathrm{a}^{\mathrm{eff}}\big]^{im}\Upsilon^m\big[\partial_{x_i}\varphi+\big(\partial_{x_i}\chi^{0,j}\big)\big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^\alpha}\big)\partial_{x_j}\varphi\big]dxdt$$
$$=\int_0^T\int_{\mathbb{R}^d}\sum_{1\le i,m\le d}\big\{a\big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^\alpha}\big)[\mathbf{I}+(\nabla\chi^0)\big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^\alpha}\big)]-a^{\mathrm{eff}}\big\}^{im}\Upsilon^m\partial_{x_i}\varphi dxdt$$
$$-\int_0^T\int_{\mathbb{R}^d}\sum_{1\le i,j,m\le d}\{a^{\mathrm{eff}}\}^{im}\Upsilon^m\big(\partial_{x_i}\chi^{0,j}\big)\big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^\alpha}\big)\partial_{x_j}\varphi dxdt.$$

The last term tends to zero as $\varepsilon \to 0$, one can easily justify it by means of integration by parts. For the other term, recalling the definition of $a^{\rm eff}$ and using the ergodic theorem we deduce that $\big\{a\big(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha}\big)\big[\mathbf{I} + \big(\nabla\chi^0\big)\big(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha}\big)\big] - a^{\rm eff}\big\} \to 0$ weakly in $L^2(\mathbb{R}^d \times (0, T))^{d^2}$, as $\varepsilon \to 0$. Therefore, this term also tends to zero. Denoting by $\mathcal{Y}^0$ the limit of $\mathcal{Y}^\varepsilon$ for a subsequence, we conclude that

$$\int_0^T \int_{\mathbb{R}^d} \mathcal{Y}^0\Big(- \partial_t\varphi - (a^{\rm eff})^{ij}\frac{\partial^2\varphi}{\partial x_i \partial x_j}\Big)dxdt = 0.$$

Therefore, $\mathcal{Y}^0 = 0$, and the whole family $\mathcal{Y}^\varepsilon$ converges a.s. to 0 in $L^2((0, T)\times\mathbb{R}^d)$. By the density argument this convergence also holds for any $\Upsilon \in L^2((0, T)\times\mathbb{R}^d)$. Since $V^\varepsilon_{\rm aux}$ converges in law in $C((0, T); H^1(\mathbb{R}^d))$, the solution of problem (33) converges to zero in probability in $L^2((0, T) \times \mathbb{R}^d)$, and the statement of the lemma follows. □

From the last lemma it follows that the solution of problem (29) converges in law, as $\varepsilon \to 0$, in $L^2(\mathbb{R}^d \times (0, T))$ equipped with strong topology, to the solution of (12). Combining this convergence with (24) and (28) we conclude that V^ε converges in law in the same space to the solution of (12). This completes the proof of Theorem 1.

3 Diffusion case

In this second setting we assume that the matrix $a(z, s)$ has the form

$$a(z, s) = \mathrm{a}(z, \xi_s), \tag{35}$$

where ξ_s is a stationary diffusion process in $\mathbb{R}^n$ with a generator

$$\mathcal{L} = \frac{1}{2}\mathrm{Tr}[q(y)D^2] + b(y).\nabla$$

(∇ stands for the gradient, D^2 for the Hessian matrix). In this case even for smooth functions $\mathrm{a}(z, y)$ the coefficients of matrix $\mathrm{a}(z, \xi_s)$ are just Hölder continuous in time and not differentiable, and the method used in the smooth case fails to work.

We still assume that Conditions **a1** and **a2** hold. Moreover we suppose that the matrix-functions $\mathrm{a}(z, y)$, $q(y)$ and vector-function $b(y)$ possess the following properties:

c1. $\mathrm{a} = \mathrm{a}(z, y)$ is periodic in z and belongs to $C^\infty(\mathbb{R}^n; C^{1,\beta}(\mathbb{T}^d))$ for some $\beta > 0$, such that for each $N > 0$ there exists $C_N > 0$ such that

$$\|\partial_y^N \mathrm{a}\|_{C^0(\mathbb{R}^n; C^{1,\beta}(\mathbb{T}^d))} \le C_N.$$

c2. The matrix $q = q(y)$ satisfies the uniform ellipticity conditions: there exist $\lambda > 0$ such that

$$\lambda^{-1}|\zeta|^2 \le q(y)\zeta\cdot\zeta \le \lambda|\zeta|^2, \quad y,\ \zeta \in \mathbb{R}^n.$$

Moreover there exists a matrix $\sigma = \sigma(y)$ such that $q(y) = \sigma^*(y)\sigma(y)$.

c3. The matrix function σ and vector-function b are smooth, for each $N > 0$ there exists $C_N > 0$ such that

$$\|\sigma\|_{C^N(\mathbb{R}^n)} \le C_N, \qquad \|b\|_{C^N(\mathbb{R}^n)} \le C_N.$$

c4. The following inequality holds for some $R > 0$ and $C_0 > 0$ and $p > -1$:

$$b(y) \cdot y \le -C_0|y|^p \quad \text{for all } y \in \{y \in \mathbb{R}^n : |y| \ge R\}.$$

We say that Condition **(C)** holds if **a1**, **a2** and **c1** – **c4** are satisfied. This case is called the *diffusive case*.

3.1 Existing results

Again according to [17], under **(C)**, the sequence u^ε converges in probability, as $\varepsilon \to 0$, to a solution u^0 of problem (2). Corrector $\chi^0 = \chi^0(z, y)$ is a periodic solution of the equation

$$\mathrm{div}_z\big(\mathrm{a}(z, y)\nabla_z\chi^0(z, y)\big) = -\mathrm{div}_z\mathrm{a}(z, y); \tag{36}$$

here $y \in \mathbb{R}^n$ is a parameter. We choose an additive constant in such a way that $\int_{\mathbb{T}^d} \chi^0(z, y)\, dz = 0$. The arguments developed to solve problem (3) can also be used here to obtain $\chi^0(\cdot, y) \in C^{2,\beta}(\mathbb{T}^d)$ and smooth in y, with the same estimate as in (13). Let us emphasize that it follows from (3) and (36) that the zero order correctors χ^0 coincide in both settings: $\chi^0(z, s) = \chi^0(z, \xi_s)$. The effective matrix is again given by (5):

$$a^{\mathrm{eff}} = \mathbf{E}\int_{\mathbb{T}^d} \mathrm{a}(z, \xi_s)\big(\mathbf{I} + \nabla_z\chi^0(z, \xi_s)\big)\, dz.$$

Let us recall that according to [27] under conditions **c2** and **c4** a diffusion process $\xi_\cdot$ with the generator $\mathcal{L}$ has an invariant measure in $\mathbb{R}^n$ that has a smooth density $\rho = \rho(y)$. For any $N > 0$ it holds

$$(1 + |y|)^N \rho(y) \le C_N$$

with some constant C_N. The function ρ is the unique up to a multiplicative constant bounded solution of the equation $\mathcal{L}^*\rho = 0$; here $*$ denotes the formally adjoint operator. We assume that the process ξ_t is stationary and distributed with the density ρ. The effective matrix can be written here as follows:

$$a^{\mathrm{eff}} = \int_{\mathbb{R}^n}\int_{\mathbb{T}^d} \Big(\mathrm{a}(z, y) + \mathrm{a}(z, y)\nabla_z\chi^0(z, y)\Big)\rho(y)\, dz dy.$$

In [19], under the condition that $d = 1$, a result similar to Theorem 1 is proved. We formulate this result under the assumption that condition **(C)** is fulfilled. As before we introduce several correctors and auxiliary quantities.

Higher order correctors are defined as periodic solutions of the equations

$$\mathrm{div}_z\big(\mathrm{a}(z, y)\nabla_z\chi^j(z, y)\big) = -\mathcal{L}_y\chi^{j-1}(z, y), \quad j = 1, 2, \dots, J^0. \tag{37}$$

Notice that $\int_{\mathbb{T}^d}\chi^{j-1}(z, y)\,dz = 0$ for all $j = 1, 2, \dots, J^0$, thus the compatibility condition is satisfied and the equations are solvable. Let us give some details for χ^1. Since χ^0 belongs to $C^\infty(\mathbb{R}^n; C^{2,\beta}(\mathbb{T}^d))$, then due to condition **a3**, $\mathcal{L}_y\chi^0(\cdot, \cdot)$ is also an element of this space.

The solutions χ^j defined by (37) satisfy the same estimate as (14): for any $N > 0$ there exists C_N such that

$$\|\chi^j\|_{C^\infty(\mathbb{R}^n;C^{2,\beta}(\mathbb{T}^d))} + \|\partial_y^N\chi^j\|_{C^0(\mathbb{R}^n;C^{2,\beta}(\mathbb{T}^d))} \le C_N.$$

Remark 4 We have already mentioned that according to (3) and (36) the zero order correctors coincide in both studied cases. It is interesting to compare the correctors defined in (37) with the ones given by (7) and to observe that the higher order correctors need not coincide.

Remark 5 In the diffusive case the regularity assumption on the initial condition φ can be relaxed as in the case studied in the previous sections, see Remark 3 for the details.

However, since $\mathcal{L}_y$ is a second order differential operator, the function $\mathrm{a}(\cdot)$ should be $2J^0 + 2$ times continuously differentiable w.r.t. y.

We introduce the matrices

$$a^{k,\mathrm{eff}} = \int_{\mathbb{R}^n}\int_{\mathbb{T}^d}\big[\mathrm{a}(z, y)\nabla_z\chi^k(z, y) + \nabla_z\big(\mathrm{a}(z, y)\chi^k(z, y)\big)\big]\rho(y)\,dzdy,$$
$$k = 1, 2, \dots,$$

and matrix valued functions

$$\begin{aligned}
\widehat{\mathrm{a}}^0(z, y) &= \mathrm{a}(z, y) + \mathrm{a}(z, y)\nabla_z\chi^0(z, y) + \nabla_z\big(\mathrm{a}(z, y)\chi^0(z, y)\big),\\
\widehat{\mathrm{a}}^k(z, y) &= \mathrm{a}(z, y)\nabla_z\chi^k(z, y) + \nabla_z\big(\mathrm{a}(z, y)\chi^k(z, y)\big), \quad k = 1, 2, \dots, \qquad (38)\\
\langle\mathrm{a}\rangle^0(y) &= \int_{\mathbb{T}^d}\big(\widehat{\mathrm{a}}^0(z, y) - a^{\mathrm{eff}}\big)dz,\\
\langle\mathrm{a}\rangle^k(y) &= \int_{\mathbb{T}^d}\big(\widehat{\mathrm{a}}^k(z, y) - a^{k,\mathrm{eff}}\big)dz, \quad k = 1, 2, \dots \qquad (39)
\end{aligned}$$

The functions $u^j = u^j(x, t)$ are defined as solutions of problems (8):

$$\frac{\partial}{\partial t}u^j = \mathrm{div}(a^{\mathrm{eff}}\nabla u^j) + \sum_{k=1}^{j}\{a^{k,\mathrm{eff}}\}^{im}\frac{\partial^2}{\partial x_i\partial x_m}u^{j-k}$$

$$u^j(x,0)=0$$

Since for each $j=1,2,\dots$ problem (8) has a unique solution, the functions u^j are uniquely defined. Finally, we consider the equation

$$\mathcal{L}Q^0(y)=\langle \mathrm{a}\rangle^0(y). \tag{40}$$

According to [28, Theorems 1 and 2], this equation has a unique up to an additive constant solution of at most polynomial growth. Denote

$$\Lambda=\{\Lambda^{ijml}\}=\int_{\mathbb{R}^n}\Big[\frac{\partial}{\partial y_{r_1}}(Q^0)^{ij}(y)\Big]q^{r_1r_2}(y)\Big[\frac{\partial}{\partial y_{r_2}}(Q^0)^{ml}(y)\Big]\rho(y)\,dy. \tag{41}$$

The matrix Λ is non-negative. Consequently its square root $\Lambda^{1/2}$ is well defined.

In the diffusive case the following result holds:

Theorem 2 *Under Condition* **(C)***, if* $d=1$ *or if* $\alpha\le 1$*, the normalized functions*

$$U^\varepsilon=\varepsilon^{-\alpha/2}\Big(u^\varepsilon-u^0-\sum_{j=1}^{J_0}\varepsilon^{j(2-\alpha)}u^j\Big)$$

converge in law, as $\varepsilon\to 0$*, in* $L^2(\mathbb{R}^d\times(0,T))$ *to the unique solution of (12) with the standard* d^2*-dimensional Brownian motion* W *and* Λ *defined in (41).*

Let us again emphasize that the case $d=1$ has been addressed in [19]. Here we only deal with the case $\alpha\le 1$.

3.2 Proof of Theorem 2 for $\alpha\le 1$

The beginning is the same as in Sect. 2.4 and is developed in [19, Section 3.1]. We consider the following expression:

$$V^\varepsilon(x,t)=\varepsilon^{-\alpha/2}\Big\{u^\varepsilon(x,t)-\sum_{k=0}^{J^0}\varepsilon^{k\delta}\Big(u^k(x,t)+\sum_{j=0}^{J^0-k}\varepsilon^{(j\delta+1)}\chi^j\Big(\frac{x}{\varepsilon},\xi_{\frac{t}{\varepsilon^\alpha}}\Big)\nabla u^k(x,t)\Big)\Big\},$$

where $\chi^j(z,y)$ and $u^k(x,t)$ are defined in (37) and (8), respectively. We substitute V^ε for u^ε in (1) using Itô's formula:

$$dV^\varepsilon-\mathrm{div}\big[a\big(\frac{x}{\varepsilon},\xi_{\frac{t}{\varepsilon^\alpha}}\big)\nabla V^\varepsilon\big]dt$$
$$=-\varepsilon^{-\frac{\alpha}{2}}\sum_{k=0}^{J^0}\varepsilon^{k\delta}\Big[\partial_t u^k+\sum_{j=0}^{J^0-k}\varepsilon^{(j\delta+1-\alpha)}\big(\mathcal{L}_y\chi^j\big)\big(\frac{x}{\varepsilon},\xi_{\frac{t}{\varepsilon^\alpha}}\big)\nabla u^k$$

$$+\sum_{j=0}^{J^0-k}\varepsilon^{(j\delta+1)}\chi^j\big(\frac{x}{\varepsilon},\xi_{\frac{t}{\varepsilon^\alpha}}\big)\partial_t\nabla u^k\Big]dt$$
$$+\sum_{k=0}^{J^0}\sum_{j=0}^{J^0-k}\varepsilon^{(1-\alpha+(k+j)\delta)}\sigma(\xi_{\frac{t}{\varepsilon^\alpha}})\nabla_y\chi^j\left(\frac{x}{\varepsilon},\xi_{\frac{t}{\varepsilon^\alpha}}\right)\nabla u^k\,dB_t$$
$$+\varepsilon^{-\frac{\alpha}{2}}\sum_{k=0}^{J^0}\varepsilon^{k\delta-1}\Big[(\mathrm{div}a)\big(\frac{x}{\varepsilon},\xi_{\frac{t}{\varepsilon^\alpha}}\big)+\sum_{j=0}^{J^0-k}\varepsilon^{j\delta}\big(\mathrm{div}(a\nabla\chi^j)\big)\big(\frac{x}{\varepsilon},\xi_{\frac{t}{\varepsilon^\alpha}}\big)\Big]\nabla u^k dt$$
$$+\varepsilon^{-\frac{\alpha}{2}}\sum_{k=0}^{J^0}\sum_{j=0}^{J^0-k}\varepsilon^{(k+j)\delta}\,\widehat{a}^{j,im}\big(\frac{x}{\varepsilon},\xi_{\frac{t}{\varepsilon^\alpha}}\big)\frac{\partial^2}{\partial x_i\partial x_m}u^k dt$$
$$+\varepsilon^{-\frac{\alpha}{2}}\sum_{k=0}^{J^0}\sum_{j=0}^{J^0-k}\varepsilon^{(k+j)\delta+1}\,(a^{im}\chi^{j,l})\big(\frac{x}{\varepsilon},\xi_{\frac{t}{\varepsilon^\alpha}}\big)\frac{\partial^3}{\partial x_i\partial x_m\partial x_l}u^k dt.$$

Here the $n\times n$ matrix $\sigma(y)$ is such that $\sigma(y)\sigma^*(y)=2q(y)$, $B_\cdot$ is a standard n-dimensional Brownian motion. Due to (36) and (37)

$$-\sum_{k=0}^{J^0}\varepsilon^{k\delta}\sum_{j=0}^{J^0-k}\varepsilon^{(j\delta+1-\alpha)}\big(\mathcal{L}_y\chi^j\big)\big(\frac{x}{\varepsilon},\xi_{\frac{t}{\varepsilon^\alpha}}\big)\nabla u^k$$
$$+\sum_{k=0}^{J^0}\varepsilon^{k\delta-1}\Big[(\mathrm{div}a)\big(\frac{x}{\varepsilon},\xi_{\frac{t}{\varepsilon^\alpha}}\big)+\sum_{j=0}^{J^0-k}\varepsilon^{j\delta}\big(\mathrm{div}(a\nabla\chi^j)\big)\big(\frac{x}{\varepsilon},\xi_{\frac{t}{\varepsilon^\alpha}}\big)\Big]\nabla u^k$$
$$=-\varepsilon^{(J^0+1)\delta-1}\sum_{k=0}^{J^0}\big(\mathcal{L}_y\chi^{J^0-k}\big)\big(\frac{x}{\varepsilon},\xi_{\frac{t}{\varepsilon^\alpha}}\big)\nabla u^k.$$

Considering Eq. (8) and the definitions of $a^{k,\mathrm{eff}}$ and $\widehat{\mathrm{a}}^k(z,y)$, we obtain an expression similar to that in (18)

$$dV^\varepsilon(x,t)-\mathrm{div}\Big[\mathrm{a}\Big(\frac{x}{\varepsilon},\xi_{\frac{t}{\varepsilon^\alpha}}\Big)\nabla V^\varepsilon\Big]dt$$
$$=\Big(\varepsilon^{-\alpha/2}\sum_{j=0}^{J^0}\sum_{k=0}^{J^0-j}\varepsilon^{(k+j)\delta}\Big[\widehat{\mathrm{a}}^k\Big(\frac{x}{\varepsilon},\xi_{\frac{t}{\varepsilon^\alpha}}\Big)-a^{k,\mathrm{eff}}\Big]^{im}\frac{\partial^2u^j}{\partial x_i\partial x_m}\Big)dt$$
$$+\sum_{k=0}^{J^0}\sum_{j=0}^{J^0-k}\varepsilon^{(1-\alpha+(k+j)\delta)}\sigma(\xi_{\frac{t}{\varepsilon^\alpha}})\nabla_y\chi^j\Big(\frac{x}{\varepsilon},\xi_{\frac{t}{\varepsilon^\alpha}}\Big)\nabla u^k(x,t)\,dB_t$$
$$+\mathcal{R}^\varepsilon(x,t)\,dt, \tag{42}$$

with $a^{0,\mathrm{eff}} = a^{\mathrm{eff}}$ and the initial condition

$$V^\varepsilon(x,0) = \varepsilon^{1-\alpha/2} \sum_{k=0}^{J^0} \sum_{j=0}^{J^0-k} \varepsilon^{j\delta} \chi^j\Big(\frac{x}{\varepsilon}, \xi_0\Big) \nabla u^k(x,0)$$

and

$$\mathcal{R}^\varepsilon(x,t) = \varepsilon^{-\alpha/2} \sum_{j=0}^{J^0} \varepsilon^{1+j\delta} \vartheta^j\Big(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}}\Big) \Phi^j(x,t) \tag{43}$$

with periodic in z smooth functions $\vartheta^j = \vartheta^j(z,y)$ of at most polynomial growth in y, and Φ^j satisfying (21).

We represent V^ε as the sum $V^\varepsilon = V_1^\varepsilon + V_2^\varepsilon + V_3^\varepsilon$ where V_1^ε and V_2^ε solve problems equivalent to (22) and (23):

$$\begin{cases} \partial_t V_1^\varepsilon - \operatorname{div}\Big[\mathrm{a}\Big(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}}\Big)\nabla V_1^\varepsilon\Big] \\ \qquad = \varepsilon^{-\alpha/2} \sum_{k=0}^{J^0} \sum_{j=0}^{J^0-k} \varepsilon^{(k+j)\delta} \Big[\widehat{\mathrm{a}}^j\Big(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha}\Big) - a^{j,\mathrm{eff}}\Big]^{im} \frac{\partial^2 u^k}{\partial x_i \partial x_m}, \\ V_1^\varepsilon(x,0) = 0, \end{cases} \tag{44}$$

and

$$\begin{cases} \partial_t V_2^\varepsilon - \operatorname{div}\Big[\mathrm{a}\Big(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}}\Big)\nabla V_2^\varepsilon\Big] = \mathcal{R}^\varepsilon(x,t), \\ V_2^\varepsilon(x,0) = V^\varepsilon(x,0). \end{cases} \tag{45}$$

We have

$$\begin{aligned} \mathbf{E}\|\mathcal{R}^\varepsilon\|^2_{L^2(\mathbb{R}^d\times(0,T))} &\le C\varepsilon^{1-\alpha/2} \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^n} (1+|y|)^{N_1}(1+|x|)^{-2n} \rho(y)\, dy dx dt \\ &\le C\varepsilon^{1-\alpha/2}. \end{aligned}$$

Similarly, $\mathbf{E}\|V_2^\varepsilon(\cdot,0)\|^2_{L^2(\mathbb{R}^d)} \le C\varepsilon^{1-\alpha/2}$. Therefore, V_2^ε still satisfies (24) and thus does not contribute in the limit.

The last term V_3^ε satisfies the SPDE:

$$\begin{aligned} &dV_3^\varepsilon(x,t) - \operatorname{div}\Big[\mathrm{a}\Big(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}}\Big)\nabla V_3^\varepsilon\Big] dt \\ &\quad = \varepsilon^{1-\alpha} \sum_{k=0}^{J^0} \sum_{j=0}^{J^0-k} \varepsilon^{(k+j)\delta} \sigma(\xi_{\frac{t}{\varepsilon^\alpha}}) \nabla_y \chi^j\Big(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}}\Big) \nabla u^k(x,t)\, dB_t \end{aligned} \tag{46}$$

with initial condition $V_3^\varepsilon(x, 0) = 0$. Let us again emphasize that the diffusive case cannot be deduced from our first case because of the presence of V_3^ε.

We turn to V_1^ε. The statement similar to that of Lemma 1 still holds. Indeed the equivalent of F^ε

$$H^\varepsilon(x,t) = \varepsilon^{-\alpha/2} \sum_{j=0}^{J^0} \sum_{k=0}^{J^0-j} \varepsilon^{(k+j)\delta} \Big[\widehat{\mathrm{a}}^k\Big(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}}\Big) - \langle \mathrm{a}\rangle^k\big(\xi_{\frac{t}{\varepsilon^\alpha}}\big)\Big]^{im} \frac{\partial^2 u^j}{\partial x_i \partial x_m}$$

admits the estimate (26):

$$\mathbf{E}\|H^\varepsilon\|^2_{L^2(0,T;H^{-1}(\mathbb{R}^d))} \le C\varepsilon^{2-\alpha}. \tag{47}$$

We split $V_1^\varepsilon = V_{1,1}^\varepsilon + V_{1,2}^\varepsilon$, where

- $V_{1,2}^\varepsilon$ solves (27) with H^ε on the right-hand side instead of F^ε, it admits estimate (28);
- $V_{1,1}^\varepsilon$ solves (29).

According to [28, Theorem 3] the processes

$$A^k(t) = \int_0^t (\langle a\rangle^k(\xi_s) - a^{k,\mathrm{eff}})ds$$

satisfy the functional central limit theorem (invariance principle), that is the process

$$A^{\varepsilon,k}(t) = \varepsilon^{\frac{\alpha}{2}} \int_0^{\varepsilon^{-\alpha}t} (\langle a\rangle^k(\xi_s) - a^{k,\mathrm{eff}})ds$$

converges in law in $C([0,T];\mathbb{R}^{d^2})$ to a d^2-dimensional Brownian motion with covariance matrix

$$(\Lambda_k) = \{(\Lambda_k)^{ijml}\} = \int_{\mathbb{R}^n} \Big[\frac{\partial}{\partial y_{r_1}}(Q^k)^{ij}(y)\Big] q^{r_1 r_2}(y) \Big[\frac{\partial}{\partial y_{r_2}}(Q^k)^{ml}(y)\Big]\rho(y)\,dy.$$

with matrix-function Q^0 defined in (40) and Q^k given by

$$\mathcal{L}Q^k(y) = \langle \mathrm{a}\rangle^k(y), \qquad k = 1, \ldots. \tag{48}$$

By the same arguments as those in the proof of Theorem 1 (see also [18, Lemma 5.1]), we obtain the same conclusions as in Lemma 1.

To finish the proof of Theorem 2, we need to control V_3^ε, solution of problem (46). Here we distinguish two cases: $\alpha < 1$ and $\alpha = 1$. As remarked in [19, Section 4.3], if $\alpha < 1$, $\mathbf{E}\|\sup_{0\le t\le T} V_3^\varepsilon(\cdot,t)\|^2_{L^2(\mathbb{R}^d)} \le C\varepsilon^{1-\alpha}$ and thus this term also does not contribute in the limit equation. Nonetheless for $\alpha = 1$, the leading term in V_3^ε solves the SPDE

$$dr^\varepsilon(x,t) - \mathrm{div}\Big[\mathrm{a}\Big(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon}}\Big)\nabla r^\varepsilon\Big]dt = \nabla_y \chi^0\Big(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon}}\Big)\nabla u^0(x,t)\sigma(\xi_{\frac{t}{\varepsilon}})\,dB_t. \tag{49}$$

Lemma 2 *r^ε converges to zero in probability in $L^2(0, T; L^2(\mathbb{R}^d))$.*

Assume for a while that this claim holds. Then due to positive powers of ε in the other terms of (46), we deduce that V_3^ε also tends to zero in the same space and the conclusion of Theorem 2 follows.

3.3 Proof of Lemma 2

Let us define

$$v_t^\varepsilon = \int_{\mathbb{R}^d} r^\varepsilon(x,t)^2 dx = \|r^\varepsilon(\cdot,t)\|^2_{L^2(\mathbb{R}^d)}$$

and

$$\Theta^\varepsilon\Big(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon}}, x, t\Big) = \nabla_y \chi^0\Big(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon}}\Big)\nabla u^0(x,t)\sigma(\xi_{\frac{t}{\varepsilon}}).$$

Note that $v_0^\varepsilon = 0$. Itô's formula and an integration by part lead to: for any $0 \le t \le T$

$$\begin{aligned} v_t^\varepsilon &+ 2\int_0^t \int_{\mathbb{R}^d} \nabla r^\varepsilon(x,s)\Big[\mathrm{a}\Big(\frac{x}{\varepsilon}, \xi_{\frac{s}{\varepsilon}}\Big)\nabla r^\varepsilon(x,s)\Big] dxds \\ &= 2\int_0^t \int_{\mathbb{R}^d} r^\varepsilon(x,s)\Theta^\varepsilon\Big(\frac{x}{\varepsilon}, \xi_{\frac{s}{\varepsilon}}, x, s\Big) dx\, dB_s \\ &\quad + \int_0^t \int_{\mathbb{R}^d} \Big\|\Theta^\varepsilon\Big(\frac{x}{\varepsilon}, \xi_{\frac{s}{\varepsilon}}, x, s\Big)\Big\|^2 dx\, ds. \end{aligned}$$

From condition **a1**, taking $t = T$ and the expectation, there exists a constant C independent of ε such that

$$\mathbf{E}\int_0^T \|\nabla r^\varepsilon(\cdot,s)\|^2_{L^2(\mathbb{R}^d)} ds \le C. \tag{50}$$

Moreover by Burkholder–Davis–Gundy, Cauchy–Schwarz and Young inequalities we have

$$\mathbf{E}\left[\sup_{t\in[0,T]} v_t^\varepsilon\right] = \mathbf{E}\left[\sup_{t\in[0,T]} \|r^\varepsilon(\cdot,t)\|^2_{L^2(\mathbb{R}^d)}\right] \le C. \tag{51}$$

Indeed

$$\begin{aligned} \mathbf{E}\left[\sup_{t\in[0,T]} v_t^\varepsilon\right] &= \mathbf{E}\left[\sup_{t\in[0,T]} \|r^\varepsilon(\cdot,t)\|^2_{L^2(\mathbb{R}^d)}\right] \\ &\le C\mathbf{E}\left[\left(\int_0^T \left|\int_{\mathbb{R}^d} r^\varepsilon(x,s)\Theta^\varepsilon\Big(\frac{x}{\varepsilon}, \xi_{\frac{s}{\varepsilon}}, x, s\Big)dx\right|^2 ds\right)^{1/2}\right] \end{aligned}$$

$$
\begin{aligned}
&+\mathbf{E}\left[\int_0^T\int_{\mathbb{R}^d}\left\|\Theta^\varepsilon\Big(\frac{x}{\varepsilon},\xi_{\frac{s}{\varepsilon}},x,s\Big)\right\|^2 dx\,ds\right]\\
&\le \frac{1}{2}\mathbf{E}\left[\sup_{t\in[0,T]} v_t^\varepsilon\right]+\frac{C}{2}\mathbf{E}\left[\int_0^T\int_{\mathbb{R}^d}\left\|\Theta^\varepsilon\Big(\frac{x}{\varepsilon},\xi_{\frac{s}{\varepsilon}},x,s\Big)\right\|^2 dx\,ds\right]\\
&+\mathbf{E}\left[\int_0^T\int_{\mathbb{R}^d}\left\|\Theta^\varepsilon\Big(\frac{x}{\varepsilon},\xi_{\frac{s}{\varepsilon}},x,s\Big)\right\|^2 dx\,ds\right].
\end{aligned}
$$

Now we prove that the sequence r^ε is tight in

$$
V_T = L^2_w(0,T;H^1(\mathbb{R}^d))\cap C(0,T;L^2_w(\mathbb{R}^d)).
$$

Remenber that the index w means that the corresponding space is equipped with the weak topology. We turn to estimating the modulus of continuity of the inner product of r^ε with a test function ϕ.

For any function $\phi\in C_0^\infty(\mathbb{R}^d)$ we define

$$
\widehat{v}_t^\varepsilon = \int_{\mathbb{R}^d} r^\varepsilon(x,t)\left(\phi(x)+\varepsilon\nabla\phi(x)\chi^0\Big(\frac{x}{\varepsilon},\xi_{\frac{t}{\varepsilon}}\Big)\right)dx = \langle r^\varepsilon(\cdot,t),\phi^\varepsilon(\cdot,t)\rangle_{L^2(\mathbb{R}^d)}.
$$

Again by Itô's formula for any $0\le t\le T$

$$
\begin{aligned}
\widehat{v}_t^\varepsilon &= \int_0^t\int_{\mathbb{R}^d}\phi^\varepsilon(x)\mathrm{div}\Big[\mathrm{a}\Big(\frac{x}{\varepsilon},\xi_{\frac{s}{\varepsilon}}\Big)\nabla r^\varepsilon(x,s)\Big]dxds\\
&+\int_0^t\int_{\mathbb{R}^d} r^\varepsilon(x,s)\nabla\phi(x)\mathcal{L}_y\chi^0\Big(\frac{x}{\varepsilon},\xi_{\frac{s}{\varepsilon}}\Big)dxds\\
&+\int_0^t\int_{\mathbb{R}^d}\phi^\varepsilon(x,s)\Theta^\varepsilon\Big(\frac{x}{\varepsilon},\xi_{\frac{s}{\varepsilon}},x,s\Big)dx\,dB_s\\
&+\varepsilon^{1/2}\int_0^t\int_{\mathbb{R}^d} r^\varepsilon(x,s)\nabla\phi(x)\nabla\chi^0\Big(\frac{x}{\varepsilon},\xi_{\frac{s}{\varepsilon}}\Big)dxdB_s\\
&+\varepsilon^{1/2}\int_0^t\int_{\mathbb{R}^d}\Theta^\varepsilon\Big(\frac{x}{\varepsilon},\xi_{\frac{s}{\varepsilon}},x,s\Big)\nabla\phi(x)\nabla\chi^0\Big(\frac{x}{\varepsilon},\xi_{\frac{s}{\varepsilon}}\Big)dxds.
\end{aligned}
$$

Evoke that $\widehat{\mathrm{a}}^0$ is defined by (38). With an integration by parts we obtain

$$
\begin{aligned}
\widehat{v}_t^\varepsilon &= \int_0^t\int_{\mathbb{R}^d} r^\varepsilon(x,s)\widehat{\mathrm{a}}^0\Big(\frac{x}{\varepsilon},\xi_{\frac{s}{\varepsilon}}\Big)\nabla^2\phi(x)dxds\\
&+\int_0^t\int_{\mathbb{R}^d} r^\varepsilon(x,s)\nabla\phi(x)\mathcal{L}_y\chi^0\Big(\frac{x}{\varepsilon},\xi_{\frac{s}{\varepsilon}}\Big)dxds\\
&+\int_0^t\int_{\mathbb{R}^d}\phi(x,s)\Theta^\varepsilon\Big(\frac{x}{\varepsilon},\xi_{\frac{s}{\varepsilon}},x,s\Big)dx\,dB_s\\
&+\varepsilon^{1/2}\int_0^t\int_{\mathbb{R}^d} r^\varepsilon(x,s)\nabla\phi(x)\nabla_z\chi^0\Big(\frac{x}{\varepsilon},\xi_{\frac{s}{\varepsilon}}\Big)dxdB_s
\end{aligned}
$$

$$+\varepsilon^{1/2}\int_0^t\int_{\mathbb{R}^d}\Theta^\varepsilon\Big(\frac{x}{\varepsilon},\xi_{\frac{s}{\varepsilon}},x,s\Big)\nabla\phi(x)\nabla_z\chi^0\Big(\frac{x}{\varepsilon},\xi_{\frac{s}{\varepsilon}}\Big)dxds$$
$$+\varepsilon\int_0^t\int_{\mathbb{R}^d}r^\varepsilon(x,s)\chi^0\Big(\frac{x}{\varepsilon},\xi_{\frac{s}{\varepsilon}}\Big)\nabla^3\phi(x)\mathrm{a}\Big(\frac{x}{\varepsilon},\xi_{\frac{s}{\varepsilon}}\Big)dxds$$
$$+\varepsilon\int_0^t\int_{\mathbb{R}^d}\nabla\phi(x,s)\chi^0\Big(\frac{x}{\varepsilon},\xi_{\frac{s}{\varepsilon}}\Big)\Theta^\varepsilon\Big(\frac{x}{\varepsilon},\xi_{\frac{s}{\varepsilon}},x,s\Big)dx\,dB_s \tag{52}$$

since from the very definition of χ^0, the two terms of order ε^{-1}

$$\varepsilon^{-1}\int_0^t\int_{\mathbb{R}^d}(\mathrm{div}_z\mathrm{a})\Big(\frac{x}{\varepsilon},\xi_{\frac{s}{\varepsilon}}\Big)\nabla\phi(x)r^\varepsilon(x,s)dxds$$
$$+\varepsilon^{-1}\int_0^t\int_{\mathbb{R}^d}\nabla\phi(x)\mathrm{div}_z\Big[\mathrm{a}\Big(\frac{x}{\varepsilon},\xi_{\frac{s}{\varepsilon}}\Big)\nabla_z\chi^0\Big(\frac{x}{\varepsilon},\xi_{\frac{s}{\varepsilon}}\Big)\Big]r^\varepsilon(x,s)dxds$$

are equal to zero.

Using BDG inequality and the estimate (51) we deduce that there exists $C>0$ such that for any $0\le t\le\tau\le T$

$$\mathbf{E}\Big[\sup_{s\in[t,\tau]}|\widehat{v}^\varepsilon_s-\widehat{v}^\varepsilon_t|\Big]=C\sqrt{\tau-t}+C\varepsilon^{1/2}.$$

Hence the sequence $\widehat{v}^\varepsilon$ is tight in $C(0,T;\mathbb{R})$, that is r^ε is tight in $C(0,T;L^2_w(\mathbb{R}^d))$.

For any $i=1,\dots,n$, since $\langle\mathcal{L}_y\chi^0\rangle=\langle(\nabla_y\chi^0)^i\rangle=0$, we can define $\zeta^{0,i}$ such that $\mathrm{div}_z\zeta^{0,i}=(\nabla_y\chi^0)^i$ and $\mathrm{div}_z\widehat{\zeta}^0=\mathcal{L}_y\chi^0$ and we have

$$\int_0^t\int_{\mathbb{R}^d}\phi(x,s)\Theta^\varepsilon\Big(\frac{x}{\varepsilon},\xi_{\frac{s}{\varepsilon}},x,s\Big)dx\,dB_s$$
$$=\int_0^t\int_{\mathbb{R}^d}\phi(x,s)\nabla_y\chi^0\Big(\frac{x}{\varepsilon},\xi_{\frac{s}{\varepsilon}}\Big)\nabla u^0(x,s)dx\sigma(\xi_{\frac{s}{\varepsilon}})\,dB_s$$
$$=\varepsilon\int_0^t\int_{\mathbb{R}^d}\phi(x,s)\mathrm{div}\zeta^0\Big(\frac{x}{\varepsilon},\xi_{\frac{s}{\varepsilon}}\Big)\nabla u^0(x,s)dx\sigma(\xi_{\frac{s}{\varepsilon}})\,dB_s$$
$$=-\varepsilon\int_0^t\int_{\mathbb{R}^d}\zeta^0\Big(\frac{x}{\varepsilon},\xi_{\frac{s}{\varepsilon}}\Big)\nabla\Big(\phi(x,s)\nabla u^0(x,s)\Big)\,dx\sigma(\xi_{\frac{s}{\varepsilon}})\,dB_s$$

and

$$\int_0^t\int_{\mathbb{R}^d}r^\varepsilon(x,s)\nabla\phi(x)\mathcal{L}_y\chi^0\Big(\frac{x}{\varepsilon},\xi_{\frac{s}{\varepsilon}}\Big)dxds$$
$$=\varepsilon\int_0^t\int_{\mathbb{R}^d}r^\varepsilon(x,s)\nabla\phi(x)\mathrm{div}\widehat{\zeta}^0\Big(\frac{x}{\varepsilon},\xi_{\frac{s}{\varepsilon}}\Big)dxds$$
$$=-\varepsilon\int_0^t\int_{\mathbb{R}^d}\nabla(r^\varepsilon(x,s)\nabla\phi(x))\widehat{\zeta}^0\Big(\frac{x}{\varepsilon},\xi_{\frac{s}{\varepsilon}}\Big)dxds.$$

From (50), these two quantities converge to zero. Therefore every term in (52), except for the first one, converges to zero.

For the first one, we have

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}^d} r^\varepsilon(x,s)\widehat{\mathrm{a}}^0\Big(\frac{x}{\varepsilon},\xi_{\frac{s}{\varepsilon}}\Big)\nabla^2\phi(x)dxds\\&=\int_0^t\int_{\mathbb{R}^d} r^\varepsilon(x,s)a^{\mathrm{eff}}\nabla^2\phi(x)dxds+\int_0^t\int_{\mathbb{R}^d} r^\varepsilon(x,s)\left(\langle\mathrm{a}\rangle^0(\xi_{\frac{s}{\varepsilon}})-a^{\mathrm{eff}}\right)\nabla^2\phi(x)dxds\\&\quad+\int_0^t\int_{\mathbb{R}^d} r^\varepsilon(x,s)\left(\widehat{\mathrm{a}}^0\Big(\frac{x}{\varepsilon},\xi_{\frac{s}{\varepsilon}}\Big)-\langle\mathrm{a}\rangle^0(\xi_{\frac{s}{\varepsilon}})\right)\nabla^2\phi(x)dxds,\end{aligned}$$

where $\langle\mathrm{a}\rangle^0$ is defined by (39). Since

$$\left\langle\widehat{\mathrm{a}}^0\Big(\frac{x}{\varepsilon},\xi_{\frac{s}{\varepsilon}}\Big)-\langle\mathrm{a}\rangle^0(\xi_{\frac{s}{\varepsilon}})\right\rangle=0,$$

the last part converges to zero. Moreover by definition of a^{eff}, we also obtain the convergence to zero of the penultimate term. Hence (52) becomes:

$$\begin{aligned}\widehat{v}_t^\varepsilon&=\int_{\mathbb{R}^d} r^\varepsilon(x,t)\left(\phi(x)+\varepsilon\nabla\phi(x)\chi^0\Big(\frac{x}{\varepsilon},\xi_{\frac{t}{\varepsilon}}\Big)\right)dx\\&=\int_0^t\int_{\mathbb{R}^d} r^\varepsilon(x,s)a^{\mathrm{eff}}\nabla^2\phi(x)dxds\\&\quad+\int_0^t\int_{\mathbb{R}^d} r^\varepsilon(x,s)\left(\langle\mathrm{a}\rangle^0(\xi_{\frac{s}{\varepsilon}})-a^{\mathrm{eff}}\right)\nabla^2\phi(x)dxds\\&\quad+\int_0^t\int_{\mathbb{R}^d} r^\varepsilon(x,s)\left(\widehat{\mathrm{a}}^0\Big(\frac{x}{\varepsilon},\xi_{\frac{s}{\varepsilon}}\Big)-\langle\mathrm{a}\rangle^0(\xi_{\frac{s}{\varepsilon}})\right)\nabla^2\phi(x)dxds+O(\varepsilon^{1/2}).\end{aligned}$$

Here $O(\varepsilon^{1/2})$ stands for functions whose $L^2(\Omega;L^\infty(0,T))$ norm is bounded by a constant times $\varepsilon^{1/2}$. On the right-hand side, the last two integrals converge to zero. Hence we have proved that the sequence r^ε converges in probability in $L^2(0,T;L^2_w(\mathbb{R}^d))\cap L^2(0,T;L^2_{loc}(\mathbb{R}^d))$ to the unique solution r^0 of the PDE (2) with initial value zero. Hence $r^0=0$.

To finish the proof of Lemma 2, we now show that the convergence holds in $L^2(0,T;L^2(\mathbb{R}^d))$. Define a non-negative function $\theta_R\in C^\infty(\mathbb{R}^d)$ equal to zero on $\{|x|\le R\}$ and equal to one on $\{|x|\ge 3R\}$ and such that $\|\nabla\theta_R\|\le 1/R$. For

$$\theta_R^\varepsilon(x,t)=\theta_R(x)+\varepsilon\nabla\theta_R(x)\chi^0\Big(\frac{x}{\varepsilon},\xi_{\frac{t}{\varepsilon}}\Big)$$

we have

$$\begin{aligned}&d(r^\varepsilon(x,t)\theta_R^\varepsilon(x))-\mathrm{div}\Big[\mathrm{a}\Big(\frac{x}{\varepsilon},\xi_{\frac{t}{\varepsilon}}\Big)\nabla(r^\varepsilon\theta_R^\varepsilon)\Big]dt\\&\quad=\theta_R^\varepsilon(x)\nabla_y\chi^0\Big(\frac{x}{\varepsilon},\xi_{\frac{t}{\varepsilon}}\Big)\nabla u^0(x,t)\sigma(\xi_{\frac{t}{\varepsilon}})\,dB_t\end{aligned}$$

$$
\begin{aligned}
&- 2a\Big(\frac{x}{\varepsilon}, \xi_{\frac{s}{\varepsilon}}\Big)\nabla r^\varepsilon(x,s)\nabla\theta_R(x)dt \\
&- r^\varepsilon(x,s)\left[a + a\nabla\chi^0 + \mathrm{div}(a\chi^0)\right]\Big(\frac{x}{\varepsilon}, \xi_{\frac{s}{\varepsilon}}\Big)\nabla^2\theta_R(x)dt \\
&+ r^\varepsilon(x,s)\nabla\theta_R(x)\mathcal{L}_y\chi^0\Big(\frac{x}{\varepsilon}, \xi_{\frac{s}{\varepsilon}}\Big)dt \\
&+ \varepsilon^{1/2} r^\varepsilon(x,s)\nabla\theta_R(x)\nabla\chi^0\Big(\frac{x}{\varepsilon}, \xi_{\frac{s}{\varepsilon}}\Big)dB_t \\
&+ \varepsilon^{1/2}\Theta^\varepsilon\Big(\frac{x}{\varepsilon}, \xi_{\frac{s}{\varepsilon}}, x, s\Big)\nabla\theta_R(x)\nabla\chi^0\Big(\frac{x}{\varepsilon}, \xi_{\frac{s}{\varepsilon}}\Big)dt \\
&- \varepsilon r^\varepsilon(x,s)a\Big(\frac{x}{\varepsilon}, \xi_{\frac{s}{\varepsilon}}\Big)\chi^0\Big(\frac{x}{\varepsilon}, \xi_{\frac{s}{\varepsilon}}\Big)\nabla^3\theta_R(x)dt.
\end{aligned}
$$

If we apply Itô's formula to

$$
v^R(t) = \|r^\varepsilon(\cdot,t)\theta_R^\varepsilon(\cdot)\|^2_{L^2(\mathbb{R}^d)}
$$

then

$$
\begin{aligned}
&v^R(t) + 2\int_0^t\int_{\mathbb{R}^d}\nabla(r^\varepsilon(x,s)\theta_R^\varepsilon(x))\Big[\mathrm{a}\Big(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon}}\Big)\nabla(r^\varepsilon(x,s)\theta_R^\varepsilon(x))\Big]dxds \\
&= 2\int_0^t\int_{\mathbb{R}^d} r^\varepsilon(x,s)\theta_R^\varepsilon(x)\theta_R^\varepsilon(x)\nabla_y\chi^0\Big(\frac{x}{\varepsilon}, \xi_{\frac{s}{\varepsilon}}\Big)\nabla u^0(x,s)\sigma(\xi_{\frac{s}{\varepsilon}})dx\,dB_s \\
&\quad - 4\int_0^t\int_{\mathbb{R}^d} r^\varepsilon(x,s)\theta_R^\varepsilon(x)a\Big(\frac{x}{\varepsilon}, \xi_{\frac{s}{\varepsilon}}\Big)\nabla r^\varepsilon(x,s)\nabla\theta_R(x)dxds \\
&\quad - 2\int_0^t\int_{\mathbb{R}^d} r^\varepsilon(x,s)\theta_R^\varepsilon(x)r^\varepsilon(x,s)\left[a + a\nabla\chi^0 + \mathrm{div}(a\chi^0)\right]\Big(\frac{x}{\varepsilon}, \xi_{\frac{s}{\varepsilon}}\Big)\nabla^2\theta_R(x)dxds \\
&\quad + 2\int_0^t\int_{\mathbb{R}^d} r^\varepsilon(x,s)\theta_R^\varepsilon(x)r^\varepsilon(x,s)\nabla\theta_R(x)\mathcal{L}_y\chi^0\Big(\frac{x}{\varepsilon}, \xi_{\frac{s}{\varepsilon}}\Big)dxds \\
&\quad + \int_0^t\int_{\mathbb{R}^d}\theta_R^\varepsilon(x)^2\left\|\nabla_y\chi^0\Big(\frac{x}{\varepsilon}, \xi_{\frac{s}{\varepsilon}}\Big)\nabla u^0(x,s)\sigma(\xi_{\frac{s}{\varepsilon}})\right\|^2 dxds + O(\varepsilon^{1/2}).
\end{aligned}
$$

Since u^0 is a Schwartz class function and $\|\nabla\theta_R^\varepsilon\| \le 1/R$, the expectation of the right-hand side does not exceed $C\left(\dfrac{1}{R} + \sqrt{\varepsilon}\right)$. It implies tightness in $L^2(0,T;L^2(\mathbb{R}^d))$.

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