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7. Unitary representations of infinite-dimensional pairs (G, K) and the formalism of R. Howe

G. I. OL'SHANSKII

Introduction

In this article we study the unitary representations of infinite-dimensional classical groups. The following problems are examined: the construction of unitary representations, proof of their irreducibility, pairwise non-equivalence, decomposition of tensor products, continuity of the representations in appropriate group topology, calculation of spherical functions, approximation of irreducible unitary representations T of a given infinite-dimensional group G by irreducible unitary representation T_n of the corresponding finite-dimensional classical groups $G(n)$. The main ideas and results were briefly given in the notes [22], [24].

1. (G, K)-PAIRS

At present it is not clear whether there exists a unique "correct" answer to the question: "What are infinite-dimensional classical groups?" It is not excluded that different problems require different definitions. In this article we have adopted the following approach. By an infinite-dimensional classical group G we mean simply the inductive limit

$$\bigcup_{n=1}^{\infty} G(n)$$

of finite-dimensional classical groups $G(n)$ and discuss the possibility of a transition to a wider topological group only after a definite stock of representations has been obtained.

A second, more important idea consists of the fact that we are studying not simply some group G or other, but the pair (G, K) where $K \subset G$. The subgroup

$$K = \bigcup_{n=1}^{\infty} K(n)$$

is distinguished by an involution and is destined to play the role of the maximal compact subgroup. It plays an important part in the definition of the "reasonable" representations of the group G, and hence there is some sense in talking about the representations of the pair (G, K).

The (G, K)-pairs enumerated in the Tables 7.1 and 7.2 are obtained as follows from the list of classical Riemannian symmetric spaces. It is assumed that $\{G(n)/K(n)\}$ is one of the classical series of symmetric spaces of compact or non-compact type while the embeddings $G(n) \to G(n+1)$ and $K(n) \to K(n+1)$ are similar to the embeddings $GL(n, \mathbb{R}) \to GL(n+1, \mathbb{R})$ and $SO(n) \to SO(n+1)$. Embeddings of the type $SO(2^n) \to SO(2^{n+1})$ are not examined in this article.

TABLE 7.1. Pairs (G, K) of finite rank $p = 1, 2, \ldots$

G (non-compact type)	G (compact type)	K
$SO_0(p, \infty)$	$SO(p+\infty) = SO(\infty)$	$SO(p) \times SO(\infty)$
$U(p, \infty)$	$U(p+\infty) = U(\infty)$	$U(p) \times U(\infty)$
$Sp(p, \infty)$	$Sp(p+\infty) = Sp(\infty)$	$Sp(p) \times Sp(\infty)$

The symbols of the groups G and K are selected in such a way that after formal replacement of the sign ∞ by n, we get $G(n)$ and $K(n)$. For example

$$SO_0(p, \infty) = \bigcup_{n=1}^{\infty} SO(p, n), \quad SO(\infty) = \bigcup_{N=1}^{\infty} SO(n), \quad U(2\infty) = \bigcup_{n=1}^{\infty} U(2n).$$

In each row of the tables there are two pairs (G, K) with the common subgroup K. Naturally they are called pairs of *non-compact and* *compact type*, since the corresponding spaces $G(n)/K(n)$ are symmetric spaces dual to one another of non-compact and compact type. In Table 7.1 are given those pairs for which the rank of the symmetric space $G(n)/K(n)$ does not change as $n \to \infty$, and in Table 7.2 those pairs for which rank $(G(n)/K(n)) \to \infty$.

TABLE 7.2. Pairs (G, K) of infinite rank

Symbol	G (non-compact type)	G (compact type)	K
$(\mathbb{R})$	$GL^+(\infty, \mathbb{R})$	$U(\infty)$	$SO(\infty)$
$(\mathbb{C})$	$GL(\infty, \mathbb{C})$	$U(\infty) \times U(\infty)$	$U(\infty)$
$(\mathbb{H})$	$GL(\infty, \mathbb{H})$	$U(2\infty)$	$Sp(\infty)$
$(\mathbb{R}_1)$	$SO_0(\infty, \infty)$	$SO(2\infty)$	$SO(\infty) \times SO(\infty)$
$(\mathbb{R}_2)$	$Sp(\infty, \mathbb{R})$	$Sp(\infty)$	$U(\infty)$
$(\mathbb{C}_1)$	$SO(\infty, \mathbb{C})$	$SO(\infty) \times SO(\infty)$	$SO(\infty)$
$(\mathbb{C}_2)$	$Sp(\infty, \mathbb{C})$	$Sp(\infty) \times Sp(\infty)$	$Sp(\infty)$
$(\mathbb{C}_3)$	$U(\infty, \infty)$	$U(2\infty)$	$U(\infty) \times U(\infty)$
$(\mathbb{H}_1)$	$Sp(\infty, \infty)$	$Sp(2\infty)$	$Sp(\infty) \times Sp(\infty)$
$(\mathbb{H}_2)$	$SO^*(2\infty)$	$SO(2\infty)$	$U(\infty)$

2. THE LANGUAGE OF ADMISSIBLE REPRESENTATIONS

Definition. A unitary representation of the group $SO(\infty)$, $U(\infty)$ or $Sp(\infty)$ is called *tame* if it is continuous in the group topology in which the descending sequence of subgroups of the type

$$\left\{\begin{bmatrix} 1_n & 0 \\ 0 & * \end{bmatrix}\right\}, \quad n = 1, 2, \ldots,$$

constitutes a fundamental system of neighbourhoods of the identity.

Any tame representation is a discrete direct sum of irreducible representations, the set (of equivalence classes) of irreducible tame representations is countable and all of them may be realized in tensor spaces. These properties of tame representations make them similar to the representations of the compact groups $SO(n)$, $U(n)$ and $Sp(n)$.

The definition of tame representations easily extends to all groups K from the Tables 7.1 and 7.2. The following definition separates from the vast category of all unitary representations of a group G a certain "reasonable" subcategory.

DEFINITION. Any unitary representation of the group G whose restriction to K is a tame representation is called an *admissible representation* of the pair (G, K).

Admissible representations of pairs of finite rank were investigated in the articles [20], [25].† It was found that there exists a continuum of irreducible admissible representations and all of them may be explicitly constructed. The restriction of any irreducible admissible representation to K has a finitely multiple discrete spectrum. An arbitrary admissible representation generates a von Neumann algebra of type I and decomposes, in essence, uniquely into irreducible admissible representations.

This article is devoted to the more difficult case of pairs of infinite rank. Here it is also possible to say a lot about the admissible representations, but the problem of their complete classification remains open. In connection with the definitions given above we observe the following:

(1) If (G, K) is one of the pairs of compact type (of infinite rank), then any tame representation of the group G is obviously an admissible representation of the pair. However, only a very small part of all admissible representations is obtained in this way.

(2) The categories of admissible representations for the pairs (G, K) of compact type with one and the same group G but different subgroups K‡ intersect only in tame representations (section 20.17).

(3) On the contrary, if we compare the pairs (G, K) with the common subgroup K (i.e., dual pairs occurring in one and the same row of Table 7.2), then a definite similarity is observed in the behaviour of their admissible representations.

†If (G, K) is a pair of compact type from Table 1, then its admissible representations are exhausted by the tame representations of the group G. In this situation K is needed only for the theory of spherical functions; hence, while discussing pairs of finite rank, we shall have in view only pairs of non-compact type.

‡For example, it is possible to distinguish three subgroups K: $SO(\infty)$, $U(\infty)\times U(\infty)$ and $Sp(\infty)$, in one and the same group $G = U(\infty) = U(2\infty)$.

(4) There are other pairs (G, K) for which there exists an interesting theory of admissible representations: (a) pairs where G/K is a symmetric space of zero curvature (the case of finite rank is discussed in [20], but the case of infinite rank is equally interesting); (b) two "non-symmetric pairs" $(SO(2\infty+1), U(\infty))$ and $(U(2\infty+1), Sp(\infty))$; (c) certain pairs connected with the infinite symmetric group $S(\infty)$ (see [24], [46]).

3. THE POSSIBILITY OF COMPLETION OF INDUCTIVE LIMITS

Any tame representation of the group $SO(\infty)$, $U(\infty)$ or $Sp(\infty)$ admits a continuous extension to the group of all unitary operators (respectively, real, complex or quaternion) endowed with the weak or, what is the same thing, strong operator topology. Conversely, any continuous unitary representation of this topological group arises from a tame representation of the corresponding inductive limit. An analogous fact is true also for admissible representations of groups of pseudounitary operators $SO_0(p,\infty)$, $U(p,\infty)$, $Sp(p,\infty)$ [25].

In the case of (G, K)-pairs of infinite rank, the situation is more delicate: it turns out that all admissible representations (constructed in this article) may be extended to a certain complete topological group $\bar{G}$. The group $\bar{G}$ contains G (or its appropriate covering) as a dense subgroup; it contains also the completion $\bar{K}$ of the group K according to the weak-strong operator topology; topology in $\bar{G}/\bar{K}$ is given by the Hilbert–Schmidt norm. For example, in the case $(G, K) = (U(\infty), SO(\infty))$, it is necessary to take the group of all complex unitary operators g such that $g(\bar{g})^{-1} - 1$ is a Hilbert–Schmidt operator; in the appropriate topology this group possesses a universal covering with the fibre $\mathbb{Z}$; this covering is just $\bar{G}$.

4. HOLOMORPHIC EXTENSIONS OF TAME REPRESENTATIONS

A peculiar analogue of H. Weyl's unitary trick is effective for all tame representations of the groups K from Table 7.2. A certain group $K^* \supset K$, which is isomorphic to $U(\infty)$ or to a product of some copies of it, acts as a complexification of the group K. For example, if K is $SO(\infty)$, $U(\infty)$ or $Sp(\infty)$, then K^* is respectively $U(\infty)$, $U(\infty) \times U(\infty)$

or $U(2\infty) = U(\infty)$. In the class of tame representations of the group K^*, a certain subclass of representations called *holomorphic* is distinguished. The "unitary trick" consists of the fact that any tame representation ρ of the group K coincides with a restriction of the uniquely defined holomorphic representation ρ^* of the group K^*; the latter generates the same von Neumann algebra and is called the *holomorphic extension* of the representation ρ. Thus the category of tame representations of the group K is equivalent to the category of holomorphic tame representations of the group K^*.

We observe that an analogous result is true also for the admissible representations of the groups $SO_0(p,\infty)$, $U(p,\infty)$ and $Sp(p,\infty)$ [25].

5. MAIN METHOD

Let (G, K) be one of the pairs of infinite rank and T be one of its admissible representations in Hilbert space H. Let us study the group G_T^* of unitary operators in H generated by the representation T and the holomorphic extension ρ^* of the tame representation $\rho = T|K$, and let us denote by T^* its identical representation in H; then T and T^* generate identical von Neumann algebras. Any time when it is necessary to prove the irreducibility of a certain concrete representation T or to decompose it into irreducible components, we may replace it by T^*. This method is extremely effective, since in many respects the structure of T^* is more simple than that of T. For example, if T is irreducible, then T^* will in a specific sense be the representation with the highest weight. Here it is also important that in concrete situations it is possible to describe the group $G_T{}^*$ explicitly. For example, for the irreducible representations T of the pair $(GL^+(\infty, \mathbb{R}), SO(\infty))$ given in this article, it is the product of a finite number of copies of the group $Sp(\infty, \mathbb{R})$. (The method described works also for pairs of finite rank; see [25].)

An important role in our theory is played by the Weil representation W of the group $Sp(\infty, \mathbb{R})$, the spinor representation S of the group $SO(2\infty)$ and allied representations of some other classical groups; the representations T^* are described in terms of these. For our purposes, it is necessary to be able to decompose tensor powers $W^{\otimes k}$ and $S^{\otimes k}$. This problem is solved by the method suggested by R. Howe [9].

6. CURRENT GROUPS

Although $G_T{}^*$ depends considerably on T, for each pair (G, K) it is possible to define a certain universal group G^* serving all the admissible representations T constructed in this article, in the sense that T^* may always be considered a representation of this group. It turns out that G^* is a certain group of currents on the real line. For example, for the pair $(GL^+(\infty, \mathbb{R}), SO(\infty))$, we get a group of functions on $\mathbb{R}$ with values in $Sp(\infty, \mathbb{R})$, and for the pair $(U(\infty), SO(\infty))$ we get a group of functions on $\mathbb{R}$ which within the segment $[-1, 1]$ take values in the group $Sp(\infty)$, outside it, in the group $Sp(\infty, \mathbb{R})$, and at the end points ± 1, in their common subgroup $U(\infty)$ (see sections 12.5 and 20.13).

We again get a kind of unitary trick: the category of admissible representations of any of the pairs (G, K) constructed in this article is equivalent to the category of "holomorphic representations" (it is possible to say further: "representations with the highest weight") of the corresponding group G^*.

7. THE LINK WITH THE THEORY OF FACTOR REPRESENTATIONS

The programme of study of the representations of inductive limits given in this article is not the only one possible. There is another approach conceptually linked with the theory of operator algebras, which has already led to a series of important results (see [2], [5], [31], [32], [36–40]). In this approach, the main objects are not irreducible representations, but factor representations.

There is a remarkably close relation between both theories. It is based on the following simple observations: whatever be the group $\mathscr{G}$, the set of the quasi-equivalence classes of its factor representations of type II_1 is in natural bijection with the set of equivalence classes of the infinite-dimensional spherical representations of the pair $(\mathscr{G} \times \mathscr{G}, \mathscr{G})$.† In particular, to any character $\chi(g)$ on $\mathscr{G}$ corresponds the spherical function $\varphi(g_1, g_2) = \chi(g_1 g_2^{-1})$ on $\mathscr{G} \times \mathscr{G}$.

†We keep in mind the irreducible unitary representations of the group $\mathscr{G} \times \mathscr{G}$ possessing a nonzero vector invariant with respect to the diagonal subgroup $\mathscr{G}$; such a vector is unique to within a factor.

It is evident from here that II_1-representations of the groups $SO(\infty)$, $U(\infty)$, $Sp(\infty)$ (or of the group $S(\infty)$) and spherical representations of the corresponding pairs $(K\times K, K)$ are, in essence, one and the same; it is important also to emphasize that the spherical representations are automatically admissible. More than that, even some other (semifinite) factor representations may be interpreted as irreducible admissible representations.

This link between the two theories is very useful. It may be used in two ways: the classification of characters obtained in [2] and [37] provides a classification of spherical functions; conversely, our results help in making more precise the realization of factor representations.

8. STRUCTURE OF THE ARTICLE

The first two Parts give the preliminary material. In Part I, the properties of tame representations are given and explicit decomposition of certain representations connected with Gaussian measure are obtained. Weil representations of the groups $Sp(\infty,\mathbb{R})^{\sim}$, $U(\infty,\infty)^{\sim}$ and $SO^*(2\infty)$ are constructed in Part II. They are realized in boson Fock spaces. The decomposition of their tensor powers is obtained with the help of R. Howe's formalism.

Part III, devoted to (G, K)-pairs of non-compact type, occupies a central place in the article. A one-parameter family $\{T_s\}$ of "fundamental" admissible representations is constructed for each such pair. The representations T_s are very simply realized in the spaces $L^2(F\times F\times \ldots, \mu)$, where F is $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, and μ indicates a Gaussian product-measure. A decomposition of tensor products of the type $T_{s_1}\otimes \ldots \otimes T_{s_p}$ is found with the help of the method described above. As a result we get a continuum family of irreducible admissible representations. Among them there are many spherical representations; the corresponding spherical functions are explicitly calculated. For the constructed irreducible representations, irreducible unitary representations of the groups $G(n)$ approximating them are found; these are the representations from the degenerate principal unitary series in the sense of Gelfand–Naimark.

In Part IV these results (except the last) are transferred to (G, K)-pairs of compact type, where considerations of analogy play a decisive role. In the compact case, the theory becomes more complicated and interesting: already there is no simple realization of representations in terms of Gaussian measure; spinor representations are used along with Weil representations; it is possible to construct not only finite but also infinite tensor products of fundamental representations.

Part V gives a few constructions for general inductive limits $G = \cup G(n)$: an analogue of the functional equation for spherical functions; the existence of approximations $T_n \to T$ for any $T \in \hat{G}$ (where $T_n \in \hat{G}(n)$); a certain correspondence between the factor representations and irreducible representations.

The section "Commentaries" gives bibliographic notes.†

9. CERTAIN PROBLEMS

Problem 1. Is it true that in this article all the irreducible admissible representations of (G, K)-pairs of infinite rank have been found?

For certain (G, K)-pairs, the affirmative answer is obtained in the particular case of spherical representations, see [2], [37], [15], [44].

Problem 2. Is it true that any admissible representation generates a von Neumann algebra of Type I?

This is proved for $SO_0(\infty, \infty)$, $U(\infty, \infty)$, $Sp(\infty, \infty)$ [44] and for certain (G, K)-pairs associated to the infinite symmetric group $S(\infty)$ [24], [46].

Problem 3. To construct an approximation of representations of the pairs of compact type in analogy with the non-compact case (**§14**).

For the characters of the group $U(\infty)$ (or, which is the same, for the spherical representations of the pair $[U(\infty) \times U(\infty), U(\infty)]$), the problem of approximation was solved by A. M. Vershik and S. V. Kerov [37].

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†After this paper had been completed (1983) the following new papers appeared: [44–51].

PART I

TAME REPRESENTATIONS OF THE GROUPS $SO(\infty)$, $U(\infty)$, $Sp(\infty)$

§1. Basic definitions and notations

1.1

We always denote by F any of the fields $\mathbb{R}$, $\mathbb{C}$ or the skew field $\mathbb{H}$ of quaternions.

$x \mapsto \bar{x}$ is the standard involution (conjugation) in F.

We write the skew field $\mathbb{H}$ as $\mathbb{C} \oplus \mathbb{C}j$, where j is taken from the standard $\mathbb{R}$-basis $\{1, i, j, k\}$ in $\mathbb{H}$. We observe that $jzj^{-1} = \bar{z}$ for all $z \in \mathbb{C}$. If $x = x^1 + x^2 j \in \mathbb{H}$ where $x^1, x^2 \in \mathbb{C}$, then $\bar{x} = \bar{x}^1 - x^2 j$.

1.2

The space of all matrices over F with k rows and n columns is denoted by $F^{k,n}$. In particular, $F^{1,n}$ and $F^{n,1}$ are spaces of row vectors and column vectors respectively.

For $x \in F^{k,n}$, we define $\bar{x} \in F^{k,n}$, $x' \in F^{n,k}$ and $x^* \in F^{n,k}$ as follows:

$$(\bar{x})_{ij} = \bar{x}_{ij}, \quad (x')_{ij} = x_{ji}, \quad (x^*)_{ij} = \overline{x_{ji}}.$$

The mapping $x \mapsto x^*$ is an involutory antiautomorphism of the ring $F^{n,n}$.

We denote by $F^{k,\infty}$ the space of all matrices over F with k rows and a countable number of columns.

1.3

We often use the following embeddings:

$$\alpha: \mathbb{C}^{k,n} \to \mathbb{R}^{2k,2n}, \quad \alpha(x) = \begin{bmatrix} \operatorname{Re} x & \operatorname{Im} x \\ -\operatorname{Im} x & \operatorname{Re} x \end{bmatrix};$$

$$\beta: \mathbb{H}^{k,n} \to \mathbb{C}^{2k,2n}, \quad \beta(x) = \begin{bmatrix} x^1 & x^2 \\ -\bar{x}^2 & \bar{x}^1 \end{bmatrix}$$

$$(x = x^1 + x^2 j \in \mathbb{H}^{k,n}, \quad x^1 \in \mathbb{C}^{k,n}, \quad x^2 \in \mathbb{C}^{k,n}).$$

If $k=n$, then α and β are morphisms of involutory rings.

1.4

The unit matrix of order n is denoted by 1_n; the unit matrix of infinite order is denoted by 1_∞. E_{ij} denotes the matrix with coefficients $(E_{ij})_{kl}=\delta_{ik}\delta_{jl}$.

1.5

We define the mapping $D: F^{n,n}\to\mathbb{R}_+$ as follows:

$$D(x)=\frac{d\mu(x\xi)}{d(\xi)} \quad \text{where } \xi\in F^{n,1}, \quad x\in F^{n,n},$$

(here μ is Lebesgue measure on $F^{n,1}$). Then

$$D(x)=|\det x| \quad \text{for} \quad F=\mathbb{R},$$

$$D(x)=\det\alpha(x)=|\det x|^2 \quad \text{for} \quad F=\mathbb{C},$$

$$D(x)=\det\alpha(\beta(x))=(\det x)^2 \quad \text{for} \quad F=\mathbb{H}$$

(the last det denotes the Dieudonné determinant).

1.6

$l^2(F)$ will denote the coordinate separable Hilbert space over F with canonical basis $e_1, e_2, \ldots,$ F^n denotes the subspace spanned by $e_i, \ldots, e_n$.

The elements of the space $l^2(F)$ will be regarded as column vectors. We shall denote the conjugate Hilbert space of row vectors by $l_2(F)$. In the case $F=\mathbb{H}$, we assume that scalar quantities act in $l^2(F)$ on the right and in $l_2(F)$ on the left.

1.7

The mark ~ indicates usually either a two-sheeted covering over a certain group or the equivalence of unitary representations.

1.8

If T is a unitary representation of a certain group, then $H(T)$ always denotes the corresponding Hilbert space. All unitary representations act in separable complex Hilbert spaces.

We shall denote by $\bar{T}$ the conjugate unitary representation. If $H=H(T)$, then $H(\bar{T})$ may be identified with $\bar{H}$ where $\bar{H}$ is obtained from H by the automorphism $z\mapsto\bar{z}$ of the field of scalars $\mathbb{C}$.

1.9

If T is a unitary representation of a certain group G, then $T(G)'$ and $T(G)''$ denote its commutant and bicommutant respectively. The latter coincides with the von Neumann algebra generated by the representation T.

1.10

If U is a compact group, then $U^\wedge$ denotes the set (of equivalence classes) of its irreducible unitary representations.

1.11

Let T be a unitary representation of a certain group G. Let us assume that in $H(T)$ unitary representation R of a certain compact group U is given, and also that R and T commute. We shall say that U is a *symmetry group* for T if $T(G)'=R(U)''$.

1.12

Let T be a unitary representation of a group G with symmetry group U. We denote by X the set of those $\pi\in U^\wedge$ which occur in the decomposition of the representation R of the group U. With any $\pi\in X$, we associate the irreducible unitary representation T_π of the group G, which is realized in the space $H(T_\pi)=\mathrm{Hom}_U(H(\pi), H(T))$. We have

$$R\otimes T\sim\bigoplus_{\pi\in X}\pi\otimes T_\pi,$$

$$T\sim\bigoplus_{\pi\in X}(\dim\pi)T_\pi,$$

where $R\otimes T$ denotes the natural representation of the group $U\times G$ in $H(T)$.

It is important to note that different representations π lead to different (non-equivalent) representations T_π.

1.13

If T is a unitary representation of a group G and $H=H(T)$, then we denote by $T^{\otimes k}$ and $H^{\otimes k}$ the k-th tensor power of the representation T and the Hilbert space H respectively; $H^{\otimes k}=H(T^{\otimes k})(k=1,2,\ldots)$.

1.14

Let T be a unitary representation of a group G. We assume that, for any $k=1,2,\ldots$, the representation $T^{\otimes k}$ possesses a symmetry group U_k. We denote by R_k the corresponding unitary representation of the group U_k and by $X_k\subseteq U_k^\wedge$ the set of irreducible representations occurring in its decomposition. Further, we assume that, for k, $l=1,2,\ldots$, there exists an embedding $U_k\times U_l\to U_{k+l}$ such that the representation $R_k\otimes R_l$ of the group $U_k\times U_l$ in $H(T^{\otimes k})\otimes H(T^{\otimes l})=H(T^{\otimes(k+l)})$ coincides with the restriction of the representation R_{k+l} to the subgroup $U_k\times U_l$.

Then, for all $\pi\in X_k$, $\sigma\in X_l$, we obviously have

$$T_\pi\otimes T_\sigma\sim\bigoplus_{\tau\in X_{k+l}} N(\tau;\pi,\sigma)T_\tau,$$

where $N(\tau;\pi,\sigma)$ is the multiplicity of entry of the representation $\pi\otimes\sigma$ in $\tau|(U_k\times U_l)$.

1.15

We shall constantly have to examine groups G that are inductive limits of their subgroups $G(n)$. This means that

$$G=\bigcup_{n=1}^{\infty} G(n),\quad \text{where}\quad G(n)\subseteq G(n+1).$$

As $G(n)$, we will have certain locally compact groups; $G(n)$ will always be a closed subgroup in $G(n+1)$. The group G is endowed with the topology of the inductive limit. All unitary representations of the group G will always be assumed to be continuous in this topology.

1.16

Let G be the inductive limit of the groups $G(n)$. We assume that for each n a unitary representation T_n of the group $G(n)$ is given and an isometric embedding $H(T_n)\to H(T_{n+1})$ commuting with the action of $G(n)$ are given.

Then in the Hilbert completion of the space

$$\bigcup_n H(T_n)$$

there arises a unitary representation T of the group G uniquely defined by

$$T(g)\xi = T_n(g)\xi, \text{ if } g\in G(n) \text{ and } \xi\in H(T_n).$$

In this situation we shall call T the *inductive limit of the sequence* $\{T_n\}$.

If T_n enters $T_{n+1}|G(n)$ exactly once, then the only arbitrariness in the selection of the embeddings $H(T_n)\to H(T_{n+1})$ is the possibility of multiplication by a scalar. The representation T does not depend on this arbitrariness.

1.17

If the representations T_n are irreducible, then even their inductive limit T is also an irreducible representation. Let us prove this simple assertion, following [12].

We denote by P_n the projector on $H(T_n)\subset H(T)$. Let $A\in T(G)'$. Then the operator $P_nA|H(T_n)$ lies in $T_n(G(n))'$ for all n and, hence, is a scalar. So A is a scalar operator.

Yet another proof is also possible. Let us select

$$\xi\in \bigcup_n H(T_n)$$

with $\|\xi\|=1$ and examine the function $\varphi(g)=(T(g)\xi, \xi)$ on G. For any n, the function $\varphi|G(n)$ is a non-decomposable positive definite function on $G(n)$. So, φ possesses the same properties. As ξ is a cyclic vector, T is irreducible.

§2. Construction of irreducible tame representations of the groups $SO(\infty)$, $U(\infty)$, $Sp(\infty)$

2.1

We shall define the groups $SO(\infty)$, $U(\infty)$, $Sp(\infty)$ as inductive limits of the groups $SO(n)$, $U(n)$, $Sp(n)$ respectively (see section 1.15). Unless otherwise stipulated, the letter K always indicates one of the groups $SO(\infty)$, $U(\infty)$, $Sp(\infty)$ and $K(n)$ is its subgroup $SO(n)$, $U(n)$, $Sp(n)$ respectively.

2.2

Let us give a more accurate definition. First we realize $K(n)$ as a group of F-linear isometric operators in $F^n \subset l^2(F)$ (see section 1.6). If $F=\mathbb{C}, \mathbb{H}$, then $K(n)$ consists of all such operators; if $F=\mathbb{R}$, it consists of operators with $\det(\cdot)=1$.

Then we regard $K(n)$ as a certain group of operators in $l^2(F)$, assuming that $K(n)$ leaves invariant the basis vectors $e_{n+1}, e_{n+2}, \ldots$. The embedding $K(n) \to K(n+1)$ is given by this. Finally, we shall get K as the union of the groups $K(n)$, $n=1, 2, \ldots$.

Thus K is a certain group of unitary operators in $l^2(F)$.

2.3

The elements $u \in K$ can conveniently be considered as infinite matrices: $u=\|u_{ij}\|$, $1 \leqslant i, j < \infty$. It is clear that $u_{ij}=\delta_{ij}$ if i or j is sufficiently large. Thus, the matrices $u-1_\infty$ are, in essence, finite.

2.4

For $n=1, 2, \ldots$, we shall examine a subgroup "complementary" to $K(n)$.

$$K_n \{u \in K \colon ue_1 = e_1, \ldots, \quad ue_n = e_n\} = \left\{ \begin{bmatrix} 1_n & 0 \\ 0 & * \end{bmatrix} \right\}.$$

For $m>n$, we put $K_n(m)=K_n\cap K(m)$. It is clear that

$$K_n=\bigcup_{m>n} K_n(m),\quad K_n(m)\simeq K(m-n),$$

Occasionally, instead of K_n, we shall write in more detail $SO_n(\infty)$, $U_n(\infty)$, $Sp_n(\infty)$.

2.5

If T is a unitary representation of the group K, then $H_n(T)$ will denote the subspace of all K_n-invariant vectors in $H(T)$. It is clear that $H_n(T)\subseteq H_{n+1}(T)$; we shall put

$$H_\infty(T)=\bigcup_{n=1}^{\infty} H_n(T).$$

This is an algebraically invariant subspace in $H(T)$.

2.6

DEFINITION. The unitary representation T of the group $K=SO(\infty)$, $U(\infty)$, $Sp(\infty)$ is called *tame*, if $H_\infty(T)$ is dense in $H(T)$.†

If T is irreducible, then it is sufficient to demand that $H_\infty(T)\neq\{0\}$.

2.7

Let $V(\lambda_1,\ldots,\lambda_n)$ denote the irreducible unitary representation of the group $U(n)$ with the highest weight $(\lambda_1,\ldots,\lambda_n)$.

Let Λ denote the set of all infinite sequences $\lambda=(\lambda_1,\lambda_2,\ldots)$, for which $\lambda_1,\lambda_2,\ldots$ are non-negative integers $\lambda_1\geqslant\lambda_2\geqslant\ldots$ and the numbers λ_i are equal to 0 if i is sufficiently large.

For any $\lambda=(\lambda_1,\lambda_2,\ldots)\in\Lambda$, we define the irreducible unitary representation ρ_λ of the group $U(\infty)$ as the inductive limit of the representations $V(\lambda_1,\ldots,\lambda_n)$ of the groups $U(n)$ (see sections 1.16, 1.17). The correctness of the definition follows from the fact that, for any n, the representation $V(\lambda_1,\ldots,\lambda_n)$ enters $V(\lambda_1,\ldots,\lambda_{n+1})|U(n)$ with multiplicity one.

†The definition given in the Introduction is equivalent to this. The proof is based on the following fact: if the unitary representation of a group only slightly shifts a certain vector ξ, then there is a fixed vector close to ξ (see the proof of theorem 20.17).

2.8

We denote by σ the identical unitary representation of the group $U(\infty)$ in Hilbert space $l^2(\mathbb{C})$. For any $m=1, 2, \ldots$, there is a natural action of the symmetric group $S(m)$ in $H(\sigma^{\otimes m})=H(\sigma)^{\otimes m}$: s^{-1}: $\xi_1 \otimes \ldots \otimes \xi_m \mapsto \xi_{s(1)} \otimes \ldots \otimes \xi_{s(m)} (s \in S(m))$.

LEMMA. For any $m=1, 2, \ldots$, the group $S(m)$ is a symmetry group (see section 1.11) for the representation $\sigma^{\otimes m}$ of the group $U(\infty)$.

PROOF. The validity of the analogous statement for the identical representation of the group $U(n)$ in $\mathbb{C}^n \subset l^2(\mathbb{C})$ is well known. To derive our lemma from this, it is sufficient to prove that $H_n(\sigma^{\otimes m})$ coincides with $(\mathbb{C}^n)^{\otimes m}$.

It is clear that the second space is contained in the first. To check the inverse inclusion, it is necessary to observe that any basis vector $e_{i_1} \otimes \ldots \otimes e_{i_m}$ from $H(\sigma)^{\otimes m}$ is an eigenvector for the subgroup $D \subset U(\infty)$ of diagonal matrices and that it is invariant with respect to $D \cap U_n(\infty)$ if and only if $i_1 \leqslant n, \ldots, i_m \leqslant n$.

2.9

COROLLARY. For any $m=1, 2, \ldots$, we have

$$\sigma^{\otimes m} \sim \bigoplus_{\lambda \in \Lambda}, \ _{\lambda_1+\lambda_2+\ldots=m} \dim[\lambda_1, \ldots, \lambda_m] \cdot \rho_\lambda ,$$

where $[\lambda_1, \ldots, \lambda_m]$ denotes the irreducible representation of the group $S(m)$ associated with the partition $m=\lambda_1+\ldots+\lambda_m$.

2.10

COROLLARY. Let $\lambda \in \Lambda$. If n is so large that $\lambda_{n+1}=\lambda_{n+2}=\ldots=0$, then the subspace $H_n(\rho_\lambda)$ is nontrivial and coincides with $H(V(\lambda_1, \ldots, \lambda_n))$.

2.11

COROLLARY. ρ_λ is a tame representation for all $\lambda \in \Lambda$. If $\lambda \neq \mu$, then ρ_λ and ρ_μ are not equivalent.

2.12

Sometimes it will be convenient to examine the group $U(\infty)$ as an inductive limit of the groups $U(2n)$; in that case we shall denote it by $U(2\infty)$. Similarly, the space $l^2(\mathbb{C})$ will be represented then as $l^2(\mathbb{C})\oplus l^2(\mathbb{C})$; the group $U(2\infty)$ will act in it. For definiteness, it may be assumed that $l^2(\mathbb{C})\oplus l^2(\mathbb{C})$ is identified with $l^2(\mathbb{C})$ by the mapping:

$$(e_i, 0)\mapsto e_{2i-1}, (0, e_i)\mapsto e_{2i} \quad (i=1, 2, \ldots).$$

The elements of the group $U(2\infty)$ will be written in the form of 2×2 matrices

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

where a, b, c, d are infinite matrices (or operators in $l^2(\mathbb{C})$).

2.13

For $K=SO(\infty)$, $U(\infty)$, $Sp(\infty)$, let us define the group K^* as $U(\infty)$, $U(\infty)\times U(\infty)$, $U(2\infty)$, respectively. Let us define an embedding $K\rightarrow K^*$ as follows.

In the case $F=\mathbb{R}$, it is the identical mapping $SO(\infty)\rightarrow U(\infty)$.

In the case $F=\mathbb{C}$, it is the mapping $g\mapsto(g, \bar{g})$.

In the case $F=\mathbb{H}$, it is the mapping β (see section 1.3).

We observe that the automorphism $g\mapsto\bar{g}$ of the group $U(\infty)$ (see section 1.2) transforms any $\rho_\lambda(\lambda\in\Lambda)$ into the conjugate representation $\bar{\rho}_\lambda$.

2.14

Let us define the representation $\rho=\rho_\lambda^{\mathbb{R}}, \rho_{\lambda,\mu}, \rho_\lambda^{\mathbb{H}}$ of the group $K=SO(\infty)$, $U(\infty)$, $Sp(\infty)$ as the restriction of the representation $\rho^*=\rho_\lambda, \rho_\lambda\otimes\rho_\mu, \rho_\lambda$ of the group K^* respectively (here $\lambda, \mu\in\Lambda$). We observe that $\rho_{\lambda,\mu}=\rho_\lambda\otimes\bar{\rho}_\mu$ and recall that $U(2\infty)$ is identified with $U(\infty)$ (see section 2.12), and hence ρ_λ may be considered as its representation.

Let us define the subgroup $K_n^*\subset K^*$ as respectively $U_n(\infty)$, $U_n(\infty)\times U_n(\infty)$, $U_{2n}(\infty)$ (see the end of section 2.4). We denote by $H_n(\rho^*)$ the subspace of all K_n^*-invariant vectors in $H(\rho^*)$.

It is obvious that $H_n(\rho)\supseteq H_n(\rho^*)$. Hence ρ is a tame representation.

2.15

THEOREM. The tame representations ρ of the groups $SO(\infty)$, $U(\infty)$, $Sp(\infty)$, constructed in section 2.14, are irreducible. Moreover, the different values of the parameter λ (or the different pairs (λ, μ)) yield nonequivalent representations.

This follows at once from theorem 2.17 (see below).

2.16

DEFINITION. A unitary representation of the group $U(\infty)$, $(U(\infty)\times U(\infty)$ respectively), is called *holomorphic* if it is a direct sum of any number of representations of the type ρ_λ, $\lambda\in\Lambda$ $(\rho_\lambda\otimes\rho_\mu$ respectively, where $\lambda, \mu\in\Lambda)$.

2.17

THEOREM. Let T^* be an arbitrary holomorphic representation of the group K^* (see sections 2.16 and 2.13) and $T=T^*|K$. Then (see section 1.9)

$$T(K)'=T^*(K^*)', \quad T(K)''=T^*(K^*)''.$$

In particular, if T^* is irreducible, then T is also irreducible.

PROOF. We examine only the case $F=\mathbb{R}$, since for $F=\mathbb{C}$, $\mathbb{H}$ the proof is identical. Thus, $K=SO(\infty)$, $K^*=U(\infty)$. We denote by Γ (respectively Γ^*) the set of all operators in $l^2(\mathbb{R})(l^2(\mathbb{C}))$ with norm $\leqslant 1$ (see section 1.6).

Γ and Γ^* are semigroups under multiplication. In addition, they are involutory semigroups (i.e., semigroups provided with an involution; conjugation of the operators is such an involution). We shall consider Γ as a subsemigroup in Γ^*.

Let us topologize Γ and Γ^* by the weak operator topology and let us note that K is dense in Γ and K^* is dense in Γ^*.

Let $H=H(T^*)$ and $\Gamma(H)$ be the involutory semigroup of all operators in H with norm $\leqslant 1$ topologized by the weak topology.

We observe that T^* can be uniquely extended to a continuous morphism τ^*: $\Gamma^*\to\Gamma(H)$ preserving the involution. In fact, if this assertion holds for a certain set of representations T^*, then it holds also for their direct sum. Hence it is sufficient to examine the case $T^* = \sigma^{\otimes m}$, where $m = 1, 2, \ldots$. But for this case our statement can be verified in a trivial way.

Similarly, T extends to a continuous morphism $\tau:\Gamma \to \Gamma(H)$.

It is clear that $T(K)'' = \tau(\Gamma)'', T^*(K^*)'' = \tau^*(\Gamma^*)''$. Hence it is sufficient to check that $\tau(\Gamma)' = \tau^*(\Gamma^*)'$.

We observe that Γ^* may be considered to be the unit sphere in the Banach space B of all bounded operators in $l^2(\mathbb{C})$. Hence the following definition makes sense. We shall say that a continuous function f on Γ^* is *holomorphic* if, for any finite dimensional subspace $M \subset B$, the function f is holomorphic (in the usual sense) in the open unit sphere of the space M. It is evident that the space of holomorphic functions on Γ^* is complete with respect to uniform convergence.

We now observe that for any $\xi, \eta \in H$ the function $f_{\xi,\eta}(\gamma) = (\tau^*(\gamma)\xi, \eta)$ is holomorphic on $\Gamma^*(\gamma \in \Gamma^*)$. In fact this is obvious if $T^* = \sigma^{\otimes m}$ and ξ and η are basis vectors of the type $e_{i_1}, \otimes \ldots, e_{i_m}$. After this, it remains to use the completeness of the space of holomorphic functions and the fact that $\|\tau^*(\gamma)\| \leqslant 1$.

Let us assume now that $H = H_1 \oplus H_2$ is a certain orthogonal decomposition, invariant with respect to $\tau(\Gamma)$. Then, for any $\xi \in H_1$, $\eta \in H_2$, we have $f_{\xi,\eta}|\Gamma \equiv 0$. But as is easily verified, Γ is a set of uniqueness for holomorphic functions on Γ^*. It means $f_{\xi,\eta} \equiv 0$. This discussion shows that our decomposition is invariant with respect to $\tau^*(\Gamma^*)$. Hence $\tau(\Gamma)' = \tau^*(\Gamma^*)'$. which proves the theorem.

2.18

Corollary. In the notations of theorem 2.17 and section 2.14, we have: $H_n(T) = H_n(T^*)$ for all $n = 1, 2, \ldots$.

In fact, $K_n^* \subset K^*$ is isomorphic to the group K^* and $T^*|K_n^*$ is its holomorphic representation.

2.19

Remark. We shall denote by $\bar{K}$ the group of all F-linear unitary operators in $l^2(F)$. $\bar{K}$ is a topological group with respect to the weak operator topology (which coincides on $\bar{K}$ with the strong operator topology). The group K is dense in $\bar{K}$.

It is evident that all the irreducible representations of the group K, constructed in section 2.14 (just like the direct sums of such representations) may be extended to continuous unitary representations of the group $\bar{K}$.

2.20

THEOREM. As usual, let $K = SO(\infty), U(\infty), Sp(\infty)$.

(i) The irreducible representations of the group K, constructed in section 2.14, exhaust all its irreducible tame representations.

(ii) An arbitrary tame representation of the group K is a discrete direct sum of irreducible tame representations.

(iii) The tame representations of the group K are precisely the restrictions of the continuous unitary representations of the group $\bar{K}$.

In fact, we shall not use these results. (i) and (ii) are proved in [20], [25]; (iii) is easily proved by the methods given in [20] (see, in particular, lemma 3.5 in [20]).

2.21

DEFINITION. We shall say that the unitary representation T^* of the group K^* is a *holomorphic extension* of the tame representation T of the group K if T^* is holomorphic in the sense of definition 2.16 and $T^*|K = T$.

The existence of the holomorphic extension follows from statements (i) and (ii) of theorem 2.20; however, for the representations constructed in section 2.14, this is obvious. The uniqueness of the holomorphic extension follows easily from theorem 2.17.

2.22

Remark. We see that the category of tame representations of the group K and the category of holomorphic representations of the group K^* are equivalent.

We observe the following remarkable property of the holomorphic representations. If T^* is a holomorphic representation of the group $U(\infty)$, P_n is the projector on $H_n(T^*) \subset H(T^*)$ and T_n^* is the natural representation of the group $U(n)$ in $H_n(T^*)$, then

$P_n T^*(U(\infty))'' | H_n(T^*) = T_n^*(U(n))''$ $(n = 1, 2, \ldots)$. (An analogous fact is true also for the group $U(\infty) \times U(\infty)$.)

This shows that it is easier to solve the problem of the decomposition of a given tame representation T of the group K if we turn to the holomorphic extension T^*. For an example of such an approach, see §4 (sections 4.6 and 4.7).

2.23

As in section 2.14, let ρ denote one of the representations $\rho_\lambda^{\mathbb{R}}$, $\rho_{\lambda,\mu}$, generally speaking, is strictly less than $H_\infty(\rho)$.

THEOREM. The representation ρ can be represented as the inductive limit of irreducible representations $\rho^{(n)}$ of the groups $K(n)$. The representations $\rho^{(n)}$ are uniquely defined to within a finite number of the integers n. If n is sufficiently large, then the highest weight of the representation $\rho^{(n)}$ is

$$(\lambda_1, \lambda_2, \ldots, 0) \text{ or } (\lambda_1, \lambda_2, \ldots, 0, \ldots, 0, -\mu_2, -\mu_1)$$

for $F = \mathbb{R}$, $\mathbb{H}$ or for $F = \mathbb{C}$ respectively; moreover, $\rho^{(n)}$ enters into $\rho | K(n)$ exactly once.

This theorem is proved in [20]. In fact, we shall not use it.

2.24

Remark. The subspace $H(\rho^{(n)})$ lies in $H_n(\rho)$, but, generally speaking, does not coincide with it. Moreover, the subspace

$$\bigcup_n H(\rho^{(n)}),$$

generally speaking, is strictly less than $H_\infty(\rho)$.

§3. Fock space (boson case)

3.1

Let H be a complex Hilbert space; $\bar{H}$ be the conjugate Hilbert space (section 1.8), $U(H) = U(\bar{H})$ be the group of all unitary operators in H;

$S^m(H)$ be the pre-Hilbert space which is an algebraic symmetric tensor power of the space H; the scalar product in $S^m(H)$ is given by the condition

$$\|\xi\otimes \ldots \otimes\xi\|^2=(m!)\|\xi\|^{2m}\ (\xi\in H).$$

DEFINITION. *By the Boson Fock space $\mathscr{F}(H)=\mathscr{F}^+(H)$ over H* we mean the completion of the space $\mathbb{C}\oplus H\oplus S^2(H)\oplus \ldots$

A natural unitary representation of the group $U(H)$ is defined in $\mathscr{F}(H)$. If $H=H_1\oplus H_2$, then $\mathscr{F}(H)=\mathscr{F}(H_1)\otimes\mathscr{F}(H_2)$.

3.2

Let H be finite-dimensional. The $\mathscr{F}(H)$ may be identified with the space of all entire functions $f(\xi)$ on $\bar{H}$ such that

$$\|f\|^2 \stackrel{\text{def}}{=} \pi^{-\dim H}\int|f(\xi)|^2\exp(-\|\xi\|^2)d\xi<+\infty,$$

where $d\xi$ is Lebesgue measure on $H\simeq\mathbb{R}^{2\dim H}$.

Let us identify H with $\mathbb{C}^{n,1}$ ($n=\dim H$) by using any orthonormal basis. Then $\bar{H}=\mathbb{C}^{1,n}$ and the identification map $\bar{H}\to H$ is $\xi\mapsto\xi^*$. Let $z_1,\ldots,z_n$ be canonical coordinates in $\mathbb{C}^{1,n}$. Then the elements of the space $\mathscr{F}(H)$ are entire functions $f(z_1,\ldots,z_n)$. The subspace $\mathbb{C}[z_1,\ldots,z_n]$ of polynomials is dense in $\mathscr{F}(H)=\mathscr{F}(\mathbb{C}^{n,1})$ and the monomials

$$(m_1!\ldots m_n!)^{-1/2}z_1^{m_1}\ldots z_n^{m_n}$$

form an orthonormal basis in $\mathscr{F}(\mathbb{C}^{n,1})$. The space $\mathscr{F}(\mathbb{C}^{n,1})$ in this concrete realization will be called the *Bargmann–Segal space of the variables* $z_1,\ldots,z_n$ and will be denoted by $\mathscr{H}(z_1,\ldots,z_n)$ or $\mathscr{H}(\mathbb{C}^{1,n})$.

We shall denote the Bargmann–Segal space of the variables $z_{\alpha,i}$ where $1\leqslant\alpha\leqslant K$, $1\leqslant i\leqslant n$, by $\mathscr{H}(\mathbb{C}^{k,n})$ or $\mathscr{H}(\{z_{\alpha,i}\},\ \alpha\leqslant k,\ i\leqslant n)$.

3.3

A canonical unitary representation $T_{\text{hol}}^{(n)}$ of the group $U(n)$ in $\mathscr{H}(\mathbb{C}^{1,n})$ is given as follows:

$$T_{\text{hol}}^{(n)}(u)f(z)=f(zu),$$

where

$$z=(z_1, \ldots, z_n)\in\mathbb{C}^{1,n}, \quad u\in U(n), \quad f\in\mathscr{H}(\mathbb{C}^{1,n})=\mathscr{F}(\mathbb{C}^{n,1}).$$

This representation is the direct sum of the irreducible representations $V(m, 0, \ldots, 0)$, $m=0,1,\ldots$ (see section 2.7).

3.4

The Fock space $\mathscr{F}(l^2(\mathbb{C}))$ may be identified with the completion of the space

$$\bigcup_{n=1}^{\infty}\mathscr{H}(z_1, \ldots, z_n);$$

the embedding $\mathscr{H}(z_1, \ldots, z_n)\rightarrow\mathscr{H}(z_1, \ldots, z_{n+1})$ is given by the fact that any function of the variables $z_1, \ldots, z_n$ can be considered as a function of the variables $z_1, \ldots, z_{n+1}$, not depending explicitly on z_{n+1}. The space of polynomials

$$\mathbb{C}[z_1, z_2, \ldots]=\bigcup_{n=1}^{\infty}\mathbb{C}[z_1, \ldots, z_n]$$

is dense in $\mathscr{F}(l^2(\mathbb{C}))$.

Another important subspace in $\mathscr{F}(l^2(\mathbb{C}))$ is

$$\bigcup_{n=1}^{\infty}\mathscr{F}(\mathbb{C}^{n,1})=\bigcup_{n=1}^{\infty}\mathscr{H}(z_1, \ldots, z_n).$$

We shall call its elements *cylindrical functions.*

We observe further that $\mathscr{F}(l^2(\mathbb{C}))$ may be identified with the tensor product (in the sense of von Neumann [16]) of a denumerable number of copies of the space $\mathscr{F}(\mathbb{C}^{1,1})$. The function $f_0\equiv 1$ is taken to be the distinguished vector $f_0\in\mathscr{F}(\mathbb{C}^{1,1})$.

3.5

We shall call the space $\mathscr{F}(l^2(\mathbb{C}))$ the *Bargmann–Segal space of the variables* $z_1, z_2, \ldots$. We shall denote it by $\mathscr{H}(z_1, z_2, \ldots)$, $\mathscr{H}(l_2(\mathbb{C}))$ or $\mathscr{H}(\mathbb{C}^{1,\infty})$. We shall denote the natural unitary representation of the group $U(\infty)$ in $\mathscr{H}(z_1, z_2, \ldots)$ by T_{hol}. It is a tame representation, equivalent to the direct sum of the representations $\rho_{(m,0,0,\ldots)}$, $(m=0, 1, \ldots)$. T_{hol} may be considered also as the inductive limit of the representations $T_{\text{hol}}^{(n)}$ of the groups $U(n)$. We observe that $H_n(T_{\text{hol}})$ coincides with $\mathscr{H}(z_1, \ldots, z_n)$.

3.6

Let us introduce the operators M_i and $D_i (1 \leqslant i < \infty)$ in the space $\mathbb{C}[z_1, z_2, \ldots] \subset \mathscr{H}(z_1, z_2, \ldots)$ as follows:

$$M_i f = z_i f, \quad D_i f = \frac{\partial}{\partial z_i} f \quad (f \in \mathbb{C}[z_1, z_2, \ldots]).$$

It is easy to check that

$$(M_i f_1, f_2) = (f_1, D_i f_2), \quad (i = 1, 2, \ldots; f_1, f_2 \in \mathbb{C}[z_1, z_2, \ldots])$$

$$[D_i, M_j] = \delta_{ij} 1 \quad (i, j = 1, 2, \ldots).$$

3.7

Consider the Lie algebra

$$\mathfrak{gl}(\infty, \mathbb{C}) = \bigcup_{n=1}^{\infty} \mathfrak{gl}(n, \mathbb{C}).$$

There is a canonical basis $\{E_{ij}\}$ (see section 1.4) in it. Let us define a representation of the Lie algebra $\mathfrak{gl}(\infty, \mathbb{C})$ in $\mathbb{C}[z_1, z_2, \ldots]$ as follows:

$$E_{ij} \mapsto M_i D_j \quad (i, j = 1, 2, \ldots).$$

This representation is the differential of the representation T_{hol}, restricted to $\mathbb{C}[z_1, z_2, \ldots]$.

3.8

The space $\mathscr{F}(l^2(\mathbb{C}) \oplus l^2(\mathbb{C}))$ will be identified with the Bargmann–Segal space of the variables $z_{\hat{1}}, z_{\hat{2}}, \ldots, z_1, z_2, \ldots$, and denoted by $\mathscr{H}(z_{\hat{1}}, z_{\hat{2}}, \ldots, z_1, z_2, \ldots))$. The representation T_{hol} of the group $U(2\infty)$ (see section 2.12) acts in it. The operators $M_{\hat{i}}$, $D_{\hat{i}}$, M_i,

D_i ($i=1, 2, \ldots$) are defined (with respect to the coordinates z_i, $\bar{z}_i$) as in section 3.6.

3.9

The space $H(T_{\mathrm{hol}}^{\otimes k})$ is $\mathscr{F}(l^2(\mathbb{C}))^{\otimes k}$. We shall identify it with the Bargmann–Segal space of the variables $z_{\alpha i}(1 \leqslant \alpha \leqslant k, i=1, 2, \ldots)$ and denote it by $\mathscr{H}(\{z_{\alpha i}\})$ or $\mathscr{H}(\mathbb{C}^{k,\infty})$. The operators $M_{\alpha i}$, $D_{\alpha i}$ have an obvious meaning. The representation of the Lie algebra $\mathfrak{gl}(\infty, \mathbb{C})$ in the subspace of polynomials $\mathbb{C}[\{z_{\alpha i}\}]$ is given as follows:

$$E_{ij} \mapsto \sum_{\alpha=1}^{k} M_{\alpha i} D_{\alpha j} \quad (i, j=1, 2, \ldots).$$

We observe that $H_n(T_{\mathrm{hol}}^{\otimes k})$ may be identified with

$$\mathscr{H}(\mathbb{C}^{k,n})=\mathscr{H}(\{z_{\alpha i}\}, \quad \alpha \leqslant k, i \leqslant n).$$

3.10

Let us define in $H(T_{\mathrm{hol}}^{\otimes k})=\mathscr{H}(\{z_{\alpha i}\})$ the representation R_k of the group $U(k)$ as follows:

$$R_k(v)f(z)=f(v^{-1}z)(v \in U(k), z=\|z_{\alpha i}\| \in \mathbb{C}^{k,\infty})$$

(here f is an arbitrary cylindrical function from $\mathscr{H}(\{z_{\alpha i}\})$.

The corresponding representation of the Lie algebra $\mathfrak{gl}(k, \mathbb{C})$ in the space of polynomials has the following form:

$$E_{\alpha\beta} \mapsto -\sum_{i=1}^{\infty} M_{\beta i} D_{\alpha i} (\alpha, \beta=1, \ldots, k).$$

3.11

Lemma. $U(k)$ is a symmetry group for the representation $T_{\mathrm{hol}}^{\otimes k}$ of the group $U(\infty)$. The irreducible components of the representation

R_k are precisely all representations of the form $V(-\lambda_k, \ldots, -\lambda_1)$, where $\lambda_k \geqslant 0$; the corresponding irreducible representations of the group $U(\infty)$ are $\rho_{(\lambda_1, \ldots, \lambda_k, 0, 0, \ldots)}$. We have

$$T_{\text{hol}}^{\otimes k} \sim \bigoplus_{(\lambda_1, \ldots, \lambda_k),\, \lambda_k \geqslant 0} \dim V(\lambda_1, \ldots, \lambda_k) \cdot \rho_{(\lambda_1, \ldots, \lambda_k, 0, \ldots)} .$$

PROOF. It is well known that an analogous statement is true for the group $U(n)$ $(n \geqslant k)$; see [43]. Since $H_n(T_{\text{hol}}^{\otimes k})$ coincides with $\mathscr{H}(\{z_{\alpha i}\}, i \leqslant n)$, the assertion of the lemma follows at once.

§4. The space $L^2(F^{k,\infty})$

4.1

Let us examine the following Gaussian probability measure on $F = \mathbb{R}, \mathbb{C}, \mathbb{H}$:

$$d\nu(x) = (2\pi)^{-d/2} \exp(-|x|^2/2)dx$$

(here dx is Lebesgue measure on F, $d = \dim_{\mathbb{R}} F$).

Let us provide the space $F^{k,n}$ (see section 1.2) with the Gaussian product-measure $\mu = \nu \otimes \ldots \otimes \nu$ (kn times); here we are identifying $F^{k,n}$ with $F \times \ldots \times F$ (kn times). We have

$$d\nu(x) = (2\pi)^{-knd/2} \exp\left(-\frac{1}{2}\sum_{\alpha=1}^{k} x_\alpha x_\alpha^*\right) dx,$$

where x_α denotes the α-th row of the matrix $x \in F^{k,n}$.

Similarly, the space $F^{k,\infty}$ (see section 1.2) is provided with a Gauss product-measure $\nu \otimes \nu \otimes \ldots$, which we again denote by μ.

4.2

Let

$$L^2(F^{k,n}) = L^2(F^{k,n}, \mu),\ L^2(F^{k,\infty}) = L^2(F^{k,\infty}, \mu).$$

We shall identify $L^2(F^{k,n})$ with the subspace of those functions $f \in L^2(F^{k,\infty})$ which depend only on the coordinates $x_{\alpha i}$ with $i \leqslant n$.

Subspace

$$\bigcup_{n=1}^{\infty} L^2(F^{k,n})$$

will be called the *subspace of cylindrical functions* in $L^2(F^{k,\infty})$; it is dense in $L^2(F^{k,\infty})$.

4.3

We note that $L^2(F^{k,n})$ may be identified with $L^2(F)^{\otimes kn}$. Similarly, $L^2(F^{k,\infty})$ may be identified with the infinite tensor product of a denumerable number of the spaces $L^2(F)$ corresponding to the coordinates $x_{\alpha i}$, where $1 \leqslant \alpha \leqslant k$, $1 \leqslant i < \infty$. The function $f_0 \equiv 1$ is selected as the distinguished vector.

4.4

For $k=1, 2, \ldots$, we shall define in $L^2(F^{k,\infty})$ the unitary representation T_F^k of the group $K=SO(\infty)$, $U(\infty)$, $Sp(\infty)$ as follows:

$$T_F^k(u)\,f(x)=f(xu), \text{ where } x\in F^{k,\infty},\ u\in K',\ f\in L^2(F^{k,\infty}).$$

We put $T_F=T_F^1$. It is clear that $T_F^k=T_F^{\otimes k}$.

4.5

Lemma. T_F^k is a tame representation and thus (see theorem 2.20 (iii)) admits a continuous extension to the group $\bar{K}$ of all unitary operators in $l^2(F)$.

Proof. $H_\infty(T_F^k)$ obviously contains the space of cylindrical functions (in fact, it coincides with it, see 4.17). Hence it is dense in $H(T_F^k)$.

4.6

By virtue of lemma 4.5, the representation T_F^k possesses a holomorphic extension $(T_F^k)^*$ to the group $K^*=U(\infty)$, $U(\infty)\times U(\infty)$, $U(2\infty)$ (see section 2.21).

THEOREM. The representation $(T^k_F)^*$ of the group K^* is equivalent to the representation

$$T^{\otimes k}_{\text{hol}},\ T^{\otimes k}_{\text{hol}} \otimes \overline{T}^{\otimes k}_{\text{hol}},\ T^{\otimes 2k}_{\text{hol}}$$

for $F = \mathbb{R}, \mathbb{C}, \mathbb{H}$ respectively.

The representation T_{hol} is defined in sections 3.5, 3.8. The proof is given in sections 4.10, 4.13, and 4.16 for $F = \mathbb{R}, \mathbb{C}, \mathbb{H}$ respectively.

4.7

COROLLARY. For $k = 1, 2, \ldots$, we have

$$T^k_{\mathbb{R}} \sim \bigoplus_{(\lambda_1, \ldots, \lambda_k),\, \lambda_k \geq 0} \dim V(\lambda_1, \ldots, \lambda_k) \cdot \rho^{\mathbb{R}}_{(\lambda_1, \ldots, \lambda_k, 0, 0, \ldots)},$$

$$T^k_{\mathbb{C}} \sim \bigoplus_{\substack{(\lambda_1, \ldots, \lambda_k),\, \lambda_k \geq 0 \\ (\mu_1, \ldots, \mu_k),\, \mu_k \geq 0}} \dim V(\lambda_1, \ldots, \lambda_k) \dim V(\mu_1, \ldots, \mu_k)$$

$$\times \rho_{(\lambda_1, \ldots, \lambda_k, 0, 0, \ldots)(\mu_1, \ldots, \mu_k, 0, 0 \ldots)}$$

$$T^k_{\mathbb{H}} \sim \bigoplus_{(\lambda_1, \ldots, \lambda_{2k}),\, \lambda_{2k} \geq 0} \dim V(\lambda_1, \ldots, \lambda_{2k})\, \rho^{\mathbb{H}}_{(\lambda_1, \ldots, \lambda_{2k}, 0, 0, \ldots)}.$$

This follows immediately from theorem 4.6 and lemma 3.11.

4.8

For k, $n = 1, 2, \ldots$, let us consider the following integral transform which sends functions on $\mathbb{R}^{k,n}$ into entire functions on $\mathbb{C}^{k,n}$:

$$(I_{\mathbb{R}}\varphi)(z) = \exp\left(-\frac{1}{2}\operatorname{tr} z z'\right) \int_{\mathbb{R}_{k,n}} \varphi(x) \exp(\operatorname{tr}(zx'))\, d\mu(x),$$

where μ is a Gaussian measure (see section 4.1).

Let $P(\mathbb{R}^{k,n})$ and $P(\mathbb{C}^{k,n})$ denote the spaces of polynomials in the coordinates $x_{\alpha i} \in \mathbb{R}$, $z_{\alpha i} \in \mathbb{C}$ respectively ($1 \leq \alpha \leq k$, $1 \leq i \leq n$).

4.9

LEMMA.

(i) The integral giving $I_{\mathbb{R}}$ converges for all $\varphi \in P(\mathbb{R}^{k,n})$.

(ii) $I_{\mathbb{R}}$ carries out a bijection of the space $P(\mathbb{R}^{k,n})$ onto the space $P(\mathbb{C}^{k,n})$.

(iii) Let us examine $P(\mathbb{R}^{k,n})$ as a subspace in $L^2(\mathbb{R}^{k,n})$ and $P(\mathbb{C}^{k,n})$ as a subspace in $\mathscr{H}(\mathbb{C}^{k,n}) = \mathscr{H}(\{z_{\alpha i}\}, i \leqslant n)$. $I_{\mathbb{R}}$ preserves scalar products and can thus be extended to the isometry

$$I_{\mathbb{R}}: L^2(\mathbb{R}^{k,n}) \to \mathscr{H}(\mathbb{C}^{k,n})$$

of Hilbert spaces.

(iv) $I_{\mathbb{R}}$ is compatible with the decompositions (into a tensor product) of $L^2(\mathbb{R}^{k,n}) = L^2(\mathbb{R})^{\otimes kn}$, $\mathscr{H}(\mathbb{C}^{k,n}) = \mathscr{H}(\mathbb{C})^{\otimes kn}$, i.e., it is a tensor product of one-dimensional transformations.

PROOF. First let $k = n = 1$. Let us consider the slightly modified Hermite polynomials

$$\varphi_m(x) = 2^{-m/2}(m!)^{-1/2} H_m(x/\sqrt{2}) = (-1)^m (m!)^{-1/2}\, e^{x^2/2} \frac{d^m}{dx^m}\, e^{-x^2/2}$$

and the monomials

$$\varphi'_m(z) = (m!)^{-1/2} z^m \; (m = 0, 1, 2, \ldots, x \in \mathbb{R}, z \in \mathbb{C}).$$

They constitute an orthonormal basis in $L^2(\mathbb{R})$ and $\mathscr{H}(\mathbb{C})$ respectively. It is easy to verify that $I_{\mathbb{R}} \varphi_m = \varphi_m'$. This proves the lemma in the case of $k = n = 1$. The general case easily follows from this.

4.10

Proof of theorem 4.6 (*the case* $F = \mathbb{R}$). Let us define the isometry

$$I_{\mathbb{R}}: L^2(\mathbb{R}^{k,\infty}) \to \mathscr{H}(\mathbb{C}^{k,\infty}) = H(T_{\text{hol}}^{\otimes k})$$

as the inductive limit of the isometries constructed in section 4.8, or, what is the same thing, as infinite tensor products of integral transforms with respect to each coordinate $x_{\alpha i}$ $(1 \leqslant \alpha \leqslant k, 1 \leqslant i < \infty)$.

It is evident that $I_{\mathbb{R}}$ commutes with the action of the group $SO(\infty)$. Thus $I_{\mathbb{R}}$ allows the identification of $T_{\mathbb{R}}^k$ with $T_{\text{hol}}^{\otimes k}|SO(\infty)$.

4.11

Let us examine the following integral transform, which sends functions on $\mathbb{C}^{k,n}$ into entire functions on $\mathbb{C}^{k,2n}=\mathbb{C}^{k,n}\times\mathbb{C}^{k,n}$:

$$(I_{\mathbb{C}}\varphi)(\hat{z}, z)=\exp(-i\operatorname{tr}(\hat{z}\, z'))$$
$$\cdot\int_{\mathbb{C}^{k,n}}\varphi(x)\exp\left(\frac{1}{\sqrt{2}}\operatorname{tr}(\hat{z}\, x'+i\, zx^*)\right)d\mu(x),$$

where $\hat{z}, z\in\mathbb{C}^{k,n}$ and μ is a Gaussian measure (section 4.1).

Let us identify $\mathbb{C}^{k,n}$ with $\mathbb{R}^{k,2n}=\mathbb{R}^{k,n}\times\mathbb{R}^{k,n}$ by means of mapping $x\mapsto(\operatorname{Re} x, \operatorname{Im} x)$.

4.12

Lemma.

(i) In this identification, we have

$$(I_{\mathbb{C}}\varphi)(\hat{z}, z)=(I_{\mathbb{R}}\varphi)((\hat{z}, z)\cdot u_n),$$

where

$$u_n=\frac{1}{\sqrt{2}}\begin{bmatrix}1_n & i1_n\\ i1_n & 1_n\end{bmatrix}\in U(2n).$$

(ii) $I_{\mathbb{C}}$ carries out a bijection of the space of polynomials in the real coordinates $\operatorname{Re} x_{ai}$, $\operatorname{Im} x_{ai}$ of the matrix $x\in\mathbb{C}^{k,n}$ onto the space of polynomials in the coordinates $\hat{z}_{ai}$, z_{ai} of the matrices $\hat{z}, z\in\mathbb{C}^{k,n}$.

(iii) $I_{\mathbb{C}}$ carries out an isometry of the space $L^2(\mathbb{C}^{k,n})$ onto the Bargmann–Segal space

$$\mathscr{H}(\mathbb{C}^{k,2n})=\mathscr{H}(\mathbb{C}^{k,n})\otimes\mathscr{H}(\mathbb{C}^{k,n}).$$

Proof. (i) is verified by a simple calculation; (ii) and (iii) follow from (i) and lemma 4.9.

4.13

Proof of theorem 4.6 (*the case* $F=\mathbb{C}$). Let us define the isometry

$$I_{\mathbb{C}}: L^2(\mathbb{C}^{k,\infty}) \to \mathscr{H}(\mathbb{C}^{k,\infty}) \otimes \mathscr{H}(\mathbb{C}^{k,\infty}) = H(T_{\text{hol}}^{\otimes k} \otimes T_{\text{hol}}^{\otimes k})$$

as the inductive limit of the isometries constructed in sections 4.11, 4.12.

We observe that $\operatorname{tr}(\hat{z}z')$ and $\operatorname{tr}(\hat{z}x' + izx^*)$ from section 4.11 do not change under the transformation

$$(\hat{z}, z, x) \mapsto (\hat{z}\bar{u}, zu, xu)(u \in U(n)).$$

This shows that the isometry

$$L^2(\mathbb{C}^{k,n}) \to \mathscr{H}(\mathbb{C}^{k,n}) \otimes \mathscr{H}(\mathbb{C}^{k,n})$$

commutes with the action of the group $U(n)$; we assume that $U(n)$ acts in the second space like the subgroup of the pairs of the type $(\bar{u}, u)$ in the group $U(n) \times U(n)$.

Hence, $I_{\mathbb{C}}$ allows the identification of $T_{\mathbb{C}}^k$ with the representation $(T_{\text{hol}}^{\otimes k} \otimes T_{\text{hol}}^{\otimes k}) | \{(\bar{u}, u); u \in U(\infty)\}$, which proves the theorem.

4.14

Let us examine the following integral transform, sending functions on $\mathbb{H}^{k,n}$ into entire functions on $\mathbb{C}^{2k,2n}$. Let us put

$$\zeta = \begin{bmatrix} \zeta^{11} & \zeta^{12} \\ \zeta^{21} & \zeta^{22} \end{bmatrix}, \quad \text{where} \quad \zeta^{pq} \in \mathbb{C}^{k,n} \text{ and } p, q = 1,2;$$

$$x = x^1 + x^2 j \in \mathbb{H}^{k,n} \text{ where } x^1, x^2 \in \mathbb{C}^{k,n}.$$

Then, by the definition,

$$(I_{\mathbb{H}}\varphi)(\zeta) = \exp(-iA(\zeta)) \int_{\mathbb{H}^{k,n}} \varphi(x) \exp\left(\frac{1}{\sqrt{2}} B(\zeta, x)\right) d\mu(x),$$

where μ is a Gaussian measure on $\mathbb{H}^{k,n}$ (section 4.1) and

$$A(\zeta)=\mathrm{tr}(\zeta^{11}(\zeta^{22})'-\zeta^{12}(\zeta^{21})'),$$

$$B(\zeta, x)=\mathrm{tr}(\zeta^{11}(x^2)'-\xi^{12}(x^1)'+i\zeta^{21}(x^1)^*+i\zeta^{22}(x^2)^*).$$

Let us identify $\mathbb{H}^{k,n}$ with $\mathbb{C}^{k,2n}=\mathbb{C}^{k,n}\times\mathbb{C}^{k,n}$ by the mapping $x\mapsto(x^1,x^2)$.

4.15

LEMMA

(i) Let us make the transformation $I_{\mathbb{C}}$ from section 4.11 on the variables x^1, $x^2\in\mathbb{C}^{k,n}$ and let us agree that to the variable x^1 corresponds the pair $(\hat{z}^1, z^1)$ and to the variable x^2 corresponds the pair $(\hat{z}^2, z^2)$. Then

$$(I_{\mathbb{H}}\varphi)\left(\begin{bmatrix}\hat{z}^1 & \hat{z}^2\\ z^1 & z^2\end{bmatrix}\right)=(I_{\mathbb{C}}\varphi)((\hat{z}^1, \hat{z}^2)w_n, (z^1, z^2)),$$

where

$$w_n=\begin{bmatrix}0 & 1_n\\ -1_n & 0\end{bmatrix}\in U(2n).$$

(ii) $I_{\mathbb{H}}$ carries out a bijection of the space of polynomials in the real coordinates Re $x^1_{\alpha i}$, Im $x^1_{\alpha i}$, Re $x^2_{\alpha i}$, Im $x^2_{\alpha i}$ of the matrix $x\in\mathbb{H}^{k,n}$ onto the space of polynomials in the coordinates $\zeta^{pq}_{\alpha i}$ of the matrix

$$\zeta=\begin{bmatrix}\zeta^{11} & \zeta^{12}\\ \zeta^{21} & \zeta^{22}\end{bmatrix}\in\mathbb{C}^{2k,2n}.$$

(iii) $I_{\mathbb{H}}$ carries out an isometry of the space $L^2(\mathbb{H}^{k,n})$ onto the Bargmann–Segal space $\mathscr{H}(\mathbb{C}^{2k,2n})$.

PROOF. (i) is verified by simple calculation; (ii) and (iii) follow from (i) and from lemma 4.12.

4.16

Proof of theorem 4.6 (*the case* $F=\mathbb{H}$). Let us recall that we must regard T_{hol} as a representation of the group $U(2\infty)$ and not $U(\infty)$.

Thus, $H(T_{\mathrm{hol}}^{\otimes 2k})$ may be identified with $\mathscr{H}(\mathbb{C}^{2k,2\infty})$, the inductive limit of the spaces $\mathscr{H}(\mathbb{C}^{2k,2n})$ as $n\to\infty$. Let us define the isometry

$$I_{\mathbb{H}}: L^2(\mathbb{H}^{k,\infty})\to\mathscr{H}(\mathbb{C}^{2k,2\infty})$$

as the inductive limit of the isometries constructed in section 4.14.

We must now show that $I_{\mathbb{H}}$ commutes with the action of the group $Sp(\infty)$. For this, it is necessary to check that the isometry defined in 4.14 commutes with the action of $Sp(n)$ (let us recall that $Sp(n)$ acts in $\mathscr{H}(\mathbb{C}^{2k,2n})$ as the subgroup $\beta(Sp(n))\subset U(2n)$. This may be done by two methods.

The first method consists of an explicit check that $A(\zeta)$ and $B(\zeta, x)$ from section 4.14 do not change under the transformations

$$(\zeta, x)\to(\zeta\,\beta(u), xu)\ (u\in Sp(n)).$$

The second method requires less calculation: it is based on lemma 4.15 (i). Let us identify $\mathbb{H}^{k,n}$ with $\mathbb{C}^{k,2n}=\mathbb{C}^{k,n}\times\mathbb{C}^{k,n}$ (see section 4.14). Then the transformation $x\mapsto xu$ becomes the transformation

$$(x^1, x^2)\mapsto(x^1, x^2)\beta(u)$$

By virtue of what is proved in section 4.13, $I_{\mathbb{C}}$ changes this transformation into the transformations

$$(\hat{z}^1, \hat{z}^2)\mapsto(\hat{z}^1, \hat{z}^2)\overline{\beta(u)}, (z^1, z^2)\mapsto(z^1, z^2)\beta(u).$$

After this it remains to note that

$$\omega_n\overline{\beta(u)}\,\omega_n^{-1}=\beta(u).$$

4.17

Corollary. *For all n, $k=1, 2,\ldots$, the space $H_n(T_F^k)$ coincides with $L^2(F^{k,n})$. So $H_\infty(T_F^k)$ coincides with the space of cylindrical functions.*

Proof. Let $F=\mathbb{R}$. We identify $H(T_{\mathbb{R}}^k)$ with $\mathscr{H}(\mathbb{C}^{k,\infty})$ by means of the isometry $I_{\mathbb{R}}$. Then the subspace $L^2(\mathbb{R}^{k,n})$ coincides with the subspace $\mathscr{H}(\mathbb{C}^{k,n})$. On the other hand, the latter subspace coincides with the subspace of all $U_n(\infty)$-invariants in $H(T_{\mathrm{hol}}^{\otimes k})=\mathscr{H}(\mathbb{C}^{k,\infty})$, which is at the same time a subspace of all $SO_n(\infty)$-invariants (corollary 2.18), i.e. $H_n(T^k_F)$.

For $F=\mathbb{C}, \mathbb{H}$, the argument is similar.

4.18

Let v be a certain element of the group $SO(k)$, $U(k)$ or $Sp(k)$ for $F=\mathbb{R}, \mathbb{C}, \mathbb{H}$ respectively.

THEOREM. The transform I_F converts the substitution of the variables $x \mapsto vx (x \in F^{k,\infty})$ into the following substitution of variables:

if $F=\mathbb{R}$, then $z \mapsto vx$, where $z \in \mathbb{C}^{k,\infty}$;

if $F=\mathbb{C}$, then $(\hat{z}, z) \mapsto (\bar{v}\hat{z}, vz)$, where $\hat{z}, z \in \mathbb{C}^{k,\infty}$,

if $F=\mathbb{H}$, then $\zeta \mapsto \beta(v_k v v_k^{-1})\zeta$, where $\zeta \in \mathbb{C}^{2k,2\infty}$, $v_k = (i^{1/2}j) \cdot 1_k \in Sp(k)$.

PROOF. For $F=\mathbb{R}, \mathbb{C}$, the assertion of the theorem follows easily from the formulae given in sections 4.8, 4.11.

In the case $F=\mathbb{H}$, we start from the definition given in section 4.14 and we check that

(1) $A(\beta(v)\zeta) = A(\zeta)$;

(2) $B(\zeta, \beta(v)x) = B(C(v)\zeta, x)$,

where

$$C(v) = C(v_1 + v_2 j) = \begin{bmatrix} v_1' & -iv_2' \\ -iv_2' & \bar{v}_1' \end{bmatrix};$$

(3) $C(v)^{-1} = \beta(v_k v v_k^{-1})$.

The assertion of the theorem for $F=\mathbb{H}$ follows from this.

4.19

Let us examine the space $F^{\infty,\infty}$ of all matrices over F of dimension $\infty \times \infty$ and provide it with the Gaussian product-measure $\mu = \nu \otimes \nu \otimes \dots$. Let us define in $L^2(F^{\infty,\infty})$ the unitary representation Π_F of the group $K \times K$ as follows:

$$\Pi_F(v, u) f(x) = f(v^{-1}xu) \quad (u, v \in K).$$

THEOREM. Π_F is equivalent to the multiplicity free sum of irreducible representations of the type $\bar{\rho}\otimes\rho$, where ρ runs through the set (of classes) of irreducible tame representations of the group K constructed in section 2.14.

We observe that $\bar{\rho}\sim\rho$ for $F=\mathbb{R}, \mathbb{H}$.

PROOF. It is obvious that Π_F is a tame representation. We shall now construct explicitly its holomorphic extension Π_F^* to the group $K^*\times K^*$ (the definition of a tame representation and its holomorphic extension for the group $K\times K$ is exactly analogous to the corresponding definitions for the group K).

Let, for example, $F=\mathbb{R}$. Consider the Bargmann–Segal space $\mathscr{H}(\mathbb{C}^{\infty,\infty})$ of variables $z_{\alpha i}$, where $1\leqslant\alpha$, $i<\infty$. It may be represented as the inductive limit of the spaces $\mathscr{H}(\mathbb{C}^{k,n})$ as $k, n\to\infty$. Let us define the representation $\Pi_{\mathbb{R}}^*$ of the group $U(\infty)\times U(\infty)$ in $\mathscr{H}(\mathbb{C}^{\infty,\infty})$ as follows:

$$\Pi_{\mathbb{R}}^*(v, u)f(z)=f(\bar{v}^{-1}zu)\ (u, v\in U(\infty), z\in\mathbb{C}^{\infty,\infty})$$

(here f is an arbitrary cylindrical function).

It is evident that $\Pi_{\mathbb{R}}^*$ decomposes into the multiplicity free sum of representations $\rho^*\otimes\rho^*$ where ρ^* runs through the set of representations of the type ρ_λ $(\lambda\in\Lambda)$.

On the other hand, we define the isometry

$$I_{\mathbb{R}}: L^2(\mathbb{R}^{\infty,\infty})\to\mathscr{H}(\mathbb{C}^{\infty,\infty})$$

as the inductive limit of the isometries $L^2(\mathbb{R}^{k,n})\to\mathscr{H}(\mathbb{C}^{k,n})$. It follows from theorems 4.6 and 4.18 that $(I_{\mathbb{R}}^{-1}\Pi_{\mathbb{R}}^*I_{\mathbb{R}})|(SO(\infty)\times SO(\infty))=\Pi_{\mathbb{R}}$. Thus the holomorphic extension of the representation $\Pi_{\mathbb{R}}$ is equivalent to $\Pi_{\mathbb{R}}^*$.

In the cases $F=\mathbb{C}, \mathbb{H}$, the argument is similar.

4.20

Remark. Theorem 4.19 may be considered as an analogue of the Peter–Weyl theorem for the groups $SO(\infty)$, $U(\infty)$, $Sp(\infty)$.

4.21

Remark. In the case $F=\mathbb{R}$, theorem 4.18 remains in force if it is assumed that $v\in O(k)$. This fact will be used later.

§5. Approximation of Gaussian measure on $F^{k,\infty}$ by invariant measures on Stiefel manifolds $\Omega^{k,n}$

5.1

LEMMA. For fixed $M>0$, $\varepsilon>0$, the limit

$$I(A)=\lim_{n\to+\infty}\int_A^n t^M\left(1-\frac{t}{n}\right)^{\varepsilon n}dt \quad (A>0)$$

exists and $I(A)\to 0$ as $A\to+\infty$.

PROOF. It is evident that

$$\lim_{n\to+\infty}\int_0^A t^M\left(1-\frac{t}{n}\right)^{\varepsilon n}dt=\int_0^A t^M e^{-\varepsilon t}dt.$$

On the other hand

$$\int_0^n t^M\left(1-\frac{t}{n}\right)^{\varepsilon n}dt=n^{M+1}B(M+1,\varepsilon n+1)=\frac{n^{M+1}\Gamma(M+1)\Gamma(\varepsilon n+1)}{\Gamma(M+1+\varepsilon n+1)}$$

$$=\frac{n^{M+1}\Gamma(M+1)}{(\varepsilon n+1)^{M+1}}\left(1+O\left(\frac{1}{n}\right)\right)\underset{n\to\infty}{\to}\varepsilon^{-(M+1)}\Gamma(M+1)=\int_0^\infty t^M e^{-\varepsilon t}dt,$$

where B and Γ are the beta-function and the gamma-function. So

$$I(A)=\int_A^\infty t^M e^{-\varepsilon t}dt,$$

which proves the lemma.

5.2

Let us recall that $K(n)$ denotes $SO(n)$, $U(n)$, $Sp(n)$ for $F=\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ respectively. Let us put $d=\dim_{\mathbb{R}} F (d=1, 2, 4)$.

Let us fix k and m where k, $m=1, 2, \ldots$. For $n \geqslant \max(k,m)$, we define the mapping $\Theta_n : K(n) \to F^{k,m}$ as follows:

$$(\Theta_n(u))_{\alpha i} = \sqrt{nd}\, u_{\alpha i} \quad (\alpha=1, \ldots, k;\ i=1, \ldots, n).$$

Here $K(n)$ is considered as a group of matrices from $F^{n,n}$.

Let χ_n be the normalized Haar measure on $K(n)$ and let μ_n denote its image with respect to Θ_n. μ_n is a probability measure on $F^{k,m}$.

Let μ be the standard Gaussian measure on $F^{k,m}$ (see section 4.1).

5.3

Lemma.

$$\lim_{n \to \infty} \mu_n = \mu$$

in the sense that, for any measurable bounded function f on $F^{k,m}$ (or even a function of polynomial growth at infinity), we have

$$\lim_{n \to \infty} \int f d\mu_n = \int f d\mu.$$

Proof. Without loss of generality, it is possible to assume $k=m$. Let

$$K(m;n) = \left\{ \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \in K(n) : x \in F^{m,m}, \quad y \in F^{n-m,n-m} \right\}$$

$$\simeq S(O(m) \times O(n-m)),\ U(m) \times U(n-m),\ Sp(m) \times Sp(n-m).$$

Let $n > 2m$. It is easy to check that any matrix $u \in K(n)$ may be written as follows:

$$u = v_1 \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1_{n-2m} \end{bmatrix} v_2, \quad \text{where } v_1, v_2 \in K(m;n),$$

$$a = d = \begin{bmatrix} r_1 & & 0 \\ & \ddots & \\ 0 & & r_m \end{bmatrix}, \quad b = -c = \begin{bmatrix} \sqrt{1-r_1^2} & & 0 \\ & \ddots & \\ 0 & & \sqrt{1-r_m^2} \end{bmatrix}$$

and $r_1 \geqslant 0, \ldots, r_m \geqslant 0$.

Here, the numbers $r_1, \ldots, r_m$ are uniquely defined to within a permutation.

Let us examine the image $\tilde{\chi}_n$ of the Haar measure χ_n under the mapping $u \mapsto (r_1, \ldots, r_m)$. This is a certain probability measure on $[0, 1] \times \ldots \times [0, 1]$ (m times); for the sake of definiteness it may be considered that $r_1 \geqslant \ldots \geqslant r_m$ or that the measure $\tilde{\chi}_n$ is invariant with respect to permutations. The measure $\tilde{\chi}_n$ is naturally called the *radial part* of the measure χ_n.

The measure $\tilde{\chi}_n$ has the following form:

$$d\tilde{\chi}_n = a_n \prod_{1 \leqslant i < j \leqslant m} |r_i^2 - r_j^2|^d \cdot \prod_{i=1}^{m} \left(r_i^{d-1} (1 - r_i^2)^{\frac{(n-2m+1)d}{2} - 1} dr_i \right),$$

where $a_n > 0$ is a certain constant. This formula is a specialization of the formula for the radial part of the invariant measure on an arbitrary compact symmetric space (see [8], chapter X, §1, number 5); it may also be proved directly.

Putting $\sqrt{nd}\, r_i = \rho_i$, we get the following probability measure on $[0, \sqrt{nd}] \times \ldots \times [0, \sqrt{nd}]$:

$$a_n' \varphi_n(\rho_1, \ldots, \rho_m) d\rho_1 \ldots d\rho_m,$$

where

$$\varphi_n(\rho_1, \ldots, \rho_m) = \prod_{1 \leqslant i < j \leqslant m} |\rho_i^2 - \rho_j^2|^d \cdot \prod_{i=1}^{m} \left(\rho_i^{d-1} \left(1 - \frac{\rho_i^2}{nd} \right)^{((n-2m+1)d/2) - 1} \right)$$

On the other hand, we observe that any matrix $x \in F^{m,m}$ may be represented as

$$x = \tilde{v}_1 \begin{bmatrix} \rho_1 & & 0 \\ & \ddots & \\ 0 & & \rho_m \end{bmatrix} \tilde{v}_2,$$

where

$$\rho_1, \ldots, \rho_m \geqslant 0 \text{ and } \tilde{v}_1, \tilde{v}_2 \in O(m), U(m), Sp(m).$$

The constructed measure coincides with the radial part of the measure μ_n in the coordinates $\rho_1, \ldots, \rho_m$.

It is obvious that

$$\lim_{n\to\infty} \varphi_n(\rho_1, \ldots, \rho_m) = \varphi(\rho_1, \ldots, \rho_m),$$

where

$$\varphi(\rho_1, \ldots, \rho_m) = \prod_{1\leqslant i<j\leqslant m} |\rho_i^2 - \rho_j^2|^d \cdot (\rho_1, \ldots, \rho_m)^{d-1}$$
$$\cdot \exp\left(- \frac{1}{2} (\rho_1^2 + \ldots + \rho_m^2) \right),$$

and that the convergence is uniform on all sets of the type

$$\{(\rho_1, \ldots, \rho_m) \in \mathbb{R}_+^m : \max_i \rho_i \leqslant A\}.$$

On the other hand, it follows from lemma 5.1 that

$$\lim_{A\to+\infty} \lim_{n\to\infty} \int_{A\leqslant \max_i \rho_i \leqslant \sqrt{(nd)}} \varphi_n(\rho_1, \ldots, \rho_m) d\rho_1 \ldots d\rho_m = 0.$$

Finally, it is easy to verify that the radial part of the measure μ in the coordinates $\rho_1, \ldots, \rho_m$ is

$$a\varphi(\rho_1, \ldots, \rho_m) d\rho_1 \ldots d\rho_m, \text{ where } a > 0.$$

Putting all these results together, we get $a_n' \to a$, $\mu_n \to \mu$ (we make considerable use of the fact that μ_n and μ are probability measures!).

5.4

For $n>k$, we put

$$\Omega^{k,n}=\{x\in F^{k,n}: xx^*=1_k\}.$$

$\Omega^{k,n}$ is a Stiefel manifold. The group $K(n)$ acts on $\Omega^{k,n}$ on the right (the element $u\in K(n)$ transforms a matrix $\omega\in\Omega^{k,n}$ into the matrix ωu) and $\Omega^{k,n}$ may be identified with $K_k(n)\backslash K(n)$ (the group $K_k(n)$ is defined in section 2.4).

We shall fix m and assume that $n>\max(k, m)$, just as in section 5.2. The mapping Θ_n is constant on the orbits of left action of the subgroup $K_k(n)$. Thus Θ_n gives a mapping

$$\Theta'_n: \Omega^{k,n}\to F^{k,m}.$$

Let χ'_n be the normalized invariant measure on $\Omega^{k,n}$. We observe that $\Theta'_n(\chi'_n)=\Theta_n(\chi_n)$. As a result, we get a corollary of the following type.

COROLLARY. $\Theta'_n(\chi'_n)\to\mu$ in the sense given in lemma 5.3.

5.5

Let us define a unitary representation $T_t^{k,n}$ of the group $K(n)$ in $L^2(\Omega^{k,n})=L^2(\Omega^{k,n}, \chi'_n)$ as follows:

$$T_t^{k,n}(u)\,f(w)=f(wu), \text{ where } f\in L^2(\Omega^{k,n}),\ w\in\Omega^{k,n},\ u\in K(n).$$

THEOREM. For any $k=1, 2, \ldots$, the representations $T_t^{k,n}$ of the group $K(n)$ approximate as $n\to\infty$ the representation T_t^k of the group K in $L^2(F^{k,\infty})$.

The representation T_t^k is defined in §4. The definition of the concept of approximation is given in §22.

For a proof of the theorem see section 5.9.

5.6

Let f be a bounded cylindrical function from $L^2(F^{k,\infty})$. The function f in fact depends on a finite number of coefficients of the matrix

$x \in F^{k,\infty}$. Hence, if n is sufficiently large, then f may be considered as a function on $F^{k,n}$. Let us put

$$f^{(n)}(\omega) = f(\sqrt{nd}\,\omega), \text{ where } \omega \in \Omega^{k,n} \subset F^{k,n}$$

This function lies in $L^2(F^{k,n})$ as it is bounded.

5.7

Lemma. In the notation of section 5.6, let n be so large that $f^{(n)}$ is defined. Then

$$(T_F^k(u)\,f)^{(n)} = T_F^{k,n}(u)\,f^{(n)} \quad (u \in K(n)).$$

This follows at once from the definitions of $f^{(n)}$ and $T_F^{k,n}$.

5.8

Lemma. For any f_1, f_2, just as in section 5.6,

$$\lim_{n\to\infty} (f_1^{(n)}, f_2^{(n)})_{L^2(\Omega^{k,n})} = (f_1, f_2)_{L^2(F^{k,\infty})}.$$

This follows at once from lemma 5.3.

5.9

Proof of theorem 5.5. It follows at once from lemmas 5.7 and 5.8 that

$$(T_F^{k,n}(u)\,f_1^{(n)}, f_2^{(n)}) \underset{n\to\infty}{\to} (T_F^k\, f_1, f_2)\,\forall u \in K.$$

Moreover the convergence is uniform on compact sets in K (see 22.1 and 22.2): to verify this assertion it is convenient to assume f_1 to be continuous. Now it remains to apply lemma 22.6.

5.10

Remark. The structure of the decomposition of the representations T_t^k and $T_t^{k,n}$ is one and the same. We obtain this from the following observations: if we write

$$T_t^k \sim \bigoplus_\rho N(\rho)\rho,$$

where ρ runs through the set (of classes) of irreducible representations of the group K that enter into T_t^k and $N(\rho)$ denotes the multiplicity, then we get

$$T_t^{k,n} \sim \bigoplus_\rho N(\rho)\rho^{(n)},$$

where $\{\rho^{(n)}\}$ is the sequence of irreducible representations of the groups $K(n)$ defined for each ρ in section 2.23.

In fact, the irreducible representation of the group $K(n)$ enters into $T_t^{k,n}$ if and only if it contains non-zero $K_k(n)$-invariant vectors. The highest weights of such representations have precisely the following form:

$(\lambda_1, \ldots, \lambda_k, 0, \ldots, 0)$ (in the case $F=\mathbb{R}$)

$(\lambda_1, \ldots, \lambda_k, 0, \ldots, 0, -\mu_k, \ldots, -\mu_1)$ (in the case $F=\mathbb{C}$)

$(\lambda_1, \ldots, \lambda_{2k}, 0, \ldots, 0)$ (in the case $F=\mathbb{H}$).

The corresponding multiplicities are found from the "ramification rules" for chains of subgroups

$$K(n) \supset K(n-1) \supset \ldots \supset K(n-k),$$

where $K(n-i)$ is identified with $K_i(n)$ $(i=1, \ldots, k)$; see [43], §§129, 66, 130, for the groups $SO(n)$, $U(n)$, $Sp(n)$, respectively. It is easy to check that, for weights of the given type, these rules lead to the same result, as do also the ramification rules for chains of subgroups

$U(n)\supset U(n-1)\supset \dots \supset U(n-k) \quad (F=\mathbb{R})$

$U(n)\times U(n)\supset U(n-1)\times U(n-1)\supset\dots\supset U(n-k)\times U(n-k)\,(F=\mathbb{C})$

$U(2n)\supset U(2n-1)\supset \dots \supset U(2n-2k) \quad (F=\mathbb{H}).$

and the weights

$$(\lambda_1,\dots,\lambda_k,0,\dots,0) \qquad (F=\mathbb{R}),$$

$$\{(\lambda_1,\dots,\lambda_k,0,\dots,0),(\mu_1,\dots,\mu_k,0,\dots,0)\} \quad (F=\mathbb{C})$$

$$(\lambda_1,\dots,\lambda_{2k},0,\dots,0) \qquad (F=\mathbb{H}).$$

respectively.

The result does not depend on n (if n is sufficiently large) and coincides with

$$\dim V(\lambda_1,\dots,\lambda_k) \quad (F=\mathbb{R}),$$

$$\dim V(\lambda_1,\dots,\lambda_k)\cdot\dim V(\mu_1,\dots,\mu_k) \quad (F=\mathbb{C}),$$

$$\dim V(\lambda_1,\dots,\lambda_{2k}) \quad (F=\mathbb{H}).$$

Comparing this with corollary 4.7, we get the assertion formulated above.

PART II

WEIL'S REPRESENTATIONS OF THE GROUPS $Sp(\infty,\mathbb{R})^{\sim}$, $U(\infty,\infty)^{\sim}$ and $SO^*(2\infty)$

§6. The formalism of R. Howe for Weil's representations of the groups $Sp(n,\mathbb{R})^{\sim}$, $U(n,n)^{\sim}$ and $SO^*(2n)$

6.1

We shall use the notation given in the following table.

Table 7.3

	$F=\mathbb{R}$	$F=\mathbb{C}$	$F=\mathbb{H}$
$\mathfrak{l}(n)$	$sp(n, \mathbb{R})$	$u(n, n)$	$so^*(2n)$
$\mathfrak{m}(n)$	$u(n)$	$u(n) \oplus u(n)$	$u(n)$
$\mathfrak{l}_{\mathbb{C}}(n)$	$sp(n, \mathbb{C})$	$\mathfrak{gl}(2n, \mathbb{C})$	$so(2n, \mathbb{C})$
$L(n)$	$Sp(n, \mathbb{R})^{\sim}$	$U(n, n)^{\sim}$	$SO^*(2n)$
$M(n)$	$U(n)^{\sim}$	$(U(n) \times U(n)^{\sim}$	$U(n)$
L	$Sp(\infty, \mathbb{R})^{\sim}$	$U(\infty, \infty)^{\sim}$	$SO^*(2\infty)$
M	$U(\infty)^{\sim}$	$(U(\infty) \times U(\infty))^{\sim}$	$U(\infty)$
$U(k, F)$	$O(k)$	$U(k)$	$Sp(k)$

The sign $\sim$ indicates a two-sheeted covering over the corresponding group. $\mathfrak{l}(n)$ and $\mathfrak{m}(n)$ are the Lie algebras of the groups $L(n)$ and $M(n)$ respectively. $\mathfrak{l}_{\mathbb{C}}(n) = \mathfrak{l}(n) \otimes_{\mathbb{R}} \mathbb{C}$. $M(n)$ is the maximal compact subgroup in $L(n)$ (see section 7.3).

$$L = \bigcup_{n=1}^{\infty} L(n) \text{ and } M = \bigcup_{n=1}^{\infty} M(n)$$

(for more details, see sections 7.9, 7.11), $U(k, F)$ is a symmetry group for certain representations of the groups $L(n)$ and L (see sections 6.12, 7.13).

6.2

We shall give an explicit realization of the Lie groups and algebras shown in Table 7.1 (the first two groups are necessary for the realization of $Sp(n, \mathbb{R})$ and $SO^*(2n)$.

$$Sp(n, \mathbb{C}) = \left\{ g \in \mathbb{C}^{2n, 2n} : g' \begin{bmatrix} 0 & 1_n \\ -1_n & 0 \end{bmatrix} g = \begin{bmatrix} 0 & 1_n \\ -1_n & 0 \end{bmatrix} \right\}.$$

$$SO(2n, \mathbb{C})=\left\{g\in\mathbb{C}^{2n,2n}: g'\begin{bmatrix}0 & 1_n\\ 1_n & 0\end{bmatrix}g=\begin{bmatrix}0 & 1_n\\ 1_n & 0\end{bmatrix}\right\}.$$

$$U(n, n)=\left\{g\in\mathbb{C}^{2n,2n}: g^*\begin{bmatrix}-1_n & 0\\ 0 & 1_n\end{bmatrix}g=\begin{bmatrix}-1_n & 0\\ 0 & 1_n\end{bmatrix}\right\}.$$

$$Sp(n, \mathbb{R})=U(n, n)\cap Sp(n, \mathbb{C}),$$

$$SO^*(2n)=U(n, n)\cap SO(2n, \mathbb{C}).$$

$$sp(n, \mathbb{C})=\left\{\begin{bmatrix}-A' & B\\ C & A\end{bmatrix}\in\mathbb{C}^{2n,2n}: B=B',\ C=C'\right\}.$$

$$so(2n, \mathbb{C})=\left\{\begin{bmatrix}-A' & B\\ C & A\end{bmatrix}\in\mathbb{C}^{2n,2n}: B=B',\ C=C'\right\}.$$

$$u(n, n)=\left\{\begin{bmatrix}A & B\\ C & D\end{bmatrix}\in\mathbb{C}^{2n,2n}: A=-A^*,\ D=-D^*,\ C=B^*\right\}.$$

$$sp(n, \mathbb{R})=\left\{\begin{bmatrix}-A' & B\\ B^* & A\end{bmatrix}\in\mathbb{C}^{2n,2n}: A=-A^*,\ B=B'\right\}.$$

$$so^*(2n)=\left\{\begin{bmatrix}-A' & B\\ B^* & A\end{bmatrix}\in\mathbb{C}^{2n,2n}: A=-A^*,\ B=-B'\right\}.$$

The group $U(n, n)^\sim$ is the two-sheeted covering over $U(n, n)$ for which the function $g\mapsto(\det g)^{1/2}$ becomes single-valued. $(U(n)\times U(n))^\sim$ is the two-sheeted covering over $U(n)\times U(n)$ for which the function $(g_1, g_2)\mapsto(\det g_1 g_2)^{1/2}$ becomes single-valued (here $g_1, g_2\in U(n)$). In all the remaining cases the two-sheeted covering over the group is determined uniquely.

6.3

Each of the three Lie algebras

$$\mathfrak{l}_{\mathbb{C}}(n)=\mathfrak{sp}(n,\mathbb{C}),\quad \mathfrak{gl}(2n,\mathbb{C}),\quad \mathfrak{so}(2n,\mathbb{C})$$

possesses the decomposition

$$\mathfrak{l}_{\mathbb{C}}(n)=\mathfrak{l}_{-1}(n)\oplus\mathfrak{l}_0(n)\oplus\mathfrak{l}_1(n),$$

where

$$\mathfrak{l}_{-1}(n)=\left\{\begin{bmatrix}0 & 0\\ * & 0\end{bmatrix}\right\},\mathfrak{l}_0(n)=\left\{\begin{bmatrix}* & 0\\ 0 & *\end{bmatrix}\right\},\mathfrak{l}_1(n)\left\{\begin{bmatrix}0 & *\\ 0 & 0\end{bmatrix}\right\}.$$

$\mathfrak{l}_1(n)$ and $\mathfrak{l}_{-1}(n)$ are abelian subspaces invariant with respect to the subalgebra $\mathfrak{l}_0(n)$; we observe further that

$$[\mathfrak{l}_{-1}(n),\mathfrak{l}_1(n)]\subseteq\mathfrak{l}_0(n).$$

$M(n)$ is the subgroup in $L(n)$ corresponding to the Lie algebra $\mathfrak{m}(n)=\mathfrak{l}_0(n)\cap\mathfrak{l}(n)$.

Let us put

$$Z=\frac{1}{2}\begin{bmatrix}1_n & 0\\ 0 & -1_n\end{bmatrix}.$$

This element lies in the centre of the algebra $\mathfrak{l}_0(n)$. We observe that

$$\operatorname{ad} Z|\mathfrak{l}_{-1}(n)=-\mathrm{id},\quad \operatorname{ad} Z|\mathfrak{l}_1(n)=\mathrm{id}.$$

6.4

We shall consider the infinite-dimensional $l_{\mathbb{C}}(n)$-modules V satisfying the following conditions:

(i) $V=V_0\oplus V_1\oplus V_2\oplus\ldots$, $\dim V_i<+\infty$, $\operatorname{ad} z|V_i=(a-i)\mathrm{id}$ (a is a certain constant).

(ii) V is provided with the structure of a pre-Hilbert space; the elements of the algebra $\mathfrak{l}(n)$ act as anti-Hermitian operators.

(iii) All irreducible $\mathfrak{l}_0(n)$-submodules in V come from (finite-dimensional) unitary representations of the group $M(n)$.

6.5

We shall give simple corollaries of these conditions:

(a) V is a semisimple $l_0(n)$-module with finite multiplicities.

(b) V is a semisimple $l_{\mathbb{C}}(n)$-module with finite multiplicities.

(c) $l_1(n)$ acts in V in a locally nilpotent way.

(d) The subspace of $l_1(n)$-invariants generates the module V.

(e) If V is irreducible, then the subspace of $l_1(n)$-invariants is irreducible as an $l_0(n)$-module; the latter uniquely defines the module V.

(f) The tensor product of a finite number of modules, satisfying conditions (i)–(iii), again satisfies these conditions.

(g) V coincides with the module of $M(n)$-finite vectors of a certain unitary representation of the group $L(n)$.

6.6

We shall define now a certain remarkable $l_{\mathbb{C}}(n)$-module satisfying conditions (i)–(iii). This module and the corresponding unitary representation of the group $L(n)$ will be denoted by $W^{(n)}$ or $W_F^{(n)}$. The representation $W_{\mathbb{R}}^{(n)}$ of the group $L(n)=Sp(n, \mathbb{R})^{\sim}$ is usually called the *Weil representation*. We shall use the same term for $W_{\mathbb{C}}^{(n)}$ and $W_{\mathbb{H}}^{(n)}$.

6.7

Let $F=\mathbb{R}$; then $l_{\mathbb{C}}(n)=sp(n, \mathbb{C})$. The module $W_{\mathbb{R}}^{(n)}$ is realized in the space $\mathbb{C}[z_1, \ldots, z_n]$ and the corresponding unitary representation of the group $L(n)=Sp(n, \mathbb{R})^{\sim}$ in $\mathscr{H}[z_1, \ldots, z_n]$ (see §3). The action of the Lie algebra $l_{\mathbb{C}}(n)$ in $\mathbb{C}[z_1, \ldots, z_n]$ is seen as follows on its generators (for notation, see sections 1.4, and 3.5):

$$\begin{bmatrix} -E_{ji} & 0 \\ 0 & E_{ij} \end{bmatrix} \mapsto M_i D_j + \frac{1}{2}\,\delta_{ij}\cdot 1 = \frac{1}{2}\,[M_i, D_j]_+,$$

$$\begin{bmatrix} 0 & E_{ij}+E_{ji} \\ 0 & 0 \end{bmatrix} \mapsto D_i D_j = \frac{1}{2}\,[D_i, D_j]_+,$$

$$\begin{bmatrix} 0 & 0 \\ E_{ij}+E_{ji} & 0 \end{bmatrix} \mapsto -M_i M_j = -\frac{1}{2}[M_i, M_j]_+,$$

Here $i, j = 1, \ldots, n$ and $[A, B]_+ = AB + BA$.

It is easy to verify that the given formulae do indeed give a certain $sp(n, \mathbb{C})$-module and that this module possesses properties (i)–(iii) from section 6.4. The graduation in our module coincides with the graduation in $\mathbb{C}[z_1, \ldots, z_n]$ in degree of polynomials.

We observe that it is precisely the presence of the factor 1/2 on the right-hand side of the first formula that causes the representation $W_{\mathbb{R}}^{(n)}$ to be not one-valued on the group $Sp(n, \mathbb{R})$.

6.8

The module $W_{\mathbb{R}}^{(h)\otimes k}$ $(k = 1, 2, \ldots)$ is realised in the space of polynomials in $z_{\alpha i}$ $(\alpha = 1, \ldots, k; i = 1, \ldots, n)$ by the following formulae:

$$\begin{bmatrix} -E_{ji} & 0 \\ 0 & E_{ij} \end{bmatrix} \mapsto \sum_{\alpha=1}^{k} M_{\alpha i} D_{\alpha j} + \frac{k}{2}\delta_{ij}\cdot 1,$$

$$\begin{bmatrix} 0 & E_{ij}+E_{ji} \\ 0 & 0 \end{bmatrix} \mapsto \sum_{\alpha=1}^{k} D_{\alpha i} D_{\alpha j},$$

$$\begin{bmatrix} 0 & 0 \\ E_{ij}+E_{ji} & 0 \end{bmatrix} \mapsto -\sum_{\alpha=1}^{k} M_{\alpha i} M_{\alpha j}.$$

The corresponding unitary representation is realized in $\mathscr{H}(\mathbb{C}^{k,n})$.

6.9

We shall define an embedding $\mathfrak{gl}(2n, \mathbb{C}) \to \mathfrak{sp}(2n, \mathbb{C})$ as follows

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \mapsto \begin{bmatrix} A & 0 & 0 & B \\ 0 & -D' & B' & 0 \\ 0 & C' & -A' & 0 \\ C & 0 & 0 & D \end{bmatrix}.$$

This mapping embeds $u(n, n)$ in $\mathfrak{sp}(2n, \mathbb{R})$ and gives an embedding of the group $U(n, n)^{\sim}$ into the group $Sp(2n, \mathbb{R})^{\sim}$.

6.10

By definition the $\mathfrak{gl}(2n, \mathbb{C})$-module $W_{\mathbb{C}}^{(n)}$ is the restriction of the $\mathfrak{sp}(2n, \mathbb{C})$-module $W_{\mathbb{H}}^{(2n)}$ to the subalgebra $\mathfrak{gl}(2n, \mathbb{C})$. This module is realized in the space of polynomials $\mathbb{C}[z_{\hat{1}}, \ldots, z_{\hat{n}}, z_1, \ldots, z_n]$ and the corresponding unitary representation in the space $\mathscr{H}(z_{\hat{1}}, \ldots, z_{\hat{n}}, z_1, \ldots, z_n)$; for this notation see section 3.8.

We immediately derive explicit formulae for the module $W_{\mathbb{C}}^{(n)\otimes k}$, $k = 1, 2, \ldots$ (for the notation, see sections 3.8, 3.9):

$$\begin{bmatrix} -E_{ji} & 0 \\ 0 & 0 \end{bmatrix} \mapsto \sum_{\alpha=1}^{k} M_{\alpha\hat{i}} D_{\alpha\hat{j}} + \frac{k}{2}\delta_{ij} \cdot 1,$$

$$\begin{bmatrix} 0 & 0 \\ 0 & E_{ij} \end{bmatrix} \mapsto \sum_{\alpha=1}^{k} M_{\alpha i} D_{\alpha j} + \frac{k}{2}\delta_{ij} \cdot 1,$$

$$\begin{bmatrix} 0 & E_{ij} \\ 0 & 0 \end{bmatrix} \mapsto \sum_{\alpha=1}^{k} D_{\alpha\hat{i}} D_{\alpha j},$$

$$\begin{bmatrix} 0 & 0 \\ E_{ij} & 0 \end{bmatrix} \mapsto -\sum_{\alpha=1}^{k} M_{\alpha i} M_{\alpha\hat{j}}.$$

Here $i, j = 1, \ldots, n$.

The unitary representation $W_{\mathbb{C}}^{(n)\otimes k}$ is realized in the space $\mathscr{H}(\mathbb{C}^{k,2n}) = \mathscr{H}(\mathbb{C}^{k,n}) \otimes \mathscr{H}(\mathbb{C}^{k,n})$.

6.11

By definition, the $\mathfrak{so}(2n, \mathbb{C})$-module $W_{\mathbb{H}}^{(n)}$ is the restriction of the module $W_{\mathbb{C}}^{(n)}$ to the subalgebra $\mathfrak{so}(2n, \mathbb{C}) \subset \mathfrak{gl}(2n, \mathbb{C})$. The module $W_{\mathbb{H}}^{(n)\otimes k}$ is realized in the same space of polynomials in $z_{\alpha\hat{i}}, z_{\alpha i}$ ($\alpha = 1, \ldots, k$; $i = 1, \ldots, n$) as the module $W_{\mathbb{C}}^{(n)\otimes k}$. However, it will be convenient for us to write here $z_{\hat{\alpha} i}$ instead of $z_{\alpha\hat{i}}$. This corresponds to the fact that we are changing from $\mathbb{C}^{k,2n}$ to $\mathbb{C}^{2k,n}$. In other words, from the matrices

$$\hat{z} = \|z_{\alpha\hat{i}}\|, \; z = \|z_{\alpha i}\| \in \mathbb{C}^{k,n},$$

we compile the matrix

$$\zeta = \begin{bmatrix} \hat{z} \\ z \end{bmatrix} \mapsto \mathbb{C}^{2k,n},$$

whose rows are numbered by the indices $\hat{1}, \ldots, \hat{k}, 1, \ldots, k$.

With this notation, $W_{\mathbb{H}}^{(n)\otimes k}$ is given by the following formulae:

$$\begin{bmatrix} -E_{ji} & 0 \\ 0 & E_{ij} \end{bmatrix} \mapsto \sum_{\alpha=1}^{k} (M_{\hat{\alpha}i} D_{\hat{\alpha}j} + M_{\alpha i} D_{\alpha j}) + k\delta_{ij} \cdot 1,$$

$$\begin{bmatrix} 0 & E_{ij} - E_{ji} \\ 0 & 0 \end{bmatrix} \mapsto \sum_{\alpha=1}^{k} (D_{\hat{\alpha}i} D_{\alpha j} - D_{\hat{\alpha}j} D_{\alpha i}),$$

$$\begin{bmatrix} 0 & 0 \\ E_{ij} - E_{ji} & 0 \end{bmatrix} \mapsto -\sum_{\alpha=1}^{k} (M_{\alpha i} M_{\hat{\alpha}j} - M_{\alpha j} M_{\hat{\alpha}i}).$$

6.12

Consider the group $U(k,F)$, (see the last row in Table 7.3 from section 6.1), and define its unitary representation $R_k^{(n)}$ in $H(W_F^{(n)\otimes k})$ as follows:

$$F=\mathbb{R}: \; R_k^{(n)}(v) f(z) = f(v^{-1}z) \quad (z \in \mathbb{C}^{k,n}, v \in O(k)).$$

$$F=\mathbb{C}: \; R_k^{(n)}(v) f(\hat{z}, z) f(v'\hat{z}, v^{-1}z) \quad (\hat{z}, z \in \mathbb{C}^{k,n}, \; v \in U(k)).$$

$$F=\mathbb{H}: \; R_k^{(n)}(v) f(\zeta) = f(\beta(v)^{-1}\zeta) \quad (\xi \in \mathbb{C}^{2k,n}, \; v \in Sp(k)).$$

Let us recall that $\beta(v) \in U(2k)$ was defined in section 1.3.

6.13

LEMMA. The representation $R_k^{(n)}$ of the group $U(k, F)$ commutes with the action of the group $L(n)$ in $H(W_F^{(n)\otimes k})$.

PROOF. It is sufficient to prove that $R_k^{(n)}$ commutes with the action of the Lie algebra $\ell(n)$, and this is easily verified.

6.14

THEOREM OF R. HOWE. The group $U(k, F) = O(k)$, $U(k)$, $Sp(k)$ is a symmetry group for the representation $W_F^{(n)\otimes k}$ of the group $L(n) = Sp(n, \mathbb{R})^{\sim}$, $U(n, n)^{\sim}$, $SO^*(2n)$ (see section 1.12).

This remarkable result is proved in [9]; see also [30].

6.15

COROLLARY. For any $\pi \in U(k, F)^{\wedge}$ occurring in the representation $R_k^{(n)}$, an irreducible unitary representation $W_\pi^{(n)}$ of the group $L(n)$ is determined (see section 1.12).

§7. Generalization to infinite-dimensional groups

7.1

LEMMA. Let $n \geqslant k$ for $F = \mathbb{R}$, $\mathbb{C}$ and $n \geqslant 2k$ for $F = \mathbb{H}$. Then the representation $R_k^{(n)}$ of the group $U(k, F)$ (see section 6.12) contains all $\pi \in U(k, F)^{\wedge}$.

PROOF. Let, for example, $F = \mathbb{R}$. It is sufficient to prove the statement of the lemma for $n = k$. Let us examine the restriction mapping $f \mapsto f|O(k)$. It transforms the algebra of polynomials $P(\mathbb{C}^{k,k})$ into a certain algebra of functions on $O(k)$. If f is a polynomial in $\{z_{ai}\}$, then the function $z \mapsto \overline{f(\bar{z})}$ is also a polynomial. On the other hand, $\bar{z} = z$ for $z \in O(k)$. Hence the algebra of functions on $O(k)$ that is obtained is stable with respect to conjugation. By the Stone–Weierstrass theorem, it is dense in $L^2(O(k))$. Our statement easily follows from this.

In the case $F = \mathbb{C}$, $\mathbb{H}$, the reasoning is similar.

7.2

COROLLARY. With the assumptions of lemma 7.1, we have: the representations $W_\pi^{(n)}$ from section 6.15 are determined for all $\pi \in U(k, F)^{\wedge}$.

7.3

Let us examine the following Lie algebra mapping for $F=\mathbb{R}, \mathbb{C}, \mathbb{H}$:

$$u(n)\ni A\mapsto\begin{bmatrix}\bar{A} & 0\\ 0 & A\end{bmatrix}=\begin{bmatrix}-A' & 0\\ 0 & A\end{bmatrix}(F=\mathbb{R}, \mathbb{H}),$$

$$u(n)\oplus u(n)\ni A_1\oplus A_2\mapsto\begin{bmatrix}\bar{A}_1 & 0\\ 0 & A_2\end{bmatrix}=\begin{bmatrix}-A'_1 & 0\\ 0 & A_2\end{bmatrix}(F=\mathbb{C}).$$

This mapping allows us to identify $\mathfrak{m}(n)$ with $u(n)$, $u(n)\oplus u(n)$ or $u(n)$ and gives an isomorphism of the group $M(n)$ with $U(n)^{\sim}$, $(U(n)\times U(n))^{\sim}$ or $U(n)$ respectively.

7.4

LEMMA. Taking into consideration the identification given in section 7.3, the representation $W_F^{(n)\otimes k}|M(n)$ has the following form:

$$f(z)\mapsto f(z\omega)(\det u)^{k/2}(F=\mathbb{R}, u\in U(n)\sim)$$

$$f(\hat{z}, z)\mapsto f(\hat{z}u_1, zu_2)(\det(u_1\ u_2))^{k/2}(F=\mathbb{C}, (u_1, u_2)\in(U(n)\times U(n))\sim),$$

$$f(\zeta)\mapsto f(\zeta u)(\det u)^{k}(F=\mathbb{H}, u\in U(n)),$$

where $z, \hat{z}\in\mathbb{C}^{k,n}$, $\zeta\in\mathbb{C}^{2k,n}$.

This follows directly from the definition of the modules $W_F^{(n)\otimes k}$.

7.5

LEMMA. Let $k_1\neq k_2$ and n be sufficiently large, ($n>\max(k_1, k_2)$ for $F\neq\mathbb{H}$; $n>\max(2k_1, 2k_2)$ for $F=\mathbb{H}$). Then the representations $W_F^{(n)\otimes k_1}|M(n)$ and $W_F^{(n)\otimes k_2}|M(n)$ are disjunct, i.e. do not have equivalent irreducible subrepresentations.

PROOF. Let, for example, $F=\mathbb{R}$. If $n>$ k, then the highest weight of any irreducible subrepresentation occurring in $W_{\mathbb{R}}^{(n)\otimes k}|U(n)^{\sim}$ has the form

$$(\mu_1, \ldots, \mu_n)=(\lambda_1, \ldots, \lambda_k, 0, \ldots, 0)+\left(\frac{k}{2}, \ldots, \frac{k}{2}\right).$$

From this, $\mu_n = k/2$. Now the assertion of the lemma becomes obvious.

For $F = \mathbb{C}, \mathbb{H}$, the proof is analogous.

7.6

THEOREM. Let $\pi_1 \in U(k_1, F)^\wedge$, $\pi_2 \in U(k_2, F)^\wedge$. Let us assume that one of the following two cases occurs:

(a) $k_1 = k_2$ and $\pi_1 \nsim \pi_2$;

(b) $k_1 \neq k_2$ and the conditions of lemma 7.5 are fulfilled.

Then the representations $W_{\pi_1}{}^{(n)}$ and $W_{\pi_2}{}^{(n)}$ of the group $L(n)$, constructed in section 6.15, are not equivalent to one another.

PROOF. In the case (a), this follows from the definition of the representations $W_\pi{}^{(n)}$ (see section 1.12); in the case (b)—from lemma 7.5.

7.7

Let us consider the outer automorphism of the Lie algebra $\mathfrak{l}(n)$.

$$\varphi: \begin{bmatrix} A & B \\ C & D \end{bmatrix} \mapsto \begin{bmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{bmatrix}, \text{ where } \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathfrak{l}(n).$$

The corresponding automorphism of the group $L(n)$ will also be denoted by φ.

THEOREM. For any $\pi \in U(k, F)^\wedge$, the representations $W_\pi{}^{(n)}$ and $W_{\bar{\pi}}{}^{(n)} \circ \varphi$ of the group $L(n)$ are conjugate to one another.

Notice that, for $F = \mathbb{R}, \mathbb{H}$, it is possible to replace $\bar{\pi}$ by π since all unitary representations of the groups $O(k)$ and $Sp(k)$ are self-conjugate.

PROOF. Let, for example, $F = \mathbb{R}$. Let us write for brevity $W = W_{\mathbb{R}}{}^{(n) \otimes k}$. Let us define an antilinear involution I of the Hilbert space $\mathscr{H}(\mathbb{C}^{k,n}) = H(W)$ as follows:

$$If(z) = \overline{f(\bar{z})} \; (z \in \mathbb{C}^{k,n}, f \in \mathscr{H}(\mathbb{C}^{k,n})).$$

I reduces to a conjugation of the coordinates in the basis consisting of monomials in the variables z_{ai}. The form

$$\langle f_1, f_2\rangle = (f_1, If_2)_{\mathscr{H}(\mathbb{C}^{k,n})}$$

gives a non-degenerate bilinear form on $\mathscr{H}(\mathbb{C}^{k,n})$ which allows the identification of $H(W)$ with the space $\overline{H(W)} = H(\bar{W})$ conjugate to it.

The representation $\bar{W}$ of the Lie algebra $\mathfrak{sp}(n, \mathbb{C})$ in the subspace of polynomials in $H(W) = H(\bar{W})$ is given by the condition

$$\langle f_1, \bar{W}(X)f_2\rangle = -\langle W(X)f_1, f_2\rangle, \text{ where } x \in \mathfrak{sp}(n, \mathbb{C})).$$

From this,

$$\bar{W}(X) = -I\, W(X)^* I (x \in \mathfrak{sp}(n, \mathbb{C})).$$

We observe that

$$-I\, W(X)^* I = W(\varphi(X)),$$

where φ denotes the $\mathbb{C}$-linear extension of the automorphism φ to $\mathfrak{sp}(n, \mathbb{C})$:

$$\varphi\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) = \begin{bmatrix} -A' & C \\ B & -D' \end{bmatrix} \quad \left(\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{sp}(n, \mathbb{C})\right).$$

In fact, for this it is necessary to use the fact that I commutes with real linear combinations of operators M_{ai}, D_{ai} and the definition of the representation W (see section 6.8).

It now remains to observe that I commutes with the representation $R_k^{(n)}$ of the group $O(k)$.

In the cases $F = \mathbb{C}, \mathbb{H}$, the reasoning is similar.

7.8

Remark. Let us investigate the group of all automorphisms of the Lie algebra $l(n)$, factorized with respect to the subgroup of inner automorphisms. When $F \neq \mathbb{C}$, this group is $\mathbb{Z}_2$ and for $F = \mathbb{C}$ this group is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. In the second case our group consists of the automorphisms id, φ, ψ, $\varphi\psi$, where

$$\psi: \begin{bmatrix} A & B \\ C & D \end{bmatrix} \mapsto \begin{bmatrix} \bar{D} & \bar{C} \\ \bar{B} & \bar{A} \end{bmatrix} = \begin{bmatrix} -D' & B' \\ C' & -A' \end{bmatrix} \left(\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in u(n, n) \right).$$

It is easy to check that ψ does not change the equivalence class of representation $W_{\mathbb{C}}^{(n)}$ (since ψ is sent into id by an appropriate inner automorphism of the Lie algebra $\mathfrak{sp}(2n, \mathbb{R}) \supset u(n, n)$). Hence in the case $F = \mathbb{C}$, it is possible, instead of φ, to take equally successfully

$$\varphi\psi: \begin{bmatrix} A & B \\ C & D \end{bmatrix} \mapsto \begin{bmatrix} D & C \\ B & A \end{bmatrix}.$$

7.9

We shall define an embedding $L(n) \to L(n+1)$ as follows. According to its definition, $L(n)$ is a group of operators in $\mathbb{C}^n \oplus \mathbb{C}^n$. Let $\{e_1, \ldots, e_{n+1}, e_1', \ldots, e_{n+1}'\}$ be the canonical basis in $\mathbb{C}^{n+1} \oplus \mathbb{C}^{n+1}$. Let us identify $\mathbb{C}^n \oplus \mathbb{C}^n$ with the subspace in $\mathbb{C}^{n+1} \oplus \mathbb{C}^{n+1}$ spanned by $e_1, \ldots, e_n, e_1', \ldots, e_n'$. We can now identify $L(n)$ with the subgroup in $L(n+1)$ that leaves invariant e_{n+1} and e_{n+1}'.

Thus we can define the group

$$L = \bigcup_{n=1}^{\infty} L(n)$$

which is $Sp(\infty, \mathbb{R})^{\sim}$, $U(\infty, \infty)^{\sim}$ or $SO^*(2\infty)$ (see section 6.1).

7.10

In section 3.4, we have defined the canonical embeddings

$$\mathscr{H}(\mathbb{C}^{1,n}) = \mathscr{H}(z_1, \ldots, z_n) \to \mathscr{H}(\mathbb{C}^{1,n+1}) = \mathscr{H}(z_1, \ldots, z_{n+1}) \quad (n = 1, 2, \ldots).$$

Let us recall that

$$H(W_F^{(n)}) = \begin{cases} \mathscr{H}(\mathbb{C}^{1,n}), & \text{if } F = \mathbb{R}, \\ \mathscr{H}(\mathbb{C}^{1,n}) \otimes \mathscr{H}(\mathbb{C}^{1,n}), & \text{if } F = \mathbb{C}, \mathbb{H}. \end{cases}$$

Thus, the given canonical embeddings determine the embeddings

$$H(W_F^{(n)}) \to H(W_F^{(n+1)}) \; (n=1, 2, \ldots, F=\mathbb{R}, \mathbb{C}, \mathbb{H}),$$

which obviously commutes with the action of the groups $L(n)$. This makes possible the following definition.

Definition. The Weil representation W_F of the group L is the inductive limit of the representations $W_F^{(n)}$ of the groups $L(n)$ as $n \to \infty$.

We observe that the representation $W_F^{\otimes k}(k=1, 2, \ldots)$ is realized in

$$\mathscr{H}(\mathbb{C}^{k,\infty}), \quad \mathscr{H}(\mathbb{C}^{k,2\infty}) = \mathscr{H}(\mathbb{C}^{k,\infty}) \otimes \mathscr{H}(\mathbb{C}^{k,\infty}), \quad \mathscr{H}(\mathbb{C}^{2k,\infty})$$

respectively, for $F=\mathbb{R}, \mathbb{C}, \mathbb{H}$. The corresponding representation of the Lie algebra

$$l(\infty) = \bigcup_{n=1}^{\infty} l(n)$$

acts in the subspace of polynomials.

7.11

Let us examine the group

$$M = \bigcup_{n=1}^{\infty} M(n),$$

which is

$$U(\infty)^{\sim}, \quad (U(\infty) \times U(\infty))^{\sim}, \quad U(\infty)$$

for $F=\mathbb{R}, \mathbb{C}, \mathbb{H}$ respectively. The embeddings $M(n) \to L(n)$ defined in section 7.3 give an embedding $M \to L$.

Lemma. For $k=1, 2, \ldots$, we have

$$W_F^{\otimes k} | M \sim \begin{cases} T_{\mathrm{hol}}{}^{\otimes k} \otimes \det(\cdot)^{k/2}, \text{ if } F=\mathbb{R}, \\ (T_{\mathrm{hol}}{}^{\otimes k} \otimes \det(\cdot)^{k/2}) \otimes (T_{\mathrm{hol}}^{\otimes k} \otimes \det(\cdot)^{k/2}), \text{ if } F=\mathbb{C}, \\ (T_{\mathrm{hol}}{}^{\otimes 2k} \otimes \det(\cdot)^{k}, \text{ if } F=\mathbb{H}. \end{cases}$$

This follows directly from lemma 7.4 and the definition of the representation T_{hol} of the group $U(\infty)$ (see sections 3.3 and 3.5).

7.12

COROLLARY. For any $n=1, 2, \ldots$, the subspace $H(W_F^{(n)\otimes k}) \subset H(W_F^{\otimes k})$ may be characterized in an intrinsic manner as a subspace of invariants of an appropriate subgroup $M_n \subset M$.

PROOF. Consider the subgroup $U_n(\infty) \subset U(\infty)$ defined in section 2.4. The subspace $\mathscr{H}(\mathbb{C}^{k,n}) \subset \mathscr{H}(\mathbb{C}^{k,\infty})$ may be characterized as the subspace of $U_n(\infty)$-invariants for the representation $T_{\mathrm{hol}}^{\otimes k}$. Further, $U_n(\infty)$ can be replaced by its subgroup $SU_n(\infty) = [U_n(\infty), U_n(\infty)]$, which is dense in it with respect to the weak operator topology. This subgroup is convenient, as the character $\det(\cdot)$ on it is trivial. Now lemma 7.11 shows that it is possible to take $SU_n(\infty)$ (if $F=\mathbb{R}, \mathbb{H}$) or $SU_n(\infty) \times SU_n(\infty)$ (if $F=\mathbb{C}$) as M_n.

7.13

By analogy with section 6.12, we shall define in $H(W_F^{\otimes k})$ the representation R_k of the group $U(k, F) = O(k), U(k), Sp(k)$. On the subspace $H(W_F^{(n)\otimes k})$, the representation R_k reduces to $R_k^{(n)}$.

THEOREM. $U(k, F)$ is a symmetry group for $W_F^{\otimes k} (k=1, 2, \ldots)$, while all the irreducible representations $\pi \in U(k, F)^\wedge$ enter into the decomposition of the representation R_k.

PROOF. The first statement follows from R. Howe's theorem (section 6.14) and corollary 7.12. The second assertion follows from lemma 7.1.

7.14

COROLLARY

(i) In conformity with the general principle (section 1.12), to any $\pi \in U(k, F)^\wedge$ corresponds a certain irreducible unitary representation W_π of the group L.

(ii) W_π is the inductive limit of the irreducible representations $W_\pi^{(n)}$ of the groups $L(n)$ defined in section 6.15. The subspace $H(W_\pi^{(n)})$ in

$H(W_\pi)$ may be characterized as the subspace of invariants of the subgroup M_n, see corollary 7.12.

(iii) Two representations W_{π_1} and W_{π_2} of the group L are equivalent if and only if π_1 and π_2 are equivalent representations of one and the same group $U(k, F)$.

(iv) The representation conjugate to W_π is equivalent to $W_\pi \circ \varphi$, where φ is the outer automorphism from section 7.7.

PROOF. The statements (i) and (ii) are obvious. Assertion (iii) follows from (ii) and theorem 7.6; it may be derived also from the fact that for different k the representations $W_F^{\otimes k}|[M, M]$ do not have equivalent irreducible components; the latter may be derived from lemma 7.11. Statement (iv) follows from theorem 7.7.

7.15

Remark. When $F=\mathbb{C}$ (and only in this case), the groups $L(n)=U(n, n)^{\sim}$ and $L=U(\infty,\infty)^{\sim}$ are not semisimple. All results in §§6–7 remain in force if these groups are replaced by $SU(n, n)$ and $SU(\infty,\infty)$ respectively. Then the group $M(n)=(U(n)\times U(n))^{\sim}$ must be replaced by its subgroup.

$$\{(u_1, u_2)\in U(n)\times U(n): \det u_1 = \det u_2\}.$$

7.16

Remark. The assertion of corollary 7.12 makes sense even for $n=0$. It indicates then that the one-dimensional subspace in $\mathscr{H}(\mathbb{C}^{k,\infty})$ generated by the function $f_0 \equiv 1$ may be characterized as the subspace of M_0-invariants, where $M_0=[M, M]$.

7.17

Remark. Let us assume that $F=\mathbb{R}, \mathbb{C}$. Then the group M coincides with a 2-covering over the group K^* and the canonical embedding $K \to K^*$ (see section 2.13) may be lifted into M. Lemma 7.11 shows that the representation $W_F^{\otimes k}|M$ differs from the holomorphic extension of the representation $W_F^{\otimes k}|K$ only by a scalar factor. Hence it is evident that as M_n one may take equally successfully the subgroup K_n (see section 2.4).

These assertions cover also the case $F=\mathbb{H}$, if the group $L=SO^*(2\infty)$ is realized as $SO^*(4\infty)$, $L(n)$ and M are taken as $SO^*(4n)$ and $U(2\infty)$ respectively, and $W_{\mathbb{H}}^{(n)}$ is understood as the Weil representation of the group $SO^*(4n)$. We act just in this manner, starting in §9.

§8. Spherical functions and properties of continuity for Weil representations

8.1

By the remark 7.16, the function $f_0\equiv 1$ from $H(W_F^{\otimes k})$ is unique to within a factor, $[M, M]$-invariant vector. We shall denote by φ_k the corresponding spherical function:

$$\varphi_k(g)=(W_F^{\otimes k}(g)f_0, f_0) \quad (g\in L).$$

Let us assign to an element $g\in L$ the matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

which either coincides with $g\in SO^*(2\infty)$ (in the case $F=\mathbb{H}$) or is the image of the element g in the groups $Sp(\infty, \mathbb{R})$, $U(\infty,\infty)$ (in the cases $F=\mathbb{R}, \mathbb{C}$).

THEOREM.

$$\varphi_k(g)=(\det a)^{-k/2}=(\overline{\det d})^{-k/2} \qquad (F=\mathbb{R}),$$
$$\varphi_k(g)=\det(a\bar{d})^{-k/2}=(\det g)^{k/2}(\det a)^{-k} \quad (F=\mathbb{C}),$$
$$\varphi_k(g)=(\det a)^{-k}=(\overline{\det d})^{-k} \qquad (F=\mathbb{H}).$$

(In the case when $F=\mathbb{R}$, $\mathbb{C}$ and k is odd, it is easy to check that the right-hand side correctly defines a function on the group L.)

PROOF. It is sufficient to prove the analogous proposition for the representation $W_F^{(n)\otimes k}$ of the group $L(n)$. The function f_0 (like any polynomial) is an analytic vector of the representation, hence φ_k is a real analytic function on the group $L(n)$. Hence it is possible, without loss of generality, to restrict ourselves to elements g close to e.

Let us extend φ_k to the holomorphic function $\tilde{\varphi}_k$ defined in the local complexification of the group $L(n)$. As f_0 is annihilated by the subalgebra $l_1(n)\subset l_{\mathbb{C}}^{(n)}$ (see section 6.3), the function $\tilde{\varphi}_k$ is left (respectively right) invariant with respect to the local complex Lie subgroup corresponding to the subalgebra $l_{-1}(n)$ (respectively $l_1(n)$). The function on the right-hand side of our formulae also possesses the same properties.

Hence it is possible to assume that g lies in the local group corresponding to the subalgebra $l_0(n)$. But $l_0(n)$ is the complexification of the subalgebra $m(n)$. Hence it is sufficient to check our formulae for $g\in M(n)$. But in this case they quickly follow from lemma 7.4.

8.2

We shall denote by $\overline{Sp}(\infty,\mathbb{R})$ the complete symplectic group, i.e., the group of all operators in $l^2(\mathbb{C})\oplus l^2(\mathbb{C})$ preserving the indefinite scalar product and symplectic form given respectively by the operator matrices

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Let $Sp_1(\infty,\mathbb{R})$ be the subgroup consisting of those $g\in\overline{Sp}(\infty,\mathbb{R})$ for which $g-1$ is a nuclear operator. We endow it with the topology induced by the nuclear norm. The group $Sp(\infty,\mathbb{R})$ is dense in it.

Just as $Sp(\infty,\mathbb{R})$, the group $Sp_1(\infty,\mathbb{R})$ possesses a unique two-sheeted covering which we shall denote by $Sp_1(\infty,\mathbb{R})^{\sim}$.

8.3

THEOREM. The Weil representation $W_{\mathbb{R}}$ of the group $Sp(\infty,\mathbb{R})^{\sim}$ extends to a continuous representation of the group $Sp_1(\infty,\mathbb{R})^{\sim}$.

PROOF (*beginning*). $W_{\mathbb{R}}$ possesses the symmetry group $O(1)=\{\pm 1\}$ and this means it is the direct sum of two irreducible representations W_1, W_{-1} which are realized in the subspaces of even and odd functions respectively.

The spherical vector $f_0 \equiv 1$ lies in $H(W_1)$. It is easy to check that the corresponding spherical function $\varphi_1(g)$ (see section 8.1) possesses a continuous extension to $Sp_1(\infty, \mathbb{R})^{\sim}$. This proves the assertion of the theorem for the subrepresentation W_1.

For the final part of the proof, see section 8.9.

8.4

DEFINITION. Let H be a complex Hilbert space. We call the space $\mathbb{R} \times H$ with the following multiplication:

$$(x, \xi)\cdot(y, \eta)=(x+y-\operatorname{Im}(\xi, \eta)_H, \xi+\eta) \text{ (here } x, y\in\mathbb{R},\ \xi, \eta\in H)$$

the *Heisenberg group* Heis (H).

8.5

THEOREM. In Fock space $\mathscr{F}(\bar{H})=\mathbb{C}\oplus\bar{H}\oplus S^2(\bar{H})\oplus\ldots$ there exists a unitary representation V of the group Heis (H), for which $\mathbb{1}$ is a cyclic vector and

$$(V(x, \xi)\cdot 1, 1)=\exp(ix-\|\xi\|^2/2).$$

This representation is irreducible.

PROOF. First let H be finite-dimensional. Let us identify it with $\mathbb{C}^{1,n}$. Let us define V as the representation in $\mathscr{H}(\mathbb{C}^{1,n})$ given by the formula

$$V(x, \xi)f(z)=\exp(ix-\xi\xi^*/2-z\xi^*)f(z+\xi),$$

where

$$z, \xi\in\mathbb{C}^{1,n}, \quad x\in\mathbb{R}, \quad f\in\mathscr{H}(\mathbb{C}^{1,n}).$$

The spherical function of this representation has the necessary form. It is easy also to check its irreducibility.

Now let H be infinite-dimensional. Let us identify it with $l_2(\mathbb{C})$. We shall examine the subgroup

$$\bigcup_{n=1}^{\infty} \operatorname{Heis}(\mathbb{C}^{1,n})\subset \operatorname{Heis}(l_2(\mathbb{C})).$$

On this subgroup the representation V may be defined as the inductive limit of the corresponding representations of the groups Heis $(\mathbb{C}^{1,n})$. This representation admits a continuous extension for the whole group Heis $(l_2(\mathbb{C}))$, since its spherical function admits such an extension.

8.6

Let heis $(\mathbb{C}^{1,n})$ be the Lie algebra of the group Heis $(\mathbb{C}^{1,n})$. Its elements are the pairs $(x, \xi)\in\mathbb{R}\oplus\mathbb{C}^{1,n}$ with the law of composition

$$[(x, \xi), (y, \eta)]=(-2\operatorname{Im}(\xi, \eta), 0).$$

All polynomials from $\mathbb{C}[z_1, \ldots, z_n]$ are analytic vectors for the representation V of the group Heis $(\mathbb{C}^{1,n})$ in $\mathscr{H}(\mathbb{C}^{1,n})$. The representation of the Lie algebra heis $(\mathbb{C}^{1,n})$ in $\mathbb{C}[z_1, \ldots, z_n]$ has the following form:

$$(1,0)\mapsto i\cdot 1,$$

$$(0, e_j)\mapsto -M_j+D_j, \quad (0, ie_j)\mapsto iM_j+iD_j$$

(here $e_1, \ldots, e_n$ is the canonical basis in $\mathbb{C}^{1,n}$).

8.7

For

$$g=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\in\overline{Sp}(\infty, \mathbb{R}) \text{ and } (x, \xi)\in \text{ Heis}(l_2(\mathbb{C}))$$

let us put

$${}^{g}(x, \xi)=(x, \xi a'-\bar{\xi}\, b').$$

LEMMA. The given formula defines the action of the group $\overline{Sp}(\infty, \mathbb{R})$ by means of automorphisms on Heis $(l_2(\mathbb{C}))$.

PROOF. It is easy to check that, for any $g \in \overline{Sp}(\infty, \mathbb{R})$, we have $a = \bar{d}$, $b = \bar{c}$. It follows from this that the representation

$$\xi \oplus \eta \mapsto (\xi \oplus \eta) g'$$

of the group $\overline{Sp}(\infty, \mathbb{R})$ in $l_2(\mathbb{C}) \oplus l_2(\mathbb{C})$ preserves the subspace of vectors of the type $\xi \oplus (-\bar{\xi})$. It is easy to check that here ξ is sent into $\xi a' - \bar{\xi} b'$. This means that our formula actually defines a certain action of the group $\overline{Sp}(\infty, \mathbb{R})$. It is easy to check that this action preserves the multiplication in Heis $(l_2(\mathbb{C}))$.

8.8

LEMMA. For any $g \in Sp(\infty, \mathbb{R})^{\sim}$ and any $h \in \mathrm{Heis}(l_2(\mathbb{C}))$, we have

$$W_{\mathbb{R}}(g)\ V(h)\, W_{\mathbb{R}}(g)^{-1} = V({}^{[g]}h),$$

where $[g]$ is the image of the element g in $Sp(\infty, \mathbb{R})$.

PROOF. Without loss of generality, we may suppose that h lies in the dense subgroup

$$\bigcup_{n=1}^{\infty} \mathrm{Heis}(\mathbb{C}^{1,n}).$$

Hence it is sufficient to prove the analogous proposition for the groups $Sp(n, \mathbb{R})^{\sim}$ and $\mathrm{Heis}(\mathbb{C}^{1,n})$. This is easily done at the level of the Lie algebras.

8.9

The final part of the proof of theorem 8.3. The vector $f_0 \in H(W_{\mathbb{R}})$ is a cyclic vector for the representation V of the group $\mathrm{Heis}(l_2(\mathbb{C}))$. Hence it is sufficient to show that any function of the type

$$g \mapsto (W_{\mathbb{R}}(g) V(h_1) f_0,\ V(h_2) f_0), \text{ where } h_1, h_2 \in \mathrm{Heis}(l_2(\mathbb{C}))$$

admits a continuous extension to $Sp_1(\infty, \mathbb{R})^{\sim}$.

By virtue of lemma 8.8, the right-hand side is

$$(W_{\mathbb{R}}(g) f_0,\ V({}^{[g]}h_1)^{-1}\, V(h_2) f_0).$$

It remains to observe that the vector-function

$$g \mapsto W_{\mathbb{R}}(g) f_0 = W_1(g) f_0$$

admits a continuous extension to $Sp_1(\infty, \mathbb{R})^{\sim}$ (see section 8.3) and that ${}^{|g|}h_1$ also depends continuously on g.

8.10

COROLLARY. The representations W_π of the group $Sp(\infty, \mathbb{R})^{\sim}$, defined in section 7.14, admit a continuous extension to the group $Sp_1(\infty, \mathbb{R})^{\sim}$.

8.11

Remark. By analogy with section 8.2, we may define the topological groups $U_1(\infty,\infty)^{\sim}$ and $SO_1^*(2\infty)$. They may be defined also as the closure of the groups $U(\infty,\infty)^{\sim}$ and $SO^*(2\infty)$ respectively in $Sp_1(2\infty, \mathbb{R})^{\sim}$. After this, theorem 8.3 and corollary 8.10 are immediately transferred to the groups $U_1(\infty,\infty)^{\sim}$ and $SO_1^*(2\infty)$.

8.12

The group $\overline{Sp}(\infty, \mathbb{R})$ acts by conjugations on its normal subgroup $Sp_1(\infty, \mathbb{R})$ and this action is lifted to $Sp_1(\infty, \mathbb{R})^{\sim}$. We shall denote this action thus:

$$g \mapsto {}^h g, \quad \text{where} \quad g \in Sp_1(\infty, \mathbb{R})^{\sim}, \quad h \in \overline{Sp}(\infty, \mathbb{R}).$$

Let us denote by $U(\infty)$ the group of all unitary operators in $l^2(\mathbb{C})$ topologized by the strong ($=$weak) operator topology (this is the group $\bar{K}$ from section 2.19 for the case $F=\mathbb{C}$). We embed it in $\overline{Sp}(\infty, \mathbb{R})$ by means of the mapping

$$u \mapsto \begin{bmatrix} \bar{u} & 0 \\ 0 & u \end{bmatrix} (u \in \bar{U}(\infty)).$$

We shall denote the extension of the representations T_{hol} and $W_{\mathbb{R}}$ of the groups $U(\infty)$ and $Sp(\infty, \mathbb{R})^{\sim}$ to $\bar{U}(\infty)$ and $Sp_1(\infty, \mathbb{R})^{\sim}$ respectively by the same letters.

8.13

LEMMA. For any $u \in \bar{U}(\infty)$ and for any $g \in Sp_1(\infty, \mathbb{R})^{\sim}$ we have:

$$W_{\mathbb{R}}({}^u g) = T_{\text{hol}}(u)\, W_{\mathbb{R}}(g)\, T_{\text{hol}}(u)^{-1}.$$

PROOF. For fixed g, the mapping $u \mapsto {}^u g$ from $\bar{U}(\infty)$ into $Sp_1(\infty, \mathbb{R})^{\sim}$ is continuous (we omit a simple verification of this fact). As $U(\infty)$ is dense in $\bar{U}(\infty)$, we may assume $u \in U(\infty)$. Further, it may be supposed that $g \in Sp(\infty, \mathbb{R})^{\sim}$. But then the assertion of the lemma follows from the fact that $T_{\text{hol}}(u)$ differs from $W_{\mathbb{R}}(u)$ only by a scalar factor (see lemma 7.11).

8.14

COROLLARY. Let u be a certain element from $\bar{U}(\infty)$ and $\mathscr{G}$ a certain subgroup in $Sp(\infty, \mathbb{R})^{\sim}$ such that ${}^u g \in Sp(\infty, \mathbb{R})^{\sim}$ for all $g \in \mathscr{G}$. Then the representations

$$g \mapsto W_{\pi}(g) \text{ and } g \mapsto W_{\pi}({}^u g) \quad (\pi \in O(k)^{\wedge},\ k = 1, 2, \dots)$$

of the group $\mathscr{G}$ are equivalent.

This result will be repeatedly used in what follows; it is easily transferred to the groups $U(\infty, \infty)^{\sim}$ and $SO^*(2\infty)$.

PART III

ADMISSIBLE REPRESENTATIONS OF THE PAIRS (*G*, *K*) OF NON-COMPACT TYPE

§9. Two constructions of fundamental representations of the groups $GL(\infty, F)$, $F = \mathbb{R}, \mathbb{C}, \mathbb{H}$

9.1

For $F = \mathbb{R}, \mathbb{C}, \mathbb{H}$, we shall use the notation given in the following table.

Table 7.4

	$F=\mathbb{R}$	$F=\mathbb{C}$	$F=\mathbb{H}$
G	$GL^+(\infty,\mathbb{R})$	$GL(\infty,\mathbb{C})$	$GL(\infty,\mathbb{H})$
$G(n)$	$GL^+(n,\mathbb{R})$	$GL(n,\mathbb{C})$	$GL(n,\mathbb{H})$
K	$SO(\infty)$	$U(\infty)$	$Sp(\infty)$
$K(n)$	$SO(n)$	$U(n)$	$Sp(n)$
L	$Sp(\infty,\mathbb{R})^{\sim}$	$U(\infty,\infty)^{\sim}$	$SO^*(4\infty)$
$L(n)$	$Sp(n,\mathbb{R})^{\sim}$	$U(n,n)^{\sim}$	$SO^*(4n)$
M	$U(\infty)^{\sim}$	$(U(\infty)\times U(\infty))^{\sim}$	$U(2\infty)$
$M(n)$	$U(n)^{\sim}$	$(U(n)\times U(n))^{\sim}$	$U(2n)$
$U(k,F)$	$O(k)$	$U(k)$	$Sp(k)$

Here $GL^+(n,\mathbb{R})$ denotes the component of unity in $GL(n,\mathbb{R})$. As usual,

$$G=\bigcup_{n=1}^{\infty} G(n),$$

etc. We observe a small divergence from Table 7.3 in the definition of the groups L and M for $F=\mathbb{H}$: instead of $SO^*(2\infty)$ it is more convenient for us to investigate the group

$$SO^*(4\infty)=\bigcup_{n=1}^{\infty} SO^*(4n)$$

isomorphic to it. We need also the Lie algebras corresponding to the groups from Table 7.4; for example,

$$\mathfrak{gl}(\infty,F)=\bigcup_{n=1}^{\infty}\mathfrak{gl}(n,F).\text{ etc.}$$

9.2

Let us define an embedding $K\to M$ by the following mapping of the corresponding Lie algebras.

In the case $F=\mathbb{R}$: the identical mapping $\mathfrak{so}(\infty)\to u(\infty)$.

In the case $F=\mathbb{C}$: the mapping $A\mapsto(\bar{A}, A)$ from $u(\infty)$ to $u(\infty)\oplus u(\infty)$. (Let us recall that the last algebra is embedded in $u(\infty,\infty)$ by means of the mapping defined in section 7.3; to sum up, the embedding $u(\infty)\to u(\infty,\infty)$ has the form

$$A\mapsto\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}.\Big)$$

In the case $F=\mathbb{H}$: the mapping β from $\mathfrak{sp}(\infty)$ to $u(2\infty)$.

9.3

Let us examine the space $F^\infty=F^{1,\infty}$ with the Gaussian measure μ (section 4.1). The measure μ is quasiinvariant with respect to the natural right action of the group G on F^∞:

$$\frac{d\mu(xg)}{d\mu(x)}=\sigma(x, g)\, D(g),$$

where

$$\sigma(x, g)=\exp\left(-\frac{1}{2}\, x(gg^*-1)\, x^*\right), \quad x\in F^\infty, \quad g\in G,$$

($D(\cdot)$ is defined in section 1.5).

9.4

For $s\in\mathbb{R}$ we define the unitary representation T_s of the group G in $L^2(F^\infty)$ as follows:

$$T_s(g)f(x)=\sigma(x, g)^{(1+is)/2}\, D(g)^{1/2} f(xg),$$

$$(f\in L^2(F^\infty)), \quad x\in F^\infty, \quad g\in G.$$

We shall call the representations T_s *fundamental.* A second construction of the fundamental representations is given below (section 9.6).

We observe that $T_s|K$ does not depend on $s\in\mathbb{R}$ and coincides with the representation T_F of the group K (section 4.4). Hence T_s is an admissible representation in the sense of the following definition.

9.5

DEFINITION. The unitary representation T of the group G is called *admissible* if the representation $T|K$ is tame.

We observe that the subspace $H_\infty(T|K)$ is algebraically invariant with respect to G. Hence for an irreducible T the admissibility condition is equivalent to the fact that $H_\infty(T|K)\neq\{0\}$.

9.6

THEOREM. There exist the embeddings $\tau_s^F:G\to L$ such that (for each $s\in\mathbb{R}$) the representations $W_F\circ\tau_s^F$ and T_s of the group G are equivalent.

By definition, $W_F\circ\tau_s^F$ denotes the representation $g\mapsto W_F(\tau_s^F(g))$. A detailed proof of this theorem is given in **§10**. In section 9.7 we shall outline the idea of the proof. The remaining part of this paragraph is devoted to the structure of the embeddings τ_s^F and to discussion.

9.7

The idea of the proof of theorem 9.6. Let, for example, $F=\mathbb{R}$. We shall examine the representation of the Lie algebra

$$\mathfrak{gl}(\infty,\mathbb{R})=\bigcup_{n=1}^{\infty}\mathfrak{gl}(n,\mathbb{R})$$

in $H(T_s)$ corresponding to the representation of the group G and denote it again by T_s. For any $A\in\mathfrak{gl}(\infty,\mathbb{R})$, the operator $T_s(A)$ is a finite linear combination of operators in $L^2(\mathbb{R}^\infty)$ of the type

$$x_i x_j 1,\quad \frac{\partial^2}{\partial x_i\,\partial x_j},\quad \frac{1}{2}\left[x_i 1,\frac{\partial}{\partial x_j}\right]_+=x_i\frac{\partial}{\partial x_j}+\delta_{ij}/2.$$

But these operators form exactly the Weil representation $W_{\mathbb{R}}$ of the Lie algebra $\mathfrak{sp}(\infty,\mathbb{R})$, which becomes clear if we move from $L^2(\mathbb{R}^\infty)$ to $\mathscr{H}(\mathbb{C}^{1,\infty})$ with the help of the transform $I_{\mathbb{R}}$ (see section 4.10).

9.8

The embedding $\tau_s^F: G \to L (s \in \mathbb{R})$ is conveniently given by the corresponding mapping of the Lie algebra which we again denote by τ_s^F:

$$\tau_s^{\mathbb{R}}: gl(\infty, \mathbb{R}) \to sp(\infty, \mathbb{R}),$$

$$\tau_s^{\mathbb{C}}: gl(\infty, \mathbb{C}) \to u(\infty, \infty),$$

$$\tau_s^{\mathbb{H}}: gl(\infty, \mathbb{H}) \to so^*(4\infty).$$

The elements $A \in \mathfrak{gl}(\infty, F)$ are conveniently written in the form

$$A = X + Y, \text{ where } X = \frac{A - A^*}{2}, \ Y = \frac{A + A^*}{2}.$$

(We observe that X lies in the Lie algebra of the group K.) In this notation, τ_s^F is given as follows:

$$\tau_s^{\mathbb{R}}: X + Y \mapsto \begin{bmatrix} X + is\,Y & (1 - is)Y \\ (1 + is)Y & X - is\,Y \end{bmatrix},$$

$$\tau_s^{\mathbb{C}}: X + Y \mapsto \begin{bmatrix} X + is\,Y & -(i + s)Y \\ (i - s)Y & X - is\,Y \end{bmatrix},$$

$$\tau_s^{\mathbb{H}}: X \mapsto \begin{bmatrix} w\beta(X)w^{-1} & 0 \\ 0 & \beta(X) \end{bmatrix},$$

$$\tau_s^{\mathbb{H}}: Y \mapsto \begin{bmatrix} is\, w\beta(Y)w^{-1} & -(i + s)w\beta(Y) \\ (i - s)\beta(Y)w^{-1} & -is\,\beta(Y) \end{bmatrix},$$

where

$$w = \begin{bmatrix} 0 & 1_\infty \\ -1_\infty & 0 \end{bmatrix} \in \bar{U}(2\infty).$$

In a more detailed manner, if $X = X_1 + X_2 j$ and $Y = Y_1 + Y_2 j$, then

$$\tau_s^{\mathbb{H}}: X \mapsto \begin{bmatrix} \bar{X}_1 & \bar{X}_2 & 0 & 0 \\ -X_2 & X_1 & 0 & 0 \\ 0 & 0 & X_1 & X_2 \\ 0 & 0 & -\bar{X}_2 & \bar{X}_1 \end{bmatrix},$$

$$\tau_s^{\mathbb{H}}: Y \mapsto \begin{bmatrix} is\begin{bmatrix} \bar{Y}_1 & \bar{Y}_2 \\ -Y_2 & Y_1 \end{bmatrix} & -(i+s)\begin{bmatrix} -\bar{Y}_2 & \bar{Y}_1 \\ -Y_1 & -Y_2 \end{bmatrix} \\ (i-s)\begin{bmatrix} Y_2 & -Y_1 \\ \bar{Y}_1 & \bar{Y}_2 \end{bmatrix} & -is\begin{bmatrix} Y_1 & Y_2 \\ -\bar{Y}_2 & \bar{Y}_1 \end{bmatrix} \end{bmatrix}.$$

9.9

In the case $F = \mathbb{R}$, we have

$$\tau_s^{\mathbb{R}}(A) = h_s \tau_0^{\mathbb{R}}(A) h_s^{-1},$$

where

$$h_s = \begin{bmatrix} \left(1 - \frac{is}{2}\right) 1_\infty & \frac{is}{2} 1_\infty \\ -\frac{is}{2} 1_\infty & \left(1 + \frac{is}{2}\right) 1_\infty \end{bmatrix} \in \overline{Sp}(\infty, \mathbb{R}).$$

This shows that $\tau_s^{\mathbb{R}}$ is in fact a morphism of the Lie algebras, because for the mapping

$$\tau_0^{\mathbb{R}}: X + Y \mapsto \begin{bmatrix} X & Y \\ Y & X \end{bmatrix}$$

this is obvious. It is clear that $\tau_s^{\mathbb{R}}(\mathfrak{gl}(\infty, \mathbb{R})) \subset \mathfrak{sp}(\infty, \mathbb{R})$.

9.10

Analogously, in the case $F=\mathbb{C}$, we have

$$\tau_s^{\mathbb{C}}(A)=\tilde{h}_s\tau_0^{\mathbb{C}}(A)\tilde{h}_s^{-1},$$

where

$$\tilde{h}_s=\begin{bmatrix}-i1_\infty & 0\\ 0 & 1_\infty\end{bmatrix}h_s\in \bar{U}(\infty,\infty).$$

This shows that $\tau_s^{\mathbb{C}}$ is in fact a morphism of the Lie algebras, because for the mapping

$$\tau_0^{\mathbb{C}}: X+Y\mapsto\begin{bmatrix}X & -iY\\ iY & X\end{bmatrix}$$

this is obvious. It is clear that $\tau_s^{\mathbb{C}}(\mathfrak{gl}(\infty,\mathbb{C}))\subset u(\infty,\infty)$.

9.11

Finally, in the case $F=\mathbb{H}$, we have

$$\tau_s^{\mathbb{H}}(A)=h\tau_s^{\mathbb{C}}(\beta(A))h^{-1},$$

where

$$h=\begin{bmatrix}w & 0\\ 0 & 1\end{bmatrix}\in \bar{U}(2\infty,2\infty).$$

This shows that $\tau_s^{\mathbb{H}}$ is in fact a morphism of the Lie algebras. From the definition of X and Y it follows that

$$X_1=-X_1^*,\quad X_2=X_2',\quad Y_1=Y_1^*,\quad Y_2=-Y_2'.$$

From this it is easy to deduce that $\tau_s^{\mathbb{H}}(\mathfrak{gl}(\infty,\mathbb{H}))\subset so^*(4\infty)$.

9.12

LEMMA. $\tau_s^F|K$ does not depend on $s\in\mathbb{R}$ and coincides with the embedding $K\to M$ defined in 9.2.

This is easily checked.

9.13

LEMMA. $\tau_s^F(G(n)) \subset L(n)$.
This is obvious.

9.14

Remark. From the definition of τ_s^F, it can be seen that, for any $n = 1, 2, \ldots,$

$$\tau_s^F \mid G(n) = \varphi_s^{(n)} \circ (\tau_0^F \mid G(n)) \quad (s \in \mathbb{R}),$$

where $\varphi_s^{(n)}$ is a certain inner automorphism of the group $L(n)$; for example, in the case $F = \mathbb{R}$, it is given by the matrix

$$h_s^{(n)} = \begin{bmatrix} \left(1 - \frac{is}{2}\right) 1_n & \frac{is}{2} 1_n \\ -\frac{is}{2} 1_n & \left(1 + \frac{is}{2}\right) 1_n \end{bmatrix} \in Sp(n, \mathbb{R}).$$

However, this does not mean that the representation $W_F \circ \tau_s^F$ does not depend on s: in **§12** we shall see that the representations $W_F \circ \tau_s^F$ for different s are pair-wise not equivalent; the simplest way of seeing this is to observe that the spherical function $\varphi_1 \circ \tau_s^F$ depends substantially on s (the function φ_1 on L was defined in section 8.1).

The fact is that the "inductive limit" of the inner automorphisms $\varphi_s^{(n)}$ as $n \to \infty$ is already an outer automorphism of the group L.

9.15

Remark. We now try to explain the "natural origin" of the family of embeddings $\{\tau_s^F\}$.

For simplicity, let $F = \mathbb{R}$. We could examine more general embeddings

$$\tau_h^{\mathbb{R}}: GL^+(\infty, \mathbb{R}) \to Sp_1(\infty, \mathbb{R})^{\sim} \text{ of the type } g \mapsto {}^h(\tau_0^{\mathbb{R}}(g)),$$

where h is an arbitrary element from $\overline{Sp}(\infty, \mathbb{R})$. We require that $\tau_h^{\mathbb{R}}|SO(\infty)$ coincide with $\tau_0^{\mathbb{R}}|SO(\infty)$; this requirement is natural as it ensures the admissibility of the representation.

But then h should have the form

$$h = \begin{bmatrix} a\cdot 1_\infty & b\cdot 1_\infty \\ c\cdot 1_\infty & d\cdot 1_\infty \end{bmatrix}, \text{ where } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Sp(1, \mathbb{R}) = SU(1, 1).$$

In view of corollary 8.14, the equivalence class of the representation $W_{\mathbb{R}} \circ \tau_h^{\mathbb{R}}$ of the group $GL^+(\infty, \mathbb{R})$ does not change if the matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Sp(1, \mathbb{R})$$

is multiplied on the left by

$$\begin{bmatrix} \bar{\theta} & 0 \\ 0 & \theta \end{bmatrix},$$

where $\theta \in \mathbb{C}$, $|\theta| = 1$. On the other hand,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

may be multiplied on the right by a matrix of the type

$$\begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix}, \quad t \in \mathbb{R},$$

as this does not change the embedding $\tau_h^{\mathbb{R}}$.

We now observe that any matrix from $Sp(1, \mathbb{R})$ may be represented in the form

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \bar{\theta} & 0 \\ 0 & \theta \end{bmatrix} \begin{bmatrix} 1 - \frac{is}{2} & \frac{is}{2} \\ -\frac{is}{2} & 1 + \frac{is}{2} \end{bmatrix} \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix}$$

(this is simply the Iwasawa decomposition). Thus, we return to the embeddings $\tau_s^{\mathbb{R}}$.

§10. Equivalence of constructions

10.1

Let us recall that in §4 we constructed a certain isometry I_F of the space $L^2(F^{k,\infty})$ on the Bargmann–Segal space

$$\mathscr{H}(\mathbb{C}^{k,\infty}),\quad \mathscr{H}(\mathbb{C}^{k,2\infty}),\quad \mathscr{H}(\mathbb{C}^{2k,2\infty})$$

for $F=\mathbb{R}, \mathbb{C}, \mathbb{H}$, respectively. Let us note now that the last space is precisely the space of the representation $W_F^{\otimes k}$ of the group L, and that $L^2(F^{k,\infty})$ coincides with $H(T_s^{\otimes k})$ for any $s\in\mathbb{R}$. We use below the notation of §9.

10.2

THEOREM. For any $s\in\mathbb{R}$ and all $k=1, 2, \ldots$, $I_F T_s^{\otimes k}(g) I_F^{-1} = W_F^{\otimes k}(\tau_s^F(g))$ $(g\in G)$.

For $k=1$, theorem 9.6 follows from this.

To prove theorem 10.2, it is possible, without loss of generality, to consider $k=1$. We shall study separately the cases $F=\mathbb{R}, \mathbb{C}, \mathbb{H}$.

10.3

PROOF OF THEOREM 10.2 ($F=\mathbb{R}$). Let us move away from the group $GL^+(\infty, \mathbb{R})$ to its Lie algebra $\mathfrak{gl}(\infty, \mathbb{R})$ and prove that

$$I_{\mathbb{R}} T_s(A) I_{\mathbb{R}}^{-1} = W_{\mathbb{R}}(\tau_s^{\mathbb{R}}(A)) \quad (s\in\mathbb{R},\ A\in\mathfrak{gl}(\infty, \mathbb{R})),$$

where both sides are considered as operators on the subspace $\mathbb{C}[z_1, z_2, \ldots]\subset\mathscr{H}(\mathbb{C}^{1,\infty})$.

This equality is evident for $A=X$ in view of lemma 9.12. Let now $A=Y$. It is sufficient to study the case $Y=E_{ij}+E_{ji}$, where $i, j=1, 2, \ldots$

It follows from the definition of the representation T_s that

$$T_s(E_{ij}+E_{ji})f=(-(is+1)x_ix_j+\delta_{ij})f+x_i\frac{\partial f}{\partial xj}+x_j\frac{\partial f}{\partial x_i},$$

where f is an arbitrary polynomial in $x_1, x_2, \ldots$).†

From the definition of the isometry $I_{\mathbb{R}}$: $L^2(\mathbb{R}^\infty)=\mathscr{H}(z_1, z_2, \ldots)$ it is easy to deduce that it transforms the operator of multiplication by x_i into M_i+D_i and the operator $\partial/\partial x_j$ into D_j. Hence

$$I_{\mathbb{R}}T_s(E_{ij}+E_{ji})I_{\mathbb{R}}^{-1}\varphi=[-is(M_iD_j+M_jD_i+\delta_{ij})$$
$$+(1-is)D_iD_j-(1+is)M_iM_j]\varphi=W_{\mathbb{R}}(\tau_s^{\mathbb{R}}(E_{ij}+E_{ji})\varphi$$

for any polynomial $\varphi\in\mathbb{C}[z_1, z_2, \ldots]$.

We now observe that, for any $A\in\mathfrak{gl}(\infty,\mathbb{R})$, the operators

$$\frac{d}{dt}I_{\mathbb{R}}T_s(\exp tA)I_{\mathbb{R}}^{-1}\bigg|_{t=0} \quad\text{and}\quad \frac{d}{dt}W_{\mathbb{R}}(\tau_s^{\mathbb{R}}(\exp tA))\bigg|_{t=0}$$

are essentially skew-adjoint on the subspace

$$\mathbb{C}[z_1, z_2, \ldots]\subset\mathscr{H}(z_1, z_2, \ldots),$$

since all polynomials are analytic vectors for the representation $W_{\mathbb{R}}$. It means that these two operators coincide, which proves the theorem.

10.4

Let us realize the group $GL^+(\infty,\mathbb{R})$ as

$$GL^+(2\infty,\mathbb{R})=\bigcup_{n=1}^{\infty} GL^+(2n,\mathbb{R}).$$

We shall denote the fundamental representations T_s of the group $GL^+(2\infty,\mathbb{R})$ and $GL(\infty,\mathbb{C})$ by $T_s^{\mathbb{R}}$ and $T_s^{\mathbb{C}}$ respectively. Let us identify the spaces

$$H(T_s^{\mathbb{C}})=L^2(\mathbb{C}^\infty) \quad\text{and}\quad H(T_s^{\mathbb{R}})=L^2(\mathbb{R}^\infty\times\mathbb{R}^\infty)$$

†The letter i here denotes simultaneously $(-1)^{1/2}$ and an index $1, 2, \ldots$.

by means of the mapping $x \mapsto (\operatorname{Re} x, \operatorname{Im} x)\ (x \in \mathbb{C}^\infty)$.

LEMMA. $T_s^{\mathbb{C}} = T_s^{\mathbb{R}} \circ \alpha$ in the sense that

$$T_s^{\mathbb{C}}(g) = T_s^{\mathbb{R}}(\alpha(g)) \quad (g \in GL(\infty, \mathbb{C}),\ s \in \mathbb{R}).$$

This is obvious. (The mapping α is defined in section 1.3.)

10.5

PROOF OF THEOREM 10.2 ($F = \mathbb{C}$). Let us realize the group $Sp(\infty, \mathbb{R})^\sim$ as $Sp(2\infty, \mathbb{R})^\sim$ and observe that

$$W_{\mathbb{C}}(g) = W_{\mathbb{R}}(\gamma(g))\ (g \in U(\infty, \infty)^\sim),$$

where γ indicates the embedding $U(\infty, \infty)^\sim \to Sp(2\infty, \mathbb{R})^\sim$ definable as in section 6.9. Along with lemma 10.4, this reduces the statement of the theorem to the following formula

$$I_{\mathbb{C}} T_s^{\mathbb{R}}(\alpha(g)) I_{\mathbb{C}}^{-1} = W_{\mathbb{R}}(\gamma(\tau_s^{\mathbb{C}}(g)) \quad (g \in GL(\infty, \mathbb{C})). \tag{1}$$

Let us recall (see section 4.12) that

$$I_{\mathbb{C}} = T_{\text{hol}}(u_\infty) I_{\mathbb{R}}, \text{ where } u_\infty = \frac{1}{\sqrt{2}} \begin{bmatrix} 1_\infty & i1_\infty \\ i1_\infty & 1_\infty \end{bmatrix}.$$

(Here $T_{\text{hol}}(u_\infty)$ makes sense, as $u_\infty \in \bar{U}(2\infty)$, and T_{hol} admits an extension to $\bar{U}(2\infty) \simeq \bar{U}(\infty)$. Since, for $F = \mathbb{R}$, the theorem is already proved, (1) is equivalent to

$$T_{\text{hol}}(u_\infty) W_{\mathbb{R}}(\tau_s^{\mathbb{R}}(\alpha(g))) T_{\text{hol}}(u_\infty)^{-1} = W_{\mathbb{R}}(\gamma(\tau_s^{\mathbb{C}}(g))) \quad (g \in GL(\infty, \mathbb{C})). \tag{2}$$

In view of corollary 8.14, the left side of (2) is

$$W_{\mathbb{R}}(u_\infty \tau_s^{\mathbb{R}}(\alpha(g))).$$

Thus we have to prove that

$$u_\infty \tau_s^{\mathbb{R}}(\alpha(g)) = \gamma(\tau_s^{\mathbb{C}}(g)) \quad (g \in GL(\infty, \mathbb{C})).$$

We observe that this statement no longer relates to any representations. It is sufficient to prove the analogous fact for the Lie algebra, i.e., that

$$\begin{bmatrix} \bar{u}_\infty & 0 \\ 0 & u_\infty \end{bmatrix} \tau_s^{\mathbb{R}}(\alpha(A)) \begin{bmatrix} \bar{u}_\infty & 0 \\ 0 & u_\infty \end{bmatrix}^{-1} = \gamma(\tau_s^{\mathbb{C}}(A)), \quad A \in gl(\infty, \mathbb{C}). \tag{3}$$

Let us assume that $A = Y$ (see section 9.8). Then $\alpha(Y)$ is again an element of the type "Y", but already for $F = \mathbb{R}$. By the definition of $\tau_s^{\mathbb{R}}$, we have

$$\tau_s^{\mathbb{R}}(\alpha(Y)) = \begin{bmatrix} is\,\alpha(Y) & (1-is)\alpha(Y) \\ (1+is)\alpha(Y) & -is\,\alpha(Y) \end{bmatrix}.$$

On the other hand, it is easy to check that, for any matrix $P \in \mathbb{C}^{\infty,\infty}$, we have

$$\bar{u}_\infty \alpha(P) u_\infty = \begin{bmatrix} P & 0 \\ 0 & \bar{P} \end{bmatrix}, \quad \bar{u}_\infty \alpha(P) \bar{u}_\infty = \begin{bmatrix} 0 & -iP \\ -i\bar{P} & 0 \end{bmatrix},$$

$$u_\infty \alpha(P) u_\infty = \begin{bmatrix} 0 & i\bar{P} \\ iP & 0 \end{bmatrix}, \quad u_\infty \alpha(P) \bar{u}_\infty = \begin{bmatrix} \bar{P} & 0 \\ 0 & P \end{bmatrix}$$

(it is sufficient to check some one of these formulae and then use the fact that

$$\bar{u}_\infty = \begin{bmatrix} 0 & 1_\infty \\ -1_\infty & 0 \end{bmatrix} u_\infty \begin{bmatrix} 0 & 1_\infty \\ -1_\infty & 0 \end{bmatrix}^{-1} \Bigg).$$

Applying these formulae to $P = Y$, we find that the left-hand side in (3) is

$$\begin{bmatrix} is\,Y & 0 & 0 & -(i+s)Y \\ 0 & is\,\bar{Y} & -(i+s)\bar{Y} & 0 \\ 0 & (i-s)\bar{Y} & -is\,\bar{Y} & 0 \\ (i-s)\,Y & 0 & 0 & -is\,Y \end{bmatrix}. \tag{4}$$

But $\tilde{Y}=Y'$, since $Y=Y^*$. From the definition of the mapping γ (section 6.9) and the embedding $\tau_s^{\mathbb{C}}$ (section 9.8) it can be seen that (4) coincides with $\gamma(\tau_s^{\mathbb{C}}(Y))$, which proves (3) for $A=Y$.

In the case $A=X$, the formula (3) can be checked in an even simpler way. Notice that it is in fact proved in section 4.13.

10.6

Proof of Theorem 10.2 ($F=\mathbb{H}$). We reason by analogy with section 10.5, using the fact that for $F=\mathbb{C}$ the theorem is already proved (compare the proof of theorem 4.6).

10.7

Remark. Let us define in the space $H(T_s^{\otimes k}=L^2(F^{k,\infty})$ the unitary representation R_k of the group $U(k, F)=O(k)$, $U(k)$, $Sp(k)$ as in section 6.12. It is obvious that R_k commutes with T_s.

Theorem 4.18 and remark 4.21 show that the isometry I_t transform R_k into the representation in the space $H(W_t^{\otimes k})$ which was defined in sections 6.12 and 7.13 and denoted in section 7.13 also by R_k (in the case $F=\mathbb{H}$, both representations differ by an inner automorphism of the group $Sp(k)$, but this is an insignificant detail).

§11. Construction of irreducible admissible representations of the groups $GL(\infty, F)$

We retain the notation introduced in **§9** and constantly use the equivalence of the representations T_s and $W\circ\tau_s^t$ of the group G, proved in theorem 10.2. Let us start by formulating the main results.

11.1

Theorem. For all $k=1, 2, \ldots$ and all $s\in\mathbb{R}$, the representation $W_t^{\otimes k}$ of the group L and its restriction to the subgroup $\tau_s^t(G)$ generate identical von Neumann algebras.

For the proof see section 11.13.

11.2

COROLLARY. For all $k=1, 2, \ldots$ and all $s\in\mathbb{R}$, the representation $T_s^{\otimes k}$ of the group G possesses a symmetry group $U(k, F)$ (its action is defined in section 10.7). The following decomposition holds:

$$T_s^{\otimes k}=\bigoplus_{\pi\in U(k, F)^\wedge}(\dim \pi)\cdot T_{\pi, s},$$

where $T_{\pi, s}$ are irreducible representations of the group G defined in conformity with the general principle (section 1.12). The isometry I_F constructed in **§10** carries out the equivalence of the representations $T_{\pi, s}$ and $W_\pi\circ\tau_s^F$.

This follows from theorems 11.1 and 7.13 and from corollary 7.14.

11.3

Let $p=1, 2, \ldots$. For an arbitrary group $\mathscr{G}$ we agree to denote by $\mathscr{G}^p$ the direct product of p copies of $\mathscr{G}$.

Let us fix the numbers $k_1, \ldots, k_p=1, 2, \ldots$ and examine the representation

$$U=(W_F^{\otimes k_1})\otimes\ldots\otimes(W_F^{\otimes k_p}) \tag{1}$$

of the group L^p. Let us fix the numbers $s_1, \ldots, s_p\in\mathbb{R}$ and let us study the embedding

$$\tau=\tau_{s_1}^I\times\ldots\times\tau_{s_p}^I: G\rightarrow L^p. \tag{2}$$

THEOREM. Let us assume that the numbers $s_1, \ldots, s_p$ are pairwise distinct. Then the representation U of the group L^p and its restriction to the subgroup $\tau(G)$ generate identical von Neumann algebras.

This theorem generalizes theorem 11.1; it is proved in section 11.13.

11.4

COROLLARY. If the numbers s_i are pairwise distinct, then any representation of the group G having the form

$$T=T_{\pi_1, s_1}\otimes\ldots\otimes T_{\pi_p, s_p}, \text{ where } \pi_i\in U(k_i, F)^\wedge \quad (i=1, \ldots, p), \tag{3}$$

is irreducible and equivalent to the representation $(W_{\pi_1} \otimes \ldots \otimes W_{\pi_p}) \circ \tau$.

11.5

THEOREM. The number p and the set $\{(\pi_i, s_i)\}$, permutations of the indices i being disregarded, are invariants of the representation T.

This theorem is proved in §**12**.

11.6

THEOREM. The representation conjugate to $T_{\pi,s}$ is equivalent to $T_{\bar{\pi},-s}$ (let us recall that at $\bar{\pi} \sim \pi$ for $F \neq \mathbb{C}$).

Two proofs are given in sections 11.19–11.20.

11.7

Remark. Along with the result of section 1.14, these results fully describe the structure of the ring of representations of the group G generated by the representations T_s, $s \in \mathbb{R}$.

11.8

Remark. The representations T of the type (3) are admissible representations in the sense of definition 9.5. In the decomposition of the representation $T|K$, all the multiplicities are finite.

11.9

Remark. As will be seen from the proof, all the results remain valid if we replace G by $[G,G] = SL(\infty, F)$.

11.10

We agree to denote by d the diagonal embedding of the group M in $M^p \subset L^p$.

THEOREM. If the numbers s_i are pairwise distinct, then the group $[L, L]^p$ is algebraically generated by its subgroups $\tau([G, G])$ and $d([M, M])$.

PROOF. For proof, see section 11.18. We note that $[L, L] = L$ for $F \neq \mathbb{C}$; $[L, L] = SU(\infty, \infty)$ for $F = \mathbb{C}$; $[M, M]$ is $SU(\infty)$, $SU(\infty) \times SU(\infty)$ or $SU(2\infty)$ for $F = \mathbb{R}, \mathbb{C}, \mathbb{H}$ respectively.

11.11

Let us proceed to the proofs. Consider the representation $T = U \circ \tau$ of the group G, where U and τ are defined by (1) and (2). Let $\rho = T|K$. This representation is equivalent to the representation $T_t{}^k$ from section 4.4, where $k = k_1 + \ldots + k_p$. Hence it is tame and possesses a holomorphic extension ρ^* to the group K^*, which was explicitly constructed in theorem 4.6.

Let us recall that M either coincides (if $F = \mathbb{H}$) or "almost coincides" (if $F = \mathbb{R}, \mathbb{C}$) with the group K^*. In any case, we can identify $[M, M]$ and $[K^*, K^*]$ (see section 7.17).

LEMMA. The representation ρ^* of the group K^* differs from the representation $U \circ d$ of the group M only by a scalar factor (which is a one-dimensional representation). In particular, both representations coincide on the group $[K^*, K^*] = [M, M]$.

PROOF. The representation $U \circ d$ of the group M coincides with $W_t{}^{\otimes k}|M$. The latter representation, coincides, to within a scalar factor, with a certain holomorphic representation of the group K^* (lemma 7.11). On the other hand, $(U \circ d)|K = \rho$, since $\tau|K = d|K$. The assertion of the lemma follows now from the uniqueness of the holomorphic extension (see section 2.21).

11.12

LEMMA. The representations $T|[K, K]$ and $(U \circ d)|[M, M]$ generate identical von Neumann algebras.

PROOF. By virtue of lemma 11.11, it is sufficient to establish that the von Neumann algebras $\rho([K, K])''$ and $\rho^*([K^*, K^*])''$ coincide. This is a general fact that is true for any tame representation ρ. In fact, $\rho(K)'' = \rho^*(K^*)''$, by virtue of theorem 2.17. Now it remains to observe that $\rho(K)'' = \rho([K, K])''$ (since the group $[K, K]$ either coincides with K or is dense in it with respect to the weak operator topology in which ρ is continuous) and that, by similar considerations, $\rho^*(K^*)'' = \rho^*([K^*, K^*])''$.

11.13

PROOF OF THEOREMS 11.1 and 11.3. Let us prove theorem 11.3 (theorem 11.1 is a particular case of it for $p=1$). First let $F \neq \mathbb{C}$; then $[L, L]=L$. By lemma 11.12, the von Neumann algebra generated by the representation $(U \circ d)([M, M])$ is contained in the von Neumann algebra generated by the representation $U \circ \tau$ of the group $[G, G]$. But, by theorem 11.10, $\tau([G, G])$ and $d([M, M])$ generate $[L, L]^p = L^p$, which proves the theorem.

When $F=\mathbb{C}$, this discussion needs to be augmented by the observation that the restriction of the representation U to the subgroup $[L, L]^p$ does not change its von Neumann algebra: this follows from the remark 7.15.

11.14

Let us proceed to prove theorem 11.10.

Let $\mathfrak{a}_1$ and $\mathfrak{a}_2$ be arbitrary Lie algebras (over an arbitrary field); $\mathfrak{a}=\mathfrak{a}_1 \oplus \mathfrak{a}_2$; χ_1 and χ_2 be the natural projections of the algebra $\mathfrak{a}$ onto $\mathfrak{a}_1$ and $\mathfrak{a}$ respectively; and let $\mathfrak{b}$ be a subalgebra in $\mathfrak{a}$ such that $\chi_1(\mathfrak{b})=\mathfrak{a}_1$ and $\chi_2(\mathfrak{b})=\mathfrak{a}_2$. Let us put $\mathfrak{b}_1=\mathfrak{b} \cap \mathfrak{a}_1$, $\mathfrak{b}_2=\mathfrak{b} \cap \mathfrak{a}_2$.

LEMMA. With these assumptions, we have:

(i) $\mathfrak{b}_1$ is an ideal in $\mathfrak{a}_1$, $\mathfrak{b}_2$ is an ideal in $\mathfrak{a}_2$,

(ii) there exists an isomorphism

$$\varphi: \mathfrak{a}_1/\mathfrak{b}_1 \to \mathfrak{a}_2/\mathfrak{b}_2$$

such that $\mathfrak{b}$ coincides with

$$\{X \in \mathfrak{a}: \varphi(\chi_1(X) \bmod \mathfrak{b}_1) = \chi_2(X) \bmod \mathfrak{b}_2\}.$$

PROOF.

(i) Let $X \in \mathfrak{b}_1$, $Y \in \mathfrak{a}_1$. Let us select $Y' \in \mathfrak{b}$ such that $\chi_1(Y')=Y$. Then

$$[X, Y']=[X, \chi_1(Y')]=[X, Y].$$

But $[X, Y'] \in \mathfrak{b}, [X, Y] \in \mathfrak{a}_1$. Thus

$$[X, Y] \in \mathfrak{b} \cap \mathfrak{a}_1 = \mathfrak{a}_1.$$

This means that $\mathfrak{b}_1$ is an ideal in $\mathfrak{a}_1$. Similarly, $\mathfrak{b}_2$ is an ideal in $\mathfrak{a}_2$.

(ii) Without loss of generality, it may be assumed that $\mathfrak{b}_1 = \mathfrak{b}_2 = \{0\}$. Then

$$\chi_1 : \mathfrak{b} \to \mathfrak{a}_1, \quad \chi_2 : \mathfrak{b} \to \mathfrak{a}_2$$

are isomorphisms and it is possible to write

$$\varphi = \chi_2 \circ (\chi_1 | \mathfrak{b})^{-1}.$$

11.15

Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_p$ be simple Lie algebras (over an arbitrary field); $\mathfrak{a}$ be their direct sum; $\chi_i : \mathfrak{a} \to \mathfrak{a}_i$ be the natural projection $(i = 1, \ldots, p)$, $\mathfrak{b}$ be a certain subalgebra in $\mathfrak{a}$ and $\chi_i(\mathfrak{b}) = \mathfrak{a}_i$ for all i.

Lemma. With these assumptions, we have: either $\mathfrak{b} = \mathfrak{a}$ or $\mathfrak{b}$ is distinguished from $\mathfrak{a}$ by some conditions of the type:

$$\varphi_{ij}(\chi_j(X)) = \chi_i(X) \quad (X \in \mathfrak{A}),$$

where $\varphi_{ij} : \mathfrak{a}_j \to \mathfrak{a}_i$ denotes a certain isomorphism of the Lie algebras $(i \neq j)$.

Proof. We carry out an induction on p. For $p = 1$, the proposition is trivial. Let $p \geqslant 2$. Let us put

$$\mathfrak{a}' = \mathfrak{a}_1 \oplus \ldots \oplus \mathfrak{a}_{p-1}, \quad b' = (\chi_1 \oplus \ldots \oplus \chi_{p-1})(\mathfrak{b}) \subseteq \mathfrak{a}'.$$

The conditions of the lemma are satisfied for $\mathfrak{a}'$ and $\mathfrak{b}'$. Hence the assertion of the lemma is valid for them. Without loss of generality, it may be assumed that $\mathfrak{b}' = \mathfrak{a}'$. Now it remains to apply lemma 11.14 to the algebra $\mathfrak{a} = \mathfrak{a}' \oplus \mathfrak{a}_p$ and its subalgebra $\mathfrak{b}$.

11.16

Let us remember that $\mathfrak{m}(n)$ denotes the Lie algebra of the group $M(n)$; it is isomorphic to $u(n)$, $u(n)\oplus u(n)$ or $u(2n)$ for $F=\mathbb{R}, \mathbb{C}, \mathbb{H}$ respectively.

LEMMA. For any $s\in\mathbb{R}$ and $n\geqslant 3$, the smallest Lie subalgebra in $l(n)$ containing $\tau_s^F(sl(n, F))$ and $[m(n), m(n)]$ coincides with $[l(n), l(n)]$.

PROOF. It is sufficient to prove that the smallest complex subalgebra $\mathfrak{b}$ in $l_{\mathbb{C}}(n)$ containing $\tau_s^F(sl(n, F))$ and $[m(n), m(n)]$ coincides with $[l_{\mathbb{C}}(n), l_{\mathbb{C}}(n)]$.

Let us consider the decomposition (see section 6.3)

$$l_{\mathbb{C}}(n)=l_{-1}(n)\oplus l_0(n)\oplus l_1(n)$$

and denote by $\mathscr{H}_{-1}, \mathscr{H}_0, \mathscr{H}_1$ the corresponding projections.

We observe that the spaces $l_{-1}(n), l_0(n), l_1(n)$, considered as modules over $[l_0(n), l_0(n)]$, are pairwise disjoint, i.e., do not have equivalent irreducible submodules. In fact, let, for example, $F=\mathbb{R}$. Then

$$l_{\mathbb{C}}(n)=sp(n, \mathbb{C}), [l_0(n), l_0(n)]=\left\{\begin{bmatrix} -A' & 0 \\ 0 & A\end{bmatrix}: A\in sl(n, \mathbb{C})\right\}.$$

It is obvious from this that $l_{-1}(n)$ is an irreducible $sl(n, \mathbb{C})$-module with the highest weight $(2, 0, \ldots, 0)$; $l_1(n)$ is the dual irreducible module with the highest weight $(0, \ldots, 0, -2)$; $l_0(n)$ is the direct sum of the one-dimensional trivial module and the adjoint representation (which has the highest weight $(1, 0, \ldots, 0, -1)$.† Such is the situation also for $F=\mathbb{C}, \mathbb{H}$: $l_{-1}(n)$ and $l_1(n)$ are irreducible and dual to one another; $l_0(n)$ is the direct sum of the adjoint representation and the trivial module of dimension 1 (for $F=\mathbb{H}$) or 2 (for $F=\mathbb{C}$).

Since $[l_0(n), l_0(n)]$ is the complexification of the subalgebra $[m(n), m(n)]$, it follows that

$$b=\mathscr{H}_{-1}(b)\oplus\mathscr{H}_0(b)\oplus\mathscr{H}_1(b).$$

†The condition $n\geqslant 3$ ensures the difference of these three weights on the algebra $sl(n, \mathbb{C})$; at $n=2$ they coincide.

In particular, the spaces

$$\mathcal{H}_{-1}(\tau_s^t(sl(n, F)) \quad \text{and} \quad \mathcal{H}_1(\tau_s^t(sl(n, F))$$

lie in b. From the definition of τ_s^t (see section 9.8), it can be seen that these two spaces are non-trivial for any $s \in \mathbb{R}$. Since the modules $l_{-1}(n)$ and $l_1(n)$ are irreducible, they are wholly contained in $\mathfrak{b}$. But $l_{-1}(n)$ and $l_1(n)$ generate $[l_{\mathbb{C}}(n), l_{\mathbb{C}}(n)]$, whence $\mathfrak{b} = [l_{\mathbb{C}}(n), l_{\mathbb{C}}(n)]$.

11.17

Let $p = 1, 2, \ldots;$ $s_1, \ldots, s_p \in \mathbb{R}$; $[l(n), l(n)]^p = [l(n), l(n)] \oplus \ldots \oplus [l(n), l(n)]$; $\tau = \tau_{s_1}^t \oplus \ldots \oplus \tau_{s_p}^t$: $sl(n, F) \to [l(n), l(n)]^p$; and d: $[\mathfrak{m}(n), \mathfrak{m}(n)] \to [l(n), l(n)]^p$ be the diagonal embedding.

LEMMA. If $n \geqslant 3$ and $s_i \neq s_j$ for $i \neq j$, then $\tau(sl(n, F))$ and $d([\mathfrak{m}(n), \mathfrak{m}(n)])$ generate $[l(n), l(n)]^p$.

PROOF. We shall use lemma 11.15. In our case $\mathfrak{a}_i = [l_{\mathbb{C}}(n), l_{\mathbb{C}}(n)]$ for $i = 1, \ldots, p$; $\mathfrak{a} = \mathfrak{a}_1 \oplus \ldots \mathfrak{a}_p$; $\mathfrak{b} \subset \mathfrak{a}$ is the smallest subalgebra over $\mathbb{C}$ containing $\tau(sl(n, F))$ and $d([\mathfrak{m}(n), \mathfrak{m}(n)])$. All the assumptions of lemma 11.15 are fulfilled: in fact, the algebras $\mathfrak{a}_i$ are simple and $\chi_i(b) = \mathfrak{a}_i$ for all i in view of lemma 11.16.

Let us assume that for a certain pair (i, j), where $i \neq j$, there exists an automorphism φ of the algebra $[l_{\mathbb{C}}(n), l_{\mathbb{C}}(n)]$ such that

$$\varphi(\chi_j(A)) = \chi_i(A), A \in b.$$

since b contains $d([\mathfrak{m}(n), \mathfrak{m}(n)]$, we have

$$\varphi \,|\, [l_0(n), l_0(n)] = \text{id}.$$

In particular, φ commutes with the projection $\mathcal{H}_0$ (see proof of lemma 11.16). So,

$$\mathcal{H}_0(\tau_{s_j}^t(A)) = \mathcal{H}_0(\tau_{s_i}^t(A))$$

for all $A \in sl(n, F)$. From the definition of the embeddings τ_s^t, it is evident that this is possible only for $s_i = s_j$, which contradicts our assumption. This contradiction shows that

$$\mathfrak{b} = \mathfrak{a} = [l_{\mathbb{C}}(n), l_{\mathbb{C}}(n)]^{p}.$$

11.18

PROOF OF THEOREM 11.10. Lemma 11.17 shows that, for any $n \geqslant 3$, the Lie algebras of the groups $\tau([G(n), G(n)])$ and $d([M(n), M(n)])$ generate the Lie algebra of the group $[L(n), L(n)]^{p}$. In view of the connectedness of the group $[L(n), L(n)]^{p}$, it is algebraically generated by the subgroups $\tau([G(n), G(n)])$ and $d([M(n), M(n)])$ (see N. Bourbaki, *Groupes et algèbres de Lie,* Ch. III, §6, ex. 25).

11.19

PROOF OF THEOREM 11.6 (first method). By theorem 10.2, we may replace $T_{\pi, s}$ by $W_{\pi} \circ \tau_s^F$. By theorem 7.7, it is sufficient to check that the representations

$$W_{\pi} \circ \varphi \circ \tau_s^F \text{ and } W_{\pi} \circ \tau_{-s}^F \tag{3}$$

of the group G are equivalent. Here φ is an automorphism of the group L defined in section 7.7.

If $F = \mathbb{R}$, then $\varphi \circ \tau_s^{\mathbb{R}} = \tau_{-s}^{\mathbb{R}}$ and this means that the representations (3) coincide. In cases $F = \mathbb{C}, \mathbb{H}$, the embedding $\varphi \circ \tau_s^F$ is transformed respectively into the embedding $\varphi \circ \tau_{-s}^F$ (see section 7.8) or into the embedding τ_{-s}^F by a suitable automorphism of the group L of the type

$$g \mapsto {}^h g \quad (g \in L),$$

where h is a certain element from $\bar{U}(\infty) \times \bar{U}(\infty)$ $(F = \mathbb{C})$ or $\bar{U}(2\infty)$ $(F = \mathbb{H})$. By corollary 8.14 and remark 7.8, the representations (3) are equivalent.

11.20

PROOF OF THEOREM 11.16 (second method). Consider the following bilinear form on $L^2(F^{k, \infty})$:

$$\langle f_1, f_2 \rangle = \int_{F^{k,\infty}} f_1(x) f_2(x) d\mu(x).$$

This form is non-degenerate and invariant with respect to the simultaneous transformations

$$f_1 \mapsto T_s^{\otimes k}(g) f_1, \quad f_2 \mapsto T_{-s}^{\otimes k}(g) f_2 \quad (g \in G).$$

The assertions of the theorem follow easily from this and from the definition of the representation $T_{\pi, s}$ (see also section 1.12).

§12. Pairwise non-equivalence of irreducible representations

The proof of theorem 11.5 is presented in this paragraph. Its analysis then leads to the construction of a certain group of currents G^*.

12.1

Let us examine two sets of numbers

$$S = \{s_1 < s_2 < \dots < s_p\},\ S' = \{s_1' < s_2' < \dots < s_q'\}$$

and two sets of representations $\{\pi_i\}$ and $\{\pi_j'\}$, where $\pi_i \in U(k_i, F)^\wedge$, $\pi_j' \in U(k_j', F)^\wedge$; $i = 1, \dots, p$; $j = 1, \dots, q$.

For the proof of theorem 11.5, it is sufficient to establish the following: if the representations

$$\bigotimes_{i=1}^{p} (W_{\pi_i} \circ \tau_{s_i}^{F}) \text{ and } \bigotimes_{j=1}^{q} (W_{\pi_j'} \circ \tau_{s_j}^{F}) \tag{1}$$

of the group $[G, G]$ are equivalent, then $p = q$, $s = s'$ and $\pi_i \sim \pi'_i$ for $i = 1, \dots, p$.

12.2

We shall write $\pi(s_i)$ and $\pi'(s_j')$ instead of π_i and π_j'. Let $t_1 < t_2 < \dots < t_r$ be all the points of the set $S \cup S'$, where $r = \text{card}\,(S \cup S')$. For $m = 1, \dots, r$, we put

$$U_m = W_{\pi(t_m)} \text{ if } t_m \in S,\ U_m = 1, \text{ if } t_m \notin S,$$
$$U_m' = W_{\pi'(t_m)} \text{ if } t_m \in S',\ U_m' = 1 \text{ if } t_m \notin S'.$$

Let us consider the irreducible representations

$$U = \bigotimes_{m=1}^{r} U_m \text{ and } U' = \bigotimes_{m=1}^{r} U_m'$$

of the group L^r and observe that $U\|[L, L]^r$ and $U'\|[L, L]^r$ are also irreducible (remark 7.15).

12.3

LEMMA. If the representations U and U' of the group $[L, L]^r$ are equivalent, then $p = q$, $S = S'$ and $\pi_i \sim \pi'_i$ for $i = 1, \ldots, p$.

PROOF. If $U \sim U'$, then $U_m \sim U_m'$ for all $m = 1, \ldots, r$. On the other hand, we observe that all the representations of the type $W_\pi\|[L, L]$, where $\pi \in U(k, F)^\wedge$, are nontrivial. In fact, lemmas 7.11 and 3.11 show that $W_t^{\otimes k}$ possesses a unique $[M, M]$-invariant vector to within a multiplier (this is the function $f_0 \equiv 1$). However, this vector is obviously not invariant with respect to $[L, L]$, so that $W_t^{\otimes k}\|[L, L]$ does not contain the trivial representation.

Finally, let us recall that from $W_\pi\|[L, L] \sim W_{\pi'}\|[L, L]$ it follows that $\pi \sim \pi'$ (corollary 7.14 (iii)). Now the assertion of the lemma becomes obvious.

12.4

PROOF OF THEOREM 11.5. Let us consider the embedding $\tau = \tau_{t_1} \times \ldots \times \tau_{t_r}$ of the group G in L^r. We observe that the representations (1) are equivalent respectively to the representations $U \circ \tau$ and $U' \circ \tau$. By lemma 12.3, it is sufficient to check that any isometry I: $H(U) \to H(U')$ commuting with the action of the group $\tau([G, G])$, commutes with the action of the group $[L, L]^r$.

By theorem 11.10, the groups $\tau([G, G])$ and $d([M, M])$ generate $[L, L]^r$. Hence it is sufficient to establish that I commutes with the action of the group $d([M, M])$. But this follows from the fact that I must intertwine the holomorphic extensions ρ^* and $(\rho')^*$ of the representations $\rho = (U \circ \tau)|K$ and $\rho' = (U' \circ \tau)|K$ respectively (here we use lemma 11.11).

12.5

The construction of section 12.2 allows any two irreducible representations (from among those constructed in **§11**) to be presented as restrictions of appropriate representations with the highest weight of one and the same group L'. It makes sense to develop this idea further and to try to construct a "universal" group catering for all representations at the same time. By analogy with the group K^* from **§2** it is natural to denote this universal group by G^*.

We shall now give a rough variant of the definition of the group G^*. We put $G^* = \cup G^*(n)$, where $G^*(n)$ is defined as the group of (all) mappings $f(s)$ of the real line $\mathbb{R}$ into the group $L(n)$ (the functions are subject to pointwise multiplication). We shall define embeddings τ: $G(n) \to G^*(n)$ and d: $M(n) \to G^*(n)$ as follows:

$$\tau(g)(s) = \tau_s^t(g),\ d(u)(s) \equiv u, \text{ where } s \in \mathbb{R},\ g \in G(n),\ u \in M(n). \qquad (2)$$

Theorem 11.10 shows that $\tau([G, G])$ and $d([M, M])$ generate inside G^* a subgroup which is dense in $[G^*, G^*]$ with respect to the topology of convergence on finite subsets in $\mathbb{R}$.

The group G possesses a family of irreducible representations with the highest weight. They have the following form

$$U_{s_1}, \ldots, s_r(f) = \bigotimes_{m=1}^{r} U_{s_m}(f(s_m))\ (s_1 < \ldots < s_r,\ f \in G^*),$$

where

$$U_{s_m} = W_{\pi_m}(\pi_m \in U(k_m, F)^\wedge).$$

By analogy with **§2**, these representations are appropriately called *holomorphic*. The structure of the ring generated by the holomorphic representations is evident.

We now see that any irreducible admissible representation of the group G (from among those constructed in **§11**) possesses a canonical holomorphic extension to the group G^*.

12.6

Let us assume for the sake of simplicity that $F = \mathbb{R}$. Let us replace in

(2) the elements $g \in G(n)$ and $u \in M(n)$ by the elements of the corresponding Lie algebras $\mathfrak{g}(n) = \mathfrak{gl}(n, \mathbb{R})$ and $\mathfrak{m}(n) = u(n)$.

THEOREM. For $F = \mathbb{R}$ and $n \geqslant 3$, the Lie algebra of matrix functions on $\mathbb{R}$ generated by the Lie algebras $\tau(sl(n, \mathbb{R}))$ and $d(su(n))$ consists of all $sp(n, \mathbb{R})$ valued functions of the type

$$s \mapsto \begin{bmatrix} -i(1+s^2)\alpha(s) \cdot 1_n + \overline{A(s)} & (1-is)B(s) \\ (1+is)\overline{B(s)} & i(1+s^2)\alpha(s) \cdot 1_n + A(s) \end{bmatrix},$$

where $\alpha(\cdot) \in \mathbb{R}[s]$, $A(\cdot) \in \mathbb{R}[s] \otimes su(n)$, $B(\cdot) \equiv B(\cdot)' \in \mathbb{R}[s] \otimes \mathbb{C}^{n,n}$.

An analogous result is true for $F = \mathbb{C}, \mathbb{H}$. This theorem is proved with the help of a certain modification of the proof of theorem 11.10 and it may be considered a refinement of it.

12.7

Remark. The definition of the group $G^*(n)$ given in section 12.5 should not be considered the final definition, but only a certain fairly rough "upper approximation". In a "correct" definition, it will be necessary to impose upon the function $f(s)$ some sort of additional restrictions, the exact form of which is not yet quite clear.

Theorem 12.6, on the contrary, gives a certain "lower approximation". We observe that it is possible to derive from it a statement like the following: $\tau([G(n), G(n)])$ and $d([M(n), M(n)]$ generate a dense subgroup in the group of continuous mappings $\mathbb{R} \to [L(n), L(n)]$ with the topology of uniform convergence on compact sets.

§13. Spherical functions and properties of continuity of admissible representations of the groups $GL(\infty, F)$

We retain the notation introduced in §9.

13.1

Any matrix $g \in G$ may be written in the form $g = ug_{t_1 t_2 \ldots}, v$, where $u, v \in K$ and

$$g_{t_1 t_2 \ldots} = \begin{bmatrix} \exp t_1 & & & 0 \\ & \exp t_2 & & \\ & & \ddots & \\ 0 & & & \ddots \end{bmatrix}, \quad t_1, t_2, \ldots \in \mathbb{R}.$$

It is assumed that $t_m = 0$ for sufficiently large m. We note that the numbers $t_1, t_2, \ldots$ are determined uniquely to within a permutation.

13.2

LEMMA. The representation

$$T = \bigotimes_{j=1}^{p} T_{\pi_j, s_j}, \quad \text{where} \quad \pi_j \in U(k_j, F)^{\wedge}, s_j \in \mathbb{R}.$$

possesses a non-zero K-invariant vector if and only if π_j is the trivial representation 1_{k_j} of the group $U(k_j, F)$ for all $j = 1, \ldots, p$. This vector is unique to within a number factor.

This follows easily from the fact that $T|K$ is a subrepresentation of the representation T_F^k defined in section 4.4, where $k = k_1 + \ldots + k_p$.

13.3

We shall write for $s \in \mathbb{R}$

$$\varphi_s(g) = (T_s(g) f_0, f_0), \quad \text{where} \quad g \in G, f_0 \in H(T_s) = L^2(F^\infty), f_0 \equiv 1.$$

THEOREM. The formula

$$\varphi_s(g_{t_1 t_2 \ldots}) = \prod_{m=1}^{\infty} (\cosh t_m + is \sinh t_m)^{-d/2} \quad (d = \dim_{\mathbb{R}} F), \tag{1}$$

holds, or, what is the same thing,

$$\varphi_s(g) = D\left(\left(\frac{1+is}{2}\right) g + \left(\frac{1-is}{2}\right)(g^*)^{-1}\right)^{-1/2} \quad (g \in G) \tag{2}$$

where $D(\cdot)$ was defined in section 1.5.

PROOF. The equivalence of formulae (1) and (2) is checked easily. Using the embeddings $GL(\infty,\mathbb{C})\to GL^+(2\infty,\mathbb{R})$ and $GL(\infty,\mathbb{H})\to GL(2\infty,\mathbb{C})$, we reduce the general case to the case $F=\mathbb{R}$ (compare with section 10.4). From here it is possible to proceed in two ways.

First method. From the definition of T_s it can be seen that

$$\varphi_s(g_{t_1t_2}\ldots)=\prod_{m=1}^{\infty}\left(e^{t_m/2}\int_{-\infty}^{\infty}\exp\left(-\frac{1}{4}x^2(e^{2t_m}-1)(1+is)\right)e^{-x^2/2}dx\right).$$

The integrals on the right-hand side are easily calculated and this leads to (1).

Second method. We shall use the fact that $T_s\sim W_{\mathbb{R}}\circ\tau_s^{\mathbb{R}}$ and that the spherical function for $W_{\mathbb{R}}$ was calculated in section 8.1. Let

$$\begin{bmatrix} a_s & b_s \\ c_s & d_s \end{bmatrix}$$

denote the image of the element $\tau_s^{\mathbb{R}}(g_{t_1t_2}\ldots)$ in the group $Sp(\infty,\mathbb{R})$ under the projection $Sp(\infty,\mathbb{R})^{\sim}\to Sp(\infty,\mathbb{R})$.

From

$$\begin{bmatrix} a_s & b_s \\ c_s & d_s \end{bmatrix}=h_s\begin{bmatrix} a_0 & b_0 \\ c_0 & d_0 \end{bmatrix}h_s^{-1}$$

(see section 9.9), we deduce that a_s is a diagonal matrix with eigenvalues $\cosh t_m+i\sinh t_m$, $m=1,2,\ldots$. Substituting a_s into the formula from section 8.1, we get (1).

13.4

THEOREM. The spherical function of the representation

$$T=\bigotimes_{j=1}^{p}T_{1_{k_j}s_j}\quad\text{is equal to}\quad\prod_{j=1}^{p}\varphi_{s_j}^{k_j}.$$

This follows from theorem 13.3.

13.5

Let us put

$$P=\{g\in G\colon g=g^*>0\}=\exp\mathscr{P},$$

where

$$\mathscr{P}=\{A\in\mathfrak{gl}(\infty, F)\colon A=A^*\}.$$

We observe that $G=KP$. If $g=up$, where $u\in K$, $p\in P$, then u and p are defined uniquely by the element g:

$$p=(g^*g)^{1/2},\quad u=gp^{-1}.$$

13.6

Let us denote by $\bar{P}$ the set of all invertible bounded positive self-adjoint operators in $l^2(F)$ of the type $1+A$, where A is a certain Hilbert–Schmidt operator. Let us endow $\bar{P}$ with the topology induced by the Hilbert–Schmidt operator norm $\|\cdot\|_2$. We observe that $\bar{P}=\exp\bar{\mathscr{P}}$, where $\bar{\mathscr{P}}$ is the space of all self-adjoint bounded Hilbert–Schmidt operators in $l^2(F)$.

Let us put $\bar{G}=\bar{K}\cdot\bar{P}$ (the topological group $\bar{K}$ was defined in section 2.19). $\bar{G}$ consists of all invertible operators g in $l^2(F)$ for which $(g^*g)^{1/2}-1$ is a Hilbert–Schmidt operator. This is equivalent to the fact that g^*g-1 is a Hilbert–Schmidt operator.

13.7

Lemma. Let us topologise $\bar{G}$ with the topology of the product $\bar{K}\times\bar{P}$. With respect to this topology $\bar{G}$ is a topological group.

A proof is given in [28]. It reduces to the fact that the mapping

$$u\mapsto uAu^{-1}\ (u\in\bar{K}),$$

where A is a fixed Hilbert–Schmidt operator in $l^2(F)$, is a continuous mapping from $\bar{K}$ to the space of Hilbert–Schmidt operators.

13.8

We observe that the group $[G, G]=SL(\infty, F)$ is dense in $\bar{G}$.

THEOREM. For any $s\in\mathbb{R}$, the representation $T_s|[G, G]$ admits a continuous extension to the group $\bar{G}$.

Proof is given in section 13.11.

13.9

COROLLARY. All irreducible representations of the group $[G, G]$ constructed in §**11** admit a continuous extension to the group $\bar{G}$.

13.10

Remark. From the proof it will be seen that in the formulation of theorem 13.8 it is possible, instead of $[G, G]$, to substitute G, if we replace T_s by $\tilde{T}_s=T_s\otimes D(\cdot)^{is/2}$ (the one-dimensional representation $g\mapsto D(g)$ was defined in section 1.5).

13.11

PROOF OF THEOREM 13.8. First of all, let us note that it is sufficient to check the statement of the theorem for $F=\mathbb{R}$ (compare section 10.4). Then $G=GL^+(\infty, \mathbb{R})$ and $D(g)=\det g$.

Further we shall proceed by analogy with the proof of theorem 8.3. We shall examine the spherical function $\tilde{\varphi}_s$ of the representation $\tilde{T}_s$ of the group $GL^+(\infty, \mathbb{R})$. Taking into consideration that $\tilde{\varphi}_s=\varphi_s \det(\cdot)^{is/2}$ and using (2), we get after simple transformations $\tilde{\varphi}_s(g)=\det h(g)$, where

$$h(g)=\left(1+\frac{1+is}{2}(g'g-1)\right)^{-1/2}(1+(g'g-1))^{(1+is)/4}.$$

We shall assume now that g does not lie in $GL^+(\infty, \mathbb{R})$ but in $\bar{G}$. We observe that $h(g)-1$ is a nuclear operator, and the mapping $g\mapsto h(g)-1$ is a continuous mapping of the topological group $\bar{G}$ into the Banach space of nuclear operators. This shows that $\tilde{\varphi}_s$ admits a continuous extension to $\bar{G}$.

We have proved, by this means, that one of the two irreducible subrepresentations in $\tilde{T}_s$ (specifically, that which is realized in the subspace of even functions) admits a continuous extension to $\bar{G}$. To complete the proof, it is necessary, as in section 8.9, to expand the group $\bar{G}$ by taking its semidirect product with the Heisenberg group Heis $(l_2(\mathbb{C}))$ (or, simpler still, with the group of translations of the space $l_2(\mathbb{R})$).

§14. Approximation of representations of the groups $GL(\infty, F)$

We retain the notation introduced in §9. Let us write also $d=\dim_{\mathbb{R}} F$.

14.1

For $K=1, 2, \ldots$, we put

$$P(k)=\left\{a\in G(k)\colon a=\begin{bmatrix} r_1 & & 0 \\ & \ddots & \\ * & & r_k \end{bmatrix}, r_1>0, \ldots, r_k>0\right\}.$$

We shall write $r_1=r_1(a), \ldots, r_k=r_k(a)$.

For $n>k$, we put

$$P^{k,n}=\left\{\begin{bmatrix} a & 0 \\ c & d \end{bmatrix}\in G(n)\colon a\in P(k)\right\}.$$

For $s_1, \ldots, s_k\in\mathbb{R}$, we examine the following character of the group $P^{k,n}$:

$$\chi^{(n)}_{s_1, \ldots, s_k}\left(\begin{bmatrix} a & 0 \\ c & d \end{bmatrix}\right)=\prod_{j=1}^{k} r_j(a)^{-ndis_j/2}.$$

We shall denote by $T^{(n)}_{s_1, \ldots, s_k}$ the unitary representation of the group $G(n)$ induced by the character $\chi^{(n)}_{s_1, \ldots, s_k}$ of the subgroup $P^{k,n}$.

14.2

THEOREM. Let us fix $k = 1, 2, \ldots$ and the arbitrary numbers $s_1, \ldots, s_k \in \mathbb{R}$. As $n \to \infty$, the representations $T^{(n)}_{s_1, \ldots, s_k}$ of the group $G(n)$ approximate the representation $T_{s_1} \otimes \ldots \otimes T_{s_k}$ of the group G in the sense of definition 22.4.†

We shall prove this theorem below.

14.3

Let us write for brevity $T^{(n)} = T^{(n)}_{s_1, \ldots, s_k}$. By the definition of the induced representation, $H(T^{(n)})$ is the space of measurable functions $f(g)$ on the group $G(n)$ satisfying the following two conditions:

$$f \mid K(n) \in L^2(K(n)),$$

$$f(pg) = \chi^{(n)}_{s_1, \ldots, s_k}(p) \mid \det \operatorname{Ad}(p)|^{1/2} f(g) \; (p \in P^{k,n}, g \in G(n)),$$

where $\operatorname{Ad}(\cdot)$ denotes the adjoint representation of the group $P^{k,n}$. The group $G(n)$ acts in $H(T^{(n)})$ by right translations

$$T^{(n)}(g) f(g_1) = f(g_1 g) \; (g, g_1 \in G(n)).$$

It will be convenient for us now to turn to another realization of the representation $T^{(n)}$, namely to realize it in $L^2(\Omega^{k,n})$ (the Stiefel manifold $\Omega^{k,n}$ was defined in section 5.4; it is provided with the normalized $K(n)$-invariant measure).

We observe that

$$P^{k,n} \cdot K(n) = G(n), \; P^{k,n} \cap K(n) = K_k(n)$$

(the subgroup $K_k(n)$ was defined in section 2.4) and that the characters $\chi^{(n)}_{s_1, \ldots, s_k}$ and $|\det \operatorname{Ad}(\cdot)|^{1/2}$ of the group $P^{k,n}$ are trivial on $K_k(n)$. This shows that $H(T^{(n)})$ may be identified with $L^2(K_k(n) \backslash K(n))$.

But $K_k(n) \backslash K(n)$ may be identified with $\Omega^{k,n}$ (by definition, the projection $K(n) \to \Omega^{k,n}$ associates with a matrix from $K(n)$ the set of

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†This is also true in a stronger sense, as defined in Ref. 44. There uniform convergence on compact sets in G is replaced by uniform convergence on so-called bounded sets in G.

its first k rows). Thus, we may identify $H(T^{(n)})$ with $L^2(\Omega^{k,n})$. In this identification a function f on $K(n)$ and the function f' on $\Omega^{k,n}$ corresponding to it are connected by the relation $f(u)=f'(\omega_0 u)$, where $u \in K(n)$ and ω_0 is a matrix of dimension $k \times n$ with elements $(\omega_0)_{ij} = \delta_{ij}$.

14.4

LEMMA. In the realization in the space $L^2(\Omega^{k,n})$, the representation $T^{(n)}$ of the group $G(n)$ is given by the formula

$$T^{(n)}(g)\,f(\omega) = D(g)^{K/2}\chi_0(a(\omega, g)) \times \prod_{j=1}^{k} r_j(\omega, g)^{-nd(1+is_j)/2} f(\omega^g) \quad (\omega \in \Omega^{k,n},\ g \in G(n)), \tag{1}$$

where the matrices $a(\omega, g)$ and ω^g are uniquely defined by the conditions

$$a(\omega, g) \in P(k), \quad \omega^g \in \Omega^{k,n}, \quad a(\omega, g)\omega^g = \omega g;$$

the real positive numbers $r_j(\omega, g)$ are the diagonal elements of the matrix $a(\omega, g)$; χ_0 is a certain character of the group $P(k)$ independent of n (let us recall that the character $D(\cdot)$ was defined in section 1.5).

PROOF. We observe that any matrix $y \in F^{k,n}$ having (maximal) rank k may be represented in the form $y = ab$ where $a \in P(k)$, $b \in \Omega^{k,n}$. The matrices a, b are uniquely determined by the conditions

$$a \in P(k), \quad aa^* = yy^*, \quad b = a^{-1}y.$$

In our case, we take $y = \omega g$ and then $a = a(\omega, g)$, $b = \omega^g$.

We observe further that for any matrix

$$p = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix} \in P^{k,n} \text{ we have } |\det \operatorname{Ad}(p)|^{1/2} = D(p)^{k/2} D(a)^{-n/2} \chi_0(a),$$

where $\chi_0(a) = |\det \operatorname{Ad}(a)|^{1/2}$ (in the last expression, $\operatorname{Ad}(\cdot)$ indicates the adjoint representation of the group $P(k)$, so that χ_0 does not actually depend on n).

Now the assertion of the lemma follows immediately from the definition of the identification $H(T^{(m)}) = L^2(\Omega^{k,n})$.

14.5

Let us recall that the matrices $a(\omega, g)$ and ω^g are given by the conditions:

$$a(\omega, g) \in P(k), \quad a(\omega, g) a(\omega, g)^* = \omega g g^* \omega^*, \tag{2}$$

$$\omega^g = a(\omega, g)^{-1} \omega g. \tag{3}$$

Lemma. The diagonal elements $r_j(\omega, g)$ of the matrix $a(\omega, g)$ are given by the following formula:

$$r_j(\omega, g)^{2d} = \frac{D(\theta_j(\omega g g^* \omega^*))}{D(\theta_{j-1}(\omega g g^* \omega^*))} \quad (j = 1, \ldots, k), \tag{4}$$

where for an arbitrary matrix $z \in F^{k,k}$ its left upper corner of dimension $j \times j$ is denoted by $\theta_j(z)$. (Let us agree that for $j = 1$ the denominator in (4) is equal to 1.)

Proof. Using (2) and the fact that $a(\omega, g)$ is a lower triangular matrix, we get

$$\theta_j(a(\omega, g))(\theta_j(a(\omega, g)))^* = \theta_j(\omega g g^* \omega^*) \quad (j = 1, \ldots, k).$$

On the other hand, it is obvious that

$$D(\theta_j(a(\omega, g))) = D(\theta_j(a(\omega, g)))^* = (r_1(\omega, g) \ldots r_j(\omega, g))^d.$$

This implies (4).

14.6

Let us consider the representation $T_{s_1} \otimes \cdots \otimes T_{s_k}$ of the group G and denote it for brevity by T. We shall fix two cylindrical functions f_1, f_2 from $H(T) = L^2(F^{k,\infty})$ (see section 4.2). Let us fix the number N to be sufficiently large that $f_1(x)$ and $f_2(x)$ depend only on the first N columns of the matrix $x \in F^{k,\infty}$. Then for any $m \geq N$ we may treat f_1 and f_2 as functions on $F^{k,m}$.

On the other hand, for any $n \geqslant N$ we define the functions $f_1^{(n)}$ and $f_2^{(n)}$ on $\Omega^{k,n}$ as in section 5.6:

$$f_i^{(n)}(\omega) = f_i(\sqrt{nd}\,\omega), \text{ where } i = 1, 2;\ \omega \in \Omega^{k,n} \subset F^{k,n},$$

If f_1 and f_2 are bounded, then $f_1^{(n)}$ and $f_2^{(n)}$ evidently lie in $L^2(\Omega^{k,n})$ for all n. We shall say that f_1 and f_2 are *continuous* if they are continuous as functions on $F^{k,N}$.

THEOREM. Let f_1 and f_2 be fixed bounded continuous cylindrical functions from $L^2(F^{k,\infty})$. Then

$$\lim_{n \to \infty} (T^{(n)}(g) f_1^{(n)}, f_2^{(n)}) = (T(g) f_1, f_2)$$

uniformly on compact sets $X \subset G$.

Theorem 14.2 immediately follows from theorem 14.6 and lemma 22.6, since the bounded continuous cylindrical functions form a dense subset in $L^2(F^{k,\infty})$.

We prove theorem 14.6 below.

14.7

According to the definition of the topology of the inductive limit, any compact set $X \subset G$ lies in $G(m)$, where m is sufficiently large. From this point, we fix $m \geqslant N$ and assume that g runs through a certain compact set $X \subset G(m)$. We assume $n > m$.

LEMMA. The following formula is true.

$$(T^{(n)}(g) f_1^{(n)}, f_2^{(n)}) = D(g)^{k/2} \int_{\Omega^{k,n}} \chi_0(a) \prod_{j=1}^{k} r_j^{-nd(1+is_j)/2} \cdot$$

$$\cdot f_1(\sqrt{nd}\, a^{-1} \omega_1 g)\, \overline{f_2(\sqrt{nd}\,\omega_1)}\, d\omega$$

where the following notation is used: $d\omega$ is the normalized $K(n)$-invariant measure on $\Omega^{k,n}$; the matrix $\omega_1 \in F^{k,m}$ consists of the first m columns of matrix ω; the matrix a depends on ω, g and is uniquely defined by the conditions

$$a \in P(k),\ aa^* = 1_k + \omega_1(gg^* - 1)\omega_1^*;$$

the numbers $r_1 > 0, \ldots, r_k > 0$ also depend on ω, g and are given by the formula

$$r_j^{2d} = \frac{D(\theta_j(1_k + \omega_1(gg^* - 1)\omega_1^*))}{D(\theta_{j-1}(1_k + \omega_1(gg^* - 1)\omega_1^*))} \quad (j = 1, \ldots, k).$$

Proof. We shall denote by $\omega_2 \in F^{k, n-m}$ the matrix composed of the last $n-m$ columns of the matrix ω. Since $\omega \in \Omega^{k,n}$, we have

$$1_k = \omega\omega^* = \omega_1\omega_1^* + \omega_2\omega_2^*,$$

whence

$$\omega_2\omega_2^* = 1_k - \omega_1\omega_1^*.$$

Taking into consideration that $g \in G(m)$, we get from this

$$\omega gg^*\omega^* = \omega_1 gg^*\omega_1^* + \omega_2\omega_2^* = 1_k + \omega_1(gg^* - 1)\omega_1^*.$$

Now the assertion of the lemma follows from (1)–(4) and the definition of the functions $f_1^{(n)}, f_2^{(n)}$.

14.8

Let us now examine the mapping $\Omega^{k,n} \to F^{k,m}$ which transforms the matrix $\omega \in \Omega^{k,n}$ into the matrix $x = \sqrt{nd}\ \omega_1$. The probability measure μ_n on $F^{k,n}$, which is the image of the measure $d\omega$, is concentrated on the set of matrices with $\|x\|^2 \leqslant nd$. This measure was studied in detail in §5. The following lemma is simply a reformulation of lemma 14.7.

Lemma. The following formula is true.

$$(T^{(n)}(g) f_1^{(n)}, f_2^{(n)}) = D(g)^{k/2} \int_{-F^{k,m}} \chi_0(a) \left(\prod_{j=1}^{k} r_j^{-nd(1+is_j)/2} \right)$$

$$\times f_1(a^{-1}xg)\, \overline{f_2(x)}\, d\mu_n(x), \qquad (5)$$

where the matrix a depends on n, x, g and is defined by

$$a \in P(k), \quad aa^* = 1_k + \frac{1}{nd} x(gg^* - 1)x^*; \tag{6}$$

the numbers $r_1 > 0, \ldots, r_k > 0$ also depend on n, x, g and are defined by

$$r_j^{2d} = \frac{D\left(\theta_j\left(1_k + \frac{1}{nd} x(gg^* - 1)x^*\right)\right)}{D\left(\theta_{j-1}\left(1_k + \frac{1}{nd} x(gg^* - 1)x^*\right)\right)} \quad (j = 1, \ldots, k). \tag{7}$$

14.9

LEMMA. The following formula is true.

$$(T(g) f_1, f_2) = D(g)^{k/2} \int_{F^{k,m}} \left(\prod_{j=1}^{k} \exp\left(-\frac{1}{4} x_j (gg^* - 1) x_j^* \times (1 + is_j) \right) \right) f_1(xg) \overline{f_2(x)}\, d\mu(x), \tag{8}$$

where $d\mu(x)$ is standard Gaussian measure on $F^{k,m}$ (see section 4.1) and x_j denotes the j-th row in x.

This quickly follows from the definition of the representation T (see section 14.6) and from the definition of fundamental representations (section 9.4).

14.10

We can now describe the proof of theorem 14.6. Let us denote by $I(n, g)$ and $I(g)$ the integrals on the right-hand side of (5) and (8) respectively. We have to prove that $I(n, g) \to I(g)$ uniformly in $g \in X \subset G(m)$.

For any $A > 0$ we may write

$$I(n, g) = I(n, g; A) + I'(n, g; A),$$

where the right-hand side is the sum of integrals of the same type as $I(n, g)$, but taken respectively over the region

$$\{x \in F^{k,m} : \|x\| < A\} \subset F^{k,m} \tag{9}$$

and over its complement.

Let $I(g; A)$ denote an integral of the type $I(g)$ but taken over the region (9).

The proof of theorem 14.6 consists of two stages.

First, we show that for any fixed $A > 0$

$$\lim_{n \to \infty} I(n, g; A) = I(g; A) \tag{10}$$

uniformly in $g \in X$. This is derived from the fact that $\mu_n \to \mu$ (see lemma 5.3).

Then we show that

$$\lim_{A \to +\infty} \lim_{n \to \infty} I'(n, g; A) = 0 \tag{11}$$

14.11

Proof of the Statement (10). We use the notation given in section 14.8. We recall that the matrix a and its diagonal elements $r_1, \ldots, r_k$ depend on n, x, g.

It is seen from the condition $\|x\| < A$ and from (6) that $a \to 1_k$ as $n \to \infty$ uniformly in x and g. Since χ_0 does not depend on n and the function f_1 is continuous on $F^{k,m}$,

$$\lim_{n \to \infty} \chi_0(a) = 1 \text{ and } \lim_{n \to \infty} f_1(a^{-1} xg) = f_1(xg)$$

uniformly in x and g.

We now observe that

$$r_j^{2d} = 1 + \frac{1}{n} x_j (gg^* - 1) x_j^* + O\left(\frac{1}{n^2}\right), \tag{12}$$

where the error estimate $O(1/n^2)$ does not depend on x and g. In fact, (12) quickly follows from (7) if we take into consideration that

$$D(1_j+z)=1+d(z_{11}+\ \dots\ +z_{jj})+O(\|z\|^2)$$

for an arbitrary matrix $z=z^*\in F^{j,j}$ with small $\|z\|$.

It follows from (12) that

$$\lim_{n\to\infty} r_j^{-nd(1+is_j)/2}=\exp\left(-\frac{1}{4}x_j(gg^*-1)x_j^*-(1+is_j)\right)(j=1,\dots,k)$$

uniformly in x and g. Now (10) becomes evident.

14.12

PROOF OF THE STATEMENT (11). Since f_1 and f_2 are bounded and

$$\left|\prod_{j=1}^{k} r_j^{-nd(1+is_j)/2}\right|=D(a)^{-n/2},$$

(11) reduces to the following statement

$$\lim_{A\to+\infty}\lim_{n\to\infty}\int_{x\in F^{k,m},\|x\|\geqslant A}\chi_0(a)\,D(a)^{-n/2}d\mu_n(x)=0 \tag{13}$$

uniformly in $g\in X$, where the matrix $a\in P(k)$ depending on n, x, g, is given by the conditions (6).

Since $aa^*=\omega gg^*\omega^*\leqslant\|g\|^2\cdot 1_k$ and $r_j\leqslant\|a\|=\|aa^*\|^{1/2}$, we see that $r_j\leqslant$const uniformly in n, x and $g\in X$ $(j=1,\dots,k)$. It easily follows that $\chi_0(a)\leqslant\text{const}\cdot D(a)^{-t}$ uniformly in n, x and $g\in X$ where $t>0$ is a constant.

Let $\rho_1\geqslant\rho_2\geqslant\dots\rho_k\geqslant 0$ be the eigenvalues of the matrix $(xx^*)^{1/2}\in F^{k,k}$. We may assume $\rho_1\leqslant\sqrt{nd}$ because $\|x\|^2\leqslant nd$ for all $x\in\operatorname{supp}\mu_n$. From (6) we now get

$$D(a)^{-n/2}=D\left(1_k+\frac{1}{nd}x(gg^*-1)x^*\right)^{-n/4}$$

$$\leqslant D\left(1_k-\frac{1}{nd}xx^*\right)^{-n/4}=\prod_{j=1}^{k}\left(1-\frac{\rho_j^2}{nd}\right)^{-nd/4}. \tag{14}$$

Let us suppose first that $k=m$. For this case, the radial part of the measure μ_n in the coordinates $\rho_1, \ldots, \rho_k$ was calculated while proving lemma 5.3. From the formula for the density of this radial part given in section 5.3, it is clear that it does not exceed

$$\text{const}\ \rho_1^M \prod_{j=1}^{k}\left(1-\frac{\rho_j^2}{nd}\right)^{\frac{(n-2k+1)d}{2}-1} \qquad (M=k^2d-k\geqslant 0), \quad (15)$$

where "const" does not depend on n.

We observe now that for large n

$$\frac{(n-2k+1)d}{2}-1-\frac{nd}{4}-\frac{td}{2}>\varepsilon n,$$

where we fix $\varepsilon>0$ such that $\varepsilon<d/4$. It follows then from (14), (15) and from the estimate on $\chi_0(a)$ given above that

$$\int_{\|x\|\geqslant A}\chi_0(a)\, D(a)^{-n/2} d\mu_n(x)$$

$$\leqslant \text{const}\int_A^{\sqrt{(nd)}} d\rho_1 \int_0^{\sqrt{(nd)}} \cdots \int_0^{\sqrt{(nd)}} \rho_1^M \prod_{j=1}^{k}\left(1-\frac{\rho_j^2}{nd}\right)^{\varepsilon n} d\rho_2 \ldots d\rho_k$$

$$=\text{const}\left(\int_A^{\sqrt{(nd)}}\rho_1^M\left(1-\frac{\rho_1^2}{nd}\right)^{\varepsilon n} d\rho_1\right)\left(\int_0^{\sqrt{(nd)}}\left(1-\frac{\rho^2}{nd}\right)^{\varepsilon n} d\rho\right)^{k-1},$$

and it remains to apply lemma 5.1.

Thus we have verified Eq. (13) for $m=k$. In the case $m\neq k$ it is still possible to calculate the radial part of μ_n and to verify (13) by the same method. But we can avoid this by making use of the following trick.

We observe that, (10) being already proved, (13) is in fact equivalent to the particular case of theorem 14.6 when $s_1=\ldots=s_k=0$ and $f_1=f_2\equiv 1$. But for $f_1=f_2\equiv 1$ we may choose $m=k$. This concludes the proof.

§15. Corollaries of approximation theorem

This paragraph is a direct continuation of §14. We retain the symbols introduced there.

15.1

THEOREM. As $n\to\infty$, the spherical functions of the representations $T^{(n)}=T^{(n)}_{s_1,\ldots,s_k}$ (corresponding to the vectors $f_0^{(n)}\equiv 1$) approximate, uniformly on compact sets, the spherical function of the representation $T=T_{s_1}\otimes\ldots\otimes T_{s_k}$ (corresponding to the vector $f_0\equiv 1$).

This follows immediately from the proof of theorem 14.6.

15.2

Let $Q(k)\supset P(k)$ be the subgroup of all lower triangular matrices in $G(k)$ and

$$Q^{k,n}=\left\{\begin{bmatrix} a & 0\\ c & d\end{bmatrix}\in G(n)\colon a\in Q(k)\right\}.$$

$Q^{k,n}$ is the parabolic subgroup in $G(n)$ preserving the flag $\{V_1,\ldots,V_k\}$, where V_i denotes the subspace in $F^{1,n}$ spanned by the first i basis vectors.

The reductive part of the group $Q(k)$ is $F^*\times\ldots\times F^*$ (k times) where $F^*=F\backslash\{0\}$. We observe that any element from F^* can be written uniquely in the form rv, where $r>0$, $v\in U(1,F)=O(1)$, $U(1)$, $Sp(1)$.

For arbitrary

$$\pi_1,\ldots,\pi_k\in U(1,F)^\wedge \text{ and } s_1,\ldots,s_k\in\mathbb{R},$$

we shall denote by $T^{(n)}_{\pi_1,s_1,\ldots,\pi_k,s_k}$ the unitary representation of the group $G(n)$ induced by the following representation of the subgroup $Q^{k,n}$ (see section 14.1):

$$\begin{bmatrix} av & 0\\ c & d\end{bmatrix}\mapsto \pi_1(v_1)\otimes\ldots\otimes\pi_k(v_k))\chi^{(n)}_{s_1,\ldots,s_n}\left(\begin{bmatrix} a & 0\\ c & d\end{bmatrix}\right),$$

where

$$a \in P(k),\ v = \begin{bmatrix} v_1 & \cdot & 0 \\ & \cdot & \\ 0 & \cdot & v_k \end{bmatrix},\ v_1, \ldots, v_k \in U(1, F).$$

It is obvious that the representation $T^{(n)}_{s_1, \ldots, s_k}$ is the direct sum of the representations $T^{(n)}_{\pi_1, s_1, \ldots, \pi_k, s_k}$ with multiplicities equal to $\dim(\pi_1 \otimes \ldots \otimes \pi_k)$; the latter are equal to 1 if $F \neq \mathbb{H}$.

15.3

Let us fix the following data.

$$k = 1, 2, \ldots;\ \pi_1, \ldots, \pi_k \in U(1, F)^\wedge;\ s_1, \ldots, s_k \in \mathbb{R}.$$

THEOREM. As $n \to \infty$, the representations $T^{(n)}_{\pi_1, s_1, \ldots, \pi_k, s_k}$ of the groups $G(n)$ approximate, in the sense of definition 22.4, the representation $T_{\pi_1, s_1} \otimes \ldots \otimes T_{\pi_k, s_k}$ of the group G.†

PROOF. The representations $T^{(n)}_{\pi_1, s_1, \ldots, \pi_k, s_k}$ appear as components of the decomposition of the space $H(T^{(n)}_{s_1, \ldots, s_k})$ with respect to the natural action of the group $U(1, F)^k = U(1, F) \times \ldots \times U(1, F)$ (k times). This action commutes with the action of the group G. In realization in the space $L^2(\Omega^{k,n})$ it has the following form:

$$((v_1, \ldots, v_k) \cdot f)(\omega) = f\left(\begin{bmatrix} v_1 & \cdot & 0 \\ & \cdot & \\ 0 & \cdot & v_k \end{bmatrix}^{-1} \omega\right),$$

where

$$f \in L^2(\Omega^{k,n}),\ \omega \in \Omega^{k,n},\ (v_1, \ldots, v_k) \in U(1, F)^k.$$

On the other hand, $U(1, F)$ is a symmetry group for each of the representations T_{s_j}, $1 \leq j \leq k$. Hence $U(1, F)^k$ acts naturally in the space

$$H(T_{s_1} \otimes \ldots \otimes T_{s_k}) = L^2(F^{k,\infty}).$$

†We may repeat again the remark given in the footnote to theorem 14.2.

This action has precisely the same form; it is necessary only to replace $\omega\in\Omega^{k,n}$ by $x\in F^{k,\infty}$.

Representations $T_{\pi_1,s_1}\otimes\ldots\otimes T_{\pi_k,s_k}$ of the group G are precisely irreducible components in decomposition with respect to this action. Now theorem 15.3 directly follows from the proof of theorem 14.2.

It is evident that theorems 15.3 and 14.2 are essentially equivalent.

15.4

Remark. The representation $T^{(n)}_{s_1,\ldots,s_k}|K(n)$ may be identified with the representation $T_F^{k,n}$ from section 5.5. Remark 5.10 shows now that the structure of the decomposition of the representations

$$T^{(n)}_{s_1,\ldots,s_k}|K(n),\ (T_{s_1}\otimes\ldots\otimes T_{s_k})|K$$

is one and the same (in the same sense as that in section 5.10). An analogous fact is true for the representations studied in theorem 15.3.

15.5

Remark. The representations $T^{(n)}_{\pi_1,s_1,\ldots,\pi_k,s_k}$ belong to a degenerate principal unitary series for the group $G(n)$. I. M. Gelfand and M. A. Naimark [4] have proved that all representations of these series are irreducible when $F=\mathbb{C}$. Their method can be transferred without any changes to the case $F=\mathbb{R}$, but not to $F=\mathbb{H}$. However, apparently the result remains valid also for $F=\mathbb{H}$. The result of the article [19] on intertwining operators gives some corroboration of this proposition.

15.6

Remark. Let T be an admissible representation of the group G and $\{T^{(n)}\}$ a sequence of representations of the groups $G(n)$, approximating T in the sense of definition 22.4. We shall say that $\{T^{(n)}\}$ approximates T *perfectly* if the structure of the decomposition of the representations $T^{(n)}|K(n)$ and $T|K$ is one and the same in the sense of section 5.10.

A "majority" of the irreducible admissible representations of the group G constructed in §**11** has the form

$$T_{\pi_1,s_1}\otimes\ldots\otimes T_{\pi_k,s_k},\ \text{where } \pi_i\in U(1,F)^\wedge,\ s_i\neq s_j \text{ for } i\neq j.$$

As follows from theorem 15.3 and remarks 15.4 and 15.5, for each such representation T we are able (at least for $F=\mathbb{R}, \mathbb{C}$) to get a sequence $\{T^{(n)}\}$ which approximates T perfectly.

For all the remaining irreducible representations T (from among those constructed in **§11**), theorem 14.2 equally provides an approximating sequence $\{T^{(n)}\}$. However, the condition of perfection is not now achieved since the "limiting representation $\lim T^{(n)}$" decomposes into a countable discrete sum of irreducible representations.

In other words, in the case when $s_i = s_j$ for at least one pair (i, j) where $i \neq j$, the finite-dimensional commutant of the representation $T^{(n)}_{s_1 \ldots s_k}$ of the group $G(n)$ expands in the limit and becomes infinite-dimensional.

It may still be possible to say that in this case the "links" between certain "$K(n)$-types" in $T^{(n)}_{s_1 \ldots s_k}$ vanish in the limit.

15.7

Remark. Among the representations of degenerate series connected with the parabolic subgroups $Q^{k,n} \subset G(n)$, there are complementary series of unitary representations (see [4], [19]). These representations also admit a passage to the limit as $n \to \infty$; however, as a result, we do not get new admissible representations of the group G.

The fact is explained as follows. The representations of the complementary series differ from the representations of the principal unitary series by the fact that in the formula for the character $\chi^{(n)}_{s_1 \ldots s_k}$ (section 14.1), certain real numbers t_j are added to purely imaginary numbers$-nd\ i\, s_j/2$. However, in all the cases known to the author, when the unitarity of the corresponding representation of the group $G(n)$ is proved, these numbers t_j are bounded by a constant not depending on n. As follows from the proof of theorem 14.2, such a modification does not have any effect on the limiting transition.

For the same reasons, nothing new is obtained from another possible generalization, that of replacing the parabolic subgroup $Q^{k,n} \subset G(n)$ (which is connected with the decomposition $n = 1 + 1 + \ldots + 1 + (n-k)$) by more general parabolic subgroups connected with decompositions of the type

$$n = k_1 + \ldots + k_p + (n-k) \quad (k = k_1 + \ldots + k_p),$$

where k does not depend on n.

§16. Generalizations to $SO_0(\infty,\infty)$ and other groups of non-compact type

In this section we shall show how to extend the results of **§§9–15** to the remaining seven pairs (G, K) of non-compact type (in Table 7.2 of the Introduction, these pairs were denoted by the symbol (F_i)).

16.1

We shall find it convenient to change the notation which we have used in **§§9–15** and to use the following (see Table 7.5).

TABLE 7.5

	G	K	G'	K'
$(\mathbb{R}_1)$	$SO_0(\infty,\infty)$	$SO(\infty)\times SO(\infty)$	$GL^+(2\infty,\mathbb{R})$	$SO(2\infty)$
$(\mathbb{R}_2)$	$Sp(\infty,\mathbb{R})$	$U(\infty)$	$GL^+(2\infty,\mathbb{R})$	$SO(2\infty)$
$(\mathbb{C}_1)$	$SO(\infty,\mathbb{C})$	$SO(\infty)$	$GL(\infty,\mathbb{C})$	$U(\infty)$
$(\mathbb{C}_2)$	$Sp(\infty,\mathbb{C})$	$Sp(\infty)$	$GL(2\infty,\mathbb{C})$	$U(2\infty)$
$(\mathbb{C}_3)$	$U(\infty,\infty)$	$U(\infty)\times U(\infty)$	$GL(2\infty,\mathbb{C})$	$U(2\infty)$
$(\mathbb{H}_1)$	$Sp(\infty,\infty)$	$Sp(\infty)\times Sp(\infty)$	$GL^+(2\infty,\mathbb{H})$	$Sp(2\infty)$
$(\mathbb{H}_2)$	$SO^*(2\infty)$	$U(\infty)$	$GL(\infty,\mathbb{H})$	$Sp(\infty)$

In all seven cases, G' is one of the groups

$$GL^+(\infty,\mathbb{R}),\quad GL(\infty,\mathbb{C}),\quad GL(\infty,\mathbb{H})$$

whose admissible representations we have studied in detail in **§§9–15**. However, in five cases it will be convenient for us to realize $GL(\infty, F)$ as $GL(2\infty, F)$; we then regard the elements of the groups G and G' as matrices of the type

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix},\quad \text{where } a, b, c, d \in F^{\infty,\infty}.$$

We shall now denote by $K'\subset G'$ that subgroup which was earlier denoted by K. We realize G as a certain subgroup in G'. Then K coincides with $G\cap K'$.

16.2

We shall describe the embedding $G\to G'$:

$$SO_0(\infty,\infty)=\left\{g=\begin{bmatrix} a & b \\ c & d\end{bmatrix}\in GL^+(2\infty,\mathbb{R}):\right.$$
$$\left. g'\begin{bmatrix} -1_\infty & 0 \\ 0 & 1_\infty\end{bmatrix}g=\begin{bmatrix} -1_\infty & 0 \\ 0 & 1_\infty\end{bmatrix},\ \det a>0\right\};$$

$$Sp(\infty,\mathbb{R})=\left\{g\in GL^+(2\infty,\mathbb{R}):\quad g'\begin{bmatrix} 0 & 1_\infty \\ -1_\infty & 0\end{bmatrix}g=\begin{bmatrix} 0 & 1_\infty \\ -1_\infty & 0\end{bmatrix}\right\};$$

$$SO(\infty,\mathbb{C})=\{g\in GL(\infty,\mathbb{C}):\quad g'g=1_\infty\};$$

$$Sp(\infty,\mathbb{C})=\left\{g\in GL(2\infty,\mathbb{C}):\quad g'\begin{bmatrix} 0 & 1_\infty \\ -1_\infty & 0\end{bmatrix}g=\begin{bmatrix} 0 & 1_\infty \\ -1_\infty & 0\end{bmatrix}\right\};$$

$$U(\infty,\infty)=\left\{g\in GL(2\infty,\mathbb{C}):\quad g^*\begin{bmatrix} -1_\infty & 0 \\ 0 & 1_\infty\end{bmatrix}g=\begin{bmatrix} -1_\infty & 0 \\ 0 & 1_\infty\end{bmatrix}\right\};$$

$$Sp(\infty,\infty)=\left\{g\in GL(2\infty,\mathbb{H}):\quad g^*\begin{bmatrix} -1_\infty & 0 \\ 0 & 1_\infty\end{bmatrix}g=\begin{bmatrix} -1_\infty & 0 \\ 0 & 1_\infty\end{bmatrix}\right\};$$

$$SO^*(2\infty)=\{g\in GL(\infty,\mathbb{H}):\quad g^*(i1_\infty)g=i1_\infty\}.$$

We observe that the groups $Sp(\infty,\mathbb{R})$ and $SO^*(2\infty)$ are not realized here as in §6. However, the transition from one realization to the other does not pose any problem.

16.3

The admissible representations of the pairs (G,K) from Table 5 are defined precisely as in section 9.5.

LEMMA. Let G be any of the seven groups defined in Table 5 and $G' \supset G$ the corresponding group. If T is an arbitrary admissible representation for (G', K'), then $T|G$ is an admissible representation for (G, K).

The proof follows trivially from the form of the embeddings $G \to G'$ and $K \to K'$.

16.4

Our problem now is to study how the admissible representations of the group G' studied above behave under restriction to G. All the propositions will be formulated generally; however, the proofs will be given only for the pair $(\mathbb{R}_1)$. Their extension to the pairs $(\mathbb{R}_2)$–$(\mathbb{H}_2)$ does not require any new ideas, although it is sometimes associated with tiresome calculations with the matrices.

16.5

THEOREM. For any fundamental representations $T_s, s \in \mathbb{R}$, of the group G', the representations $T_s|G$ and $T_{-s}|G$ are equivalent.

For proof see section 16.17. The validity of this result may be guessed after noting that $\varphi_s|G = \varphi_{-s}|G$, where φ_s is the spherical function of the representation T_s (see section 13.3).

16.6

THEOREM. Let us assume that $s \neq 0$. For any $k = 1, 2, \ldots$, the representation $T_s^{\otimes k}$ of the group G' and its restriction to G have common commutants. Thus, for all $k = 1, 2, \ldots$ and $\pi \in U(k, F)^\wedge$, the representation $T_{\pi,s}|G$ is irreducible (we shall denote it again by $T_{\pi,s}$.

For proof see section 16.18.

16.7

In the case $s = 0$, the picture is different (up to a certain extent this has already been seen from theorem 16.5). We need new notations (see Table 7.6).

TABLE 7.6

	L'	L	$U_G(k, F)$	$W_F^{\otimes k}\mid L$
$(\mathbb{R}_1)$	$Sp(2\infty, \mathbb{R})^{\sim}$	$U(\infty,\infty)$	$U(k)$	$W_1^{\otimes k}$
$(\mathbb{R}_2)$	$Sp(2\infty, \mathbb{R})^{\sim}$	$[Sp(\infty, \mathbb{R}) \times Sp(\infty, \mathbb{R})]^{\sim}$	$O(k) \times O(k)$	$(W_2 \otimes W_2)^{\otimes k}$
$(\mathbb{C}_1)$	$U(\infty,\infty)^{\sim}$	$SO^*(2\infty)$	$Sp(k)$	$W_2^{\otimes k}$
$(\mathbb{C}_2)$	$U(2\infty, 2\infty)^{\sim}$	$Sp(2\infty, \mathbb{R})$	$O(2k)$	$W_2^{\otimes 2k}$
$(\mathbb{C}_3)$	$U(2\infty, 2\infty)^{\sim}$	$[U(\infty,\infty) \times U(\infty,\infty)]^{\sim}$	$U(k) \times U(k)$	$(W_1 \otimes W_1)^{\otimes k}$
$(\mathbb{H}_1)$	$SO^*(8\infty)$	$U(2\infty, 2\infty)$	$U(2k)$	$W_1^{\otimes 2k}$
$(\mathbb{H}_2)$	$SO^*(4\infty)$	$SO^*(2\infty) \times SO^*(2\infty)$	$Sp(k) \times Sp(k)$	$(W_2 \otimes W_2)^{\otimes k}$

In this table L' denotes the same group that was denoted in Table 7.4 by L. In addition, in those cases when the definition of the group G' involves a "doubling of infinity", the realization of the group L' also changes in a corresponding manner.

16.8

Let τ_0^t denote the embedding τ_s^t: $G' \to L'$ (section 9.8) at the point $s = 0$.

THEOREM. There exists an embedding $L \to L'$ possessing the following properties:

(i) $\tau_0^t(G) \subset L$;

(ii) the restriction of the representation $W_F^{\otimes k}$ of the group L' to the subgroup L has the form given in the last column of Table 7.6.

We observe that the representation $W_F^{\otimes k}|L$ possesses a symmetry group $U_G(k, F)$.

For proof see section 16.19.

16.9

Let us indicate explicit the form of the embedding γ:$G \to L$ given by the mapping τ_0^t; below we return to the "complex" realization of the groups $Sp(\infty, \mathbb{R})$ and $SO^*(2\infty)$ (§6).

In the case $(\mathbb{R}_1)$, γ is the identical embedding $SO_0(\infty,\infty)\to U(\infty,\infty)$ lifted to a two-sheeted covering over $U(\infty,\infty)$.

In the case $(\mathbb{C}_1)$, γ is the mapping

$$g\mapsto\begin{bmatrix}\frac{1}{2}(g+(g^*)^{-1}) & \frac{1}{2}(g-(g^*)^{-1})\\ \frac{1}{2}(g-(g^*)^{-1}) & \frac{1}{2}(g+(g^*)^{-1})\end{bmatrix}$$

from $SO(\infty,\mathbb{C})$ into $SO^*(2\infty)$.

In the case $(\mathbb{C}_2)$, γ is the mapping

$$g\mapsto\begin{bmatrix}\frac{1}{2}w(g+(g^*)^{-1})w^{-1} & \frac{1}{2}w(g-(g^*)^{-1})\\ \frac{1}{2}(g-(g^*)^{-1})w^{-1} & \frac{1}{2}(g+(g^*)^{-1})\end{bmatrix},\text{ where } w=\begin{bmatrix}0 & 1_\infty\\ -1_\infty & 0\end{bmatrix},$$

from $Sp(\infty,\mathbb{C})$ into $Sp(2\infty,\mathbb{R})$ lifted to the two-sheeted covering.

In the case $(\mathbb{H}_1)$, γ is the mapping

$$g=\begin{bmatrix}a & b\\ c & d\end{bmatrix}\mapsto\begin{bmatrix}\beta(a) & \beta(b)\\ \beta(c) & \beta(d)\end{bmatrix}$$

from $Sp(\infty,\infty)$ into $U(2\infty,2\infty)$, lifted to a two-sheeted covering (β is defined in section 1.3).

In the cases $(\mathbb{R}_2)$, $(\mathbb{C}_3)$ and $(\mathbb{H}_2)$, the group L is either $G\times G$ or its 2-covering and then γ is the mapping $g\mapsto(g,\bar{g})$, lifted, if necessary to a two-sheeted covering.

16.10

THEOREM. For any $k=1, 2, \ldots,$ the representation $W_F^{\otimes k}|L$ and its restriction to the subgroup $\gamma(G)$ have a common commutant. Thus

$$T_0^{\otimes k}|G\sim\bigoplus_{\pi\in U_0(k,F)^\wedge}(\dim\pi)\cdot T_{\pi,0},$$

where $T_{\pi,0}$ are irreducible admissible representations of the group G.
For proof see section 16.20.

16.12

Remark. We observe particularly the pairs $(\mathbb{R}_2)$, $(\mathbb{C}_3)$ and $(\mathbb{H}_2)$. If G is one of the groups of this type, then the representation $T_0|G$ coincides with $W_F \otimes \bar{W}_F$, where $F=\mathbb{R}, \mathbb{C}, \mathbb{H}$. This fact arises very naturally: from lemma 7.11, it is evident that $W_F \otimes \bar{W}_F$ is an admissible representation.

Let $p=1, 2, \ldots$; $k_1, \ldots, k_p = 1, 2, \ldots$; $s_1 \geqslant 0, \ldots, s_1 \geqslant 0$ be pairwise distinct numbers; π_i be an irreducible representation of the group $U(k_i, F)$ (if $s_i > 0$) or group $U_{ci}(k_i, F)$ (if $s_i = 0$).

THEOREM (compare corollary 11.4). Under these assumptions, a representation of the group G of the type

$$T_{\pi_1, s_1} \otimes \ldots \otimes T_{\pi_p, s_p}$$

is irreducible.
For proof see section 16.21.

16.13

THEOREM (compare with theorem 11.5). The irreducible representations of the type

$$T_{\pi_1, s_1} \otimes \ldots \otimes T_{\pi_p, s_p}, \; T_{\pi'_1, s'_1} \otimes \ldots \otimes T_{\pi'_q, s'_q}$$

are equivalent if and only if $p=q$ and the sets

$$\{(\pi_1, s_1), \ldots, (\pi_p, S_p)\}, \quad \{(\pi'_1, s'_1), \ldots, (\pi'_q, s'_q)\}$$

coincide to within their order of enumeration.
For proof see section 16.22.

16.14

Remark. The results formulated above fully describe the structure of the ring of admissible representations generated by the representations $T_s|G$. The presence of a symmetry $s \leftrightarrow -s$ and the effect of the extension of the symmetry group at the point $s=0$ cause the difference of this ring from the corresponding ring for the group G'.

16.15

Remark. Theorems 11.6 (on conjugate representation), 13.3 and 13.4 (on spherical functions) and 13.8 (on continuity in the appropriate topology) are easily transferred to the groups G from Table 7.5.

Theorem 14.2 and 15.3 (on the approximation of representations) also transfer without any great difficulty to the groups G from Table 7.5. As the representations $T^{(n)}$ of the groups $G(n)$, it is necessary to take the representations of the principal degenerate series connected with parabolic subgroups which preserve the flags $V_1 \subset V_2 \subset \ldots \subset V_k$, where the V_i are isotropic subspaces.

16.16

Remark. The groups G coincide with their own derived groups $[G, G]$ in all cases except $(\mathbb{C}_3)$, when $G=U(\infty,\infty)$. All the results remain in force if this group is replaced by $SU(\infty,\infty)$.

16.17

Proof of Theorem 16.5. Let us identify T_s with $W_F \circ \tau_s^F$. For each of the seven groups it is individually checked that there exists an automorphism ψ_s of the group L' preserving the equivalence class of the representation W_F and sending $\tau_s^F|G$ into $\tau_{-s}^F|G$.

Let us examine in detail the case $(\mathbb{R}_1)$. The Lie algebra $so(\infty,\infty)$ of the group $G=SO_0(\infty,\infty)$ consists of real matrices of the type

$$A=X+Y=\begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}+\begin{bmatrix} 0, & Y \\ Y' & 0 \end{bmatrix},$$

where $X_1=-X_1'$, $X_2=-X_2'$.

It is embedded in the Lie algebra $sp(2\infty,\mathbb{R})$ of the group $L' = Sp(2\infty,\mathbb{R})^{\sim}$ by means of mapping $\tau_s^{\mathbb{R}}$ (section 9.8). Let us examine the matrix

$$h=\left[\begin{array}{cc|cc} \bar{\theta}.1_\infty & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 0 & -\bar{\theta}.1_\infty & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & \theta.1_\infty & 0 \\ & & 0 & -\theta.1_\infty \end{array}\right], \text{ where } \theta^2=\frac{is-1}{1+is}.$$

This is an element of the group $\bar{U}(2\infty)\subset\overline{Sp}(2\infty,\mathbb{R})$. It is easy to check that

$$h\tau_s^{\mathbb{R}}(A)h^{-1}=\tau_{-s}^{\mathbb{R}}(A), \quad A\in so(\infty,\infty).$$

Hence it is possible to write (see sections 8.12, 8.14)

$$\psi_s(g)={}^hg, \quad g\in Sp(\infty,\mathbb{R})^{\sim}.$$

We observe further that in the cases $(\mathbb{C}_1)$, $(\mathbb{C}_2)$ it is necessary to use the existence of an outer automorphism of the group $U(\infty,\infty)^{\sim}$ that does not change the Weil representation (see section 7.8).

16.18

The proof of theorem 16.6 follows the proof of theorem 11.1. In the group L', a subgroup M' containing the group K and in essence coinciding with K^* is selected. Let us put

$$T=(W_F^{\otimes k}\circ\tau_s^F)|G, \; \rho=T|K.$$

The representation $W_F^{\otimes k}|M'$, in essence, coincides with ρ^*. It is then checked that $\tau_s^F(G)$ and M' generate $[L', L']$ (analogue of theorem 11.10).

We shall examine in detail the case $(\mathbb{R}_1)$. Then

$$G=SO_0(\infty,\infty),\; K=SO(\infty)\times SO(\infty),\; K^*=U(\infty)\times U(\infty).$$

We put

$$M' = SU(\infty)\times SU(\infty)\subset SU(2\infty)\subset U(2\infty)^{\sim}\subset Sp(2\infty,\mathbb{R})^{\sim},$$

$$M'(n) = SU(n)\times SU(n)\subset L'(n) = Sp(2n,\mathbb{R})^{\sim},$$

$$\mathfrak{m}'(n) = su(n)+su(n)\subset \mathfrak{l}'(n) = sp(2n,\mathbb{R}).$$

We observe that

$$\mathfrak{m}'(n)=\left\{\left[\begin{array}{cc|cc} \bar{A}_1 & 0 & \multicolumn{2}{c}{0} \\ 0 & \bar{A}_2 & & \\ \hline \multicolumn{2}{c|}{0} & A_1 & 0 \\ & & 0 & A_2 \end{array}\right] : A_1, A_2\in su(n)\right\}.$$

We have to establish the analogue of lemma 11.16. It is formulated as follows: if $s\neq 0$ and $n\geqslant 3$, then the smallest complex subalgebra $\mathfrak{b}$ in the algebra $\mathfrak{l}_{\mathbb{C}}'(n)=sp(2n,\mathbb{C})$ containing $\tau_s^{\mathbb{R}}(so(n,n))$ and $\mathfrak{m}'(n)$ coincides with $\mathfrak{l}_{\mathbb{C}}'$.

Let us examine the decomposition (see section 6.3).

$$l'_{\mathbb{C}}(n)=l'_{-1}(n)\oplus l'_0(n)\oplus l'_1(n)$$

and the corresponding projections $\mathscr{H}_{-1}, \mathscr{H}_0, \mathscr{H}_1$.

First we shall check that the components of this decomposition are disjunct as $m'(n)$-modules. This shows that b is stable with respect to $\mathscr{H}_{-1}, \mathscr{H}_0, \mathscr{H}_1$.

Then we shall check that $\mathscr{H}_0(\tau_s^{\mathbb{R}}(so(n,n))$ and $m'(n)$ generate $[l_0'(n), l_0'(n)]=sl(2n,\mathbb{C})$. It is just at this point that the condition $s\neq 0$ is used for $s=0$, the space $\mathscr{H}_0(\tau_s^{\mathbb{R}}(so(n,n))$ reduces to a subspace of $\mathfrak{m}'(n)$).

Now we use the irreducibility of the $[l_0'(n), l_0'(n)]$-modules $l_1'(n)$ and $l_{-1}'(n)$ for checking that b contains $l_{-1}'(n)$ and $l_1'(n)$. After this it is evident that $b=sp(2n,\mathbb{C})$.

16.19

Proof of Theorem 16.8. Let us study the case $(\mathbb{R}_1)$. Then

$$L'=Sp(2\infty,\mathbb{R})^{\sim},\ L=U(\infty,\infty)^{\sim}.$$

The embedding $L \to L'$ is, by definition, a canonical embedding defined in section 6.9 and this proves (ii).

In the notation of section 16.17, we have for $A \in so(\infty,\infty)$:

$$\tau_0^{\mathbb{R}}(A) = \tau_0^{\mathbb{R}}\left(\begin{bmatrix} X_1 & Y \\ Y' & X_2 \end{bmatrix}\right) = \left[\begin{array}{cc|cc} X_1 & 0 & 0 & Y \\ 0 & X_2 & Y' & 0 \\ \hline 0 & Y & X_1 & 0 \\ Y' & 0 & 0 & X_2 \end{array}\right].$$

since $X_1 = -X_1'$, $X_2 = -X_2'$, this matrix lies in the Lie algebra $u(\infty,\infty)$ of the group L embedded in $sp(2\infty, \mathbb{R})$ canonically. This proves (i).

16.20

PROOF OF THEOREM 16.10. Let us study the case $(\mathbb{R}_1)$. Then $G = SO_0(\infty,\infty)$, $K = SO(\infty) \times SO(\infty)$, $L = U(\infty,\infty)^{\sim}$. The embedding $G \to L$ is generated by the identical mapping $so(\infty,\infty) \to u(\infty,\infty)$.

Now the proof reduces to a check of the fact that $SO_0(\infty,\infty)$ and $SU(\infty) \times SU(\infty)$ generate the group $[L, L] = SU(\infty,\infty)$.

16.21

PROOF OF THEOREM 16.12 follows the proof of theorem 11.3 and reduces to a check of an assertion analogous to lemma 11.17. We shall formulate and prove this assertion for the case $(\mathbb{R}_1)$.

Among the numbers $s_1, \ldots, s_p$ a zero is found not more than once. If it occurs, it will be assumed for definiteness that it is s_p. Let us fix $n \geqslant 3$ and put

$$\mathfrak{m} = \begin{cases} sp(2n, \mathbb{C})^p = sp(2n, \mathbb{C}) \oplus \ldots \oplus sp(2n, \mathbb{C}), \text{ if } s_p \neq 0, \\ sp(2n, \mathbb{C})^{p-1} \oplus sl(2n, \mathbb{C}) \ \textit{if}\ s_p = 0; \end{cases}$$

$$\tau = \tau_{s_1}^{\mathbb{R}} \oplus \ldots \oplus \tau_{s_p}^{\mathbb{R}}: so(n, n) \to \mathfrak{m};$$

$$d: su(n) \oplus su(n) \to \mathfrak{m},$$

the diagonal embedding.

Let $\mathfrak{b}\subset\mathfrak{a}$ be the smallest complex subalgebra containing $\tau(so(n, n))$ and $d(su(n)\oplus su(n))$. Our assertion is that $b=\mathfrak{a}$.

Just as in section 11.17, we use lemma 11.15; its prerequisites are fulfilled by virtue of what is proved in sections 16.18, 16.19. Just as in section 11.17, we have to exclude the existence of an automorphism ϕ of the algebra $l'_{\mathbb{C}}(n)=sp(2n, \mathbb{C})$ such that

$$\phi \mid su(n)\oplus su(n)=\mathrm{id}, \tag{1}$$

$$\phi(\tau^{\mathbb{R}}_{s_j}(A))=\tau^{\mathbb{R}}_{s_i}(A)(A\in so(n, n) \tag{2}$$

for certain i, j, where

$$i\neq j, \quad s_i>0, \quad s_j>0, \quad s_i\neq s_j. \tag{3}$$

From (1) it follows that ϕ commutes with the projections $\mathscr{H}_{-1}, \mathscr{H}_0$, $\mathscr{H}_1$. In particular, ϕ leaves invariant the subalgebra

$$[l'_0(n), l'_0(n)]=sl(2n, \mathbb{C}).$$

Any automorphism of this algebra that is identical on the subalgebra $sl(n, \mathbb{C})\oplus sl(n, \mathbb{C})$ has the form

$$\psi_t:\begin{bmatrix} P & Q\\ R & S\end{bmatrix}\mapsto\begin{bmatrix} P & tQ\\ t^{-1}R & S\end{bmatrix}, \quad t\in\mathbb{C}^*.$$

From (2) and the fact that ϕ commutes with $\mathscr{H}_0$, it follows that

$$\psi_t(\mathscr{H}_0(\tau^{\mathbb{R}}_{s_j}(A)))=\mathscr{H}_0(\tau^{\mathbb{R}}_{s_i}(A)), A\in so(n, n).$$

From this and from the definition of the embedding $\tau^{\mathbb{R}}_s$, it is easy to obtain

$$t=\pm 1, s_i=\pm s_j.$$

But this contradicts (3).

16.22

Proof of Theorem 16.13. After proving theorem 16.12, we may argue in precisely the same way as in §**12**. As $G^*(n)$, one may take the group of $L'(n)$-valued functions $f(s)$ on the half-line $s \geqslant 0$, such that $f(0) \in L(n) \subset L'(n)$.

PART IV
ADMISSIBLE REPRESENTATIONS OF THE PAIRS (*G*, *K*) OF COMPACT TYPE

§17. The formalism of R. Howe for spinor representations of the groups $Sp(\infty)$, $U(2\infty)^{\sim}$ and $SO(2\infty)^{\sim}$

17.1

We adopt the notation indicated in Table 7.7 (compare with Table 7.3 of §**6**).

TABLE 7.7

	$F=\mathbb{R}$	$F=\mathbb{C}$	$F=\mathbb{H}$
$\mathfrak{l}(n)$	$sp(n)$	$u(2n)$	$so(2n)$
$\mathfrak{m}(n)$	$u(n)$	$u(n)\oplus u(n)$	$u(n)$
$\mathfrak{l}_c(n)$	$sp(n, \mathbb{C})$	$\mathfrak{gl}(2n, \mathbb{C})$	$so(2n, \mathbb{C})$
$L(n)$	$Sp(n)$	$U(2n)^{\sim}$	$SO(2n)^{\sim}$
$M(n)$	$U(n)$	$(U(n)\times U(n))^{\sim}$	$U(n)^{\sim}$
L	$Sp(\infty)$	$U(2\infty)^{\sim}$	$SO(2\infty)^{\sim}$
M	$U(\infty)$	$(U(\infty)\times U(\infty))^{\sim}$	$U(\infty)^{\sim}$
$U(k, F)$	$Sp(K)$	$U(k)$	$O(k)$

As usual,

$$L=\bigcup_{n=1}^{\infty} L(n), \quad M=\bigcup_{n=1}^{\infty} M(n).$$

17.2

We realize Lie algebras $l(n)$ as follows:

$$sp(n)=\left\{\begin{bmatrix} -A' & B \\ -B^* & A \end{bmatrix}\in\mathbb{C}^{2n,2n}: A=-A^*, B=B'\right\}$$

$$=u(2n)\cap sp(n,\mathbb{C});$$

$$u(2n)=\left\{\begin{bmatrix} A & B \\ -B^* & D \end{bmatrix}\in\mathbb{C}^{2n,2n}: A=-A^*, D=-D^*\right\};$$

$$so(2n)=\left\{\begin{bmatrix} -A' & B \\ -B^* & A \end{bmatrix}\in\mathbb{C}^{2n,2n}: A=-A^*, B=-B'\right\}$$

$$=u(2n)\cap so(2n,\mathbb{C}).$$

The pairs $(l(n), \mathfrak{m}(n))$ studied here are dual in the sense of Cartan to the pairs $(l(n), \mathfrak{m}(n))$ from **§6**.

17.3

For the groups

$$L(n)=Sp(n),\ U(2n)^{\sim},\ SO(2n)^{\sim},$$

there exist representations which are analogues of the Weil representations $W_F^{(n)}$. We shall call them *spinor representations* and denote them by $S_F^{(n)}$, where $F=\mathbb{R}, \mathbb{C}, \mathbb{H}$. These representations are easily generalized to the groups $L=Sp(\infty)$, $U(2\infty)^{\sim}$, $SO(2\infty)^{\sim}$ (see below). We call the corresponding representations spinor representations also and denote by S_F.

The analogues of all the statements of **§§6–8** on Weil representations are valid for spinor representations. Hence our account here will be less detailed.

17.4

Let us recall that Weil representations $W_F^{(n)}$ are constructed first for $F=\mathbb{R}$ and then for $F=\mathbb{C}$ and $F=\mathbb{H}$. In this construction the following embeddings are used:

$$U(n, n)^{\sim} \subset Sp(2n, \mathbb{R})^{\sim}, \quad SO^*(2n) \subset U(n, n)^{\sim}.$$

For spinor representations $S_F^{(n)}$, the reverse order is used: first $F=\mathbb{H}$ and then $F=\mathbb{C}$ and $F=\mathbb{R}$. Now the following embeddings are used

$$U(2n)^{\sim} \subset SO(4n)^{\sim}, \quad Sp(n) \subset U(2n)^{\sim}.$$

The group $U(k, F)$ shown in the last row of Table 7.7 is the symmetry group for $S_F^{(n)\otimes k}$.

17.5

Let H be a complex Hilbert space and $\wedge^m(H)$ the pre-Hilbert space which is the algebraic exterior m-th power of space H. The scalar product in $\wedge^m(H)$ is given as follows:

$$(\xi_1 \wedge \ldots \wedge \xi_m, \eta_1 \wedge \ldots \wedge \eta_m) = \det\|(\xi_i, \eta_j)\|.$$

If $\{e_1, e_2, \ldots\}$ is an orthonormal basis in H, then the multivectors

$$e_{i_1} \ldots i_m = e_{i_1} \wedge \ldots \wedge e_{i_m} \; (i_1 < \ldots < i_m)$$

form a basis in $\wedge^m(H)$.

17.6

Definition (compare definition 3.1) The completion of the pre-Hilbert space

$$\mathbb{C} \oplus H \oplus \wedge^2(H) \oplus \ldots$$

is called *Fermion Fock space* $\mathcal{F}(H) = \mathcal{F}^-(H)$.

If H is finite-dimensional, then $\mathcal{F}(H)$ is also finite-dimensional. If $H = H_1 \oplus H_2$, then $\mathcal{F}(H) = \mathcal{F}(H) \otimes \mathcal{F}(H_2)$.

In $\mathcal{F}(H)$, a unitary representation of the group $U(H) = U(\bar{H})$ of all unitary operators in the space H is defined.

As in the boson case, instead of $\mathcal{F}(\bar{H})$, we shall write $\mathcal{H}(H)$.

17.7

For $\xi \in H$, we define in $\mathscr{F}(H)$ the operators M_ξ and D_ξ as follows:

$$M_\xi x = \xi \wedge x, \quad D_\xi = M_\xi^*.$$

These operators are bounded:

$$\|M_\xi\|^2 = \|D_\xi\|^2 = (\xi, \xi).$$

They satisfy the canonical anticommutation relations

$$[M_\xi, M_\eta]_+ = [D_\xi, D_\eta]_+ = 0,$$

$$[M_\xi, D_\eta]_+ = (\xi, \eta)\cdot 1 \; (\xi, \eta \in H).$$

If a basis $\{e_1, e_2, \ldots\}$ is selected in H, then we shall put

$$M_i = M_{e_i}, \; D_i = D_{e_i} \; (i = 1, 2, \ldots).$$

17.8

Now we shall define the representations $S_F^{(n)}$. First let $F = \mathbb{H}$: then $l(n) = so(2n)$.

The spinor representation $S_{\mathbb{H}}^{(n)}$ of the group $L(n) = SO(2n)^-$ acts in the finite-dimensional space $\mathscr{H}(\mathbb{C}^{1,n})$. The corresponding representation of the Lie algebra $l_{\mathbb{C}}(n) = so(2n, \mathbb{C})$ is also denoted by $S_{\mathbb{H}}^{(n)}$.

Let us give explicit formulae for the representation $S_{\mathbb{H}}^{(n)\otimes k}$ $(k = 1, 2, \ldots)$, which acts in the space $\mathscr{H}(\mathbb{C}^{k,n})$:

$$\begin{bmatrix} -E_{ji} & 0 \\ 0 & E_{ij} \end{bmatrix} \mapsto \left(\sum_{\alpha=1}^{k} M_{\alpha i} D_{\alpha j} \right) - \frac{k}{2} \delta_{ij} \cdot 1$$

$$= \frac{1}{2} \sum_{\alpha=1}^{k} [M_{\alpha i}, D_{\alpha j}],$$

$$\begin{bmatrix} 0 & E_{ij} - E_{ji} \\ 0 & 0 \end{bmatrix} \mapsto \sum_{\alpha=1}^{k} D_{\alpha i} D_{\alpha j} = \frac{1}{2} \sum_{\alpha=1}^{k} [D_{\alpha i}, D_{\alpha j}].$$

$$\begin{bmatrix} 0 & 0 \\ E_{ij}-E_{ji} & 0 \end{bmatrix} \mapsto \sum_{\alpha=1}^{k} M_{\alpha i} M_{\alpha j} = \frac{1}{2} \sum_{\alpha=1}^{k} [M_{\alpha i}, M_{\alpha j}].$$

Here $i, j = 1, \ldots, n$; the operators $M_{\alpha i}, D_{\alpha i}$ have an obvious meaning.

The representation $S_{\mathbb{H}}^{(n)}|M(n)$ of the group $M(n) = U(n)^{\sim}$ is the multiplicity free direct sum of irreducible representations with the highest weights

$$(\underbrace{1, \ldots, 1}_{m}, \underbrace{0 \ldots, 0}_{n-m}) - (1/2, \ldots, 1/2),$$

where $m = 0, 1, \ldots, n$.

17.9

Let us study now the case $F = \mathbb{C}$. Then $\mathfrak{l}(n) = u(2n)$, $\mathfrak{l}_{\mathbb{C}}(n) = \mathfrak{gl}(2n, \mathbb{C})$.

We shall define the embedding $\mathfrak{gl}(2n, \mathbb{C}) \to so(4n, \mathbb{C})$ as follows (compare section 6.9):

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \mapsto \left[\begin{array}{cc|cc} A & 0 & 0 & B \\ 0 & -D' & -B' & 0 \\ \hline 0 & -C' & -A' & 0 \\ C & 0 & 0 & D \end{array}\right].$$

This mapping embeds $u(2n)$ into $so(4n)$ and defines an embedding $U(2n)^{\sim} \to SO(4n)^{\sim}$.

Let us define the representation $S_{\mathbb{C}}^{(n)}$ of the group $L(n) = U(2n)^{\sim}$ as the restriction to it of the representation $S_{\mathbb{H}}^{(2n)}$. The representation $S_{\mathbb{C}}^{(n)\otimes k}$ of the Lie algebra $\mathfrak{gl}(2n, \mathbb{C})$ acts in the space:

$$\mathscr{H}(\mathbb{C}^{k,2n}) = \mathscr{H}(\mathbb{C}^{k,n} \oplus \mathbb{C}^{k,n})$$

as follows (compare section 6.10).

$$\begin{bmatrix} -E_{ji} & 0 \\ 0 & 0 \end{bmatrix} \mapsto \left(\sum_{\alpha=1}^{k} M_{\alpha i} D_{\alpha j} \right) - \frac{k}{2} \delta_{ij} \cdot 1,$$

$$\begin{bmatrix} 0 & 0 \\ 0 & E_{ij} \end{bmatrix} \mapsto \left(\sum_{\alpha=1}^{k} M_{\alpha i} D_{\alpha j} \right) - \frac{k}{2} \delta_{ij} \cdot 1,$$

$$\begin{bmatrix} 0 & E_{ij} \\ 0 & 0 \end{bmatrix} \mapsto \sum_{\alpha=1}^{k} D_{\alpha i} D_{\alpha j},$$

$$\begin{bmatrix} 0 & 0 \\ E_{ij} & 0 \end{bmatrix} \mapsto \sum_{\alpha=1}^{k} M_{\alpha i} M_{\alpha j}.$$

The representation $S_{\mathbb{C}}^{(n)}|M(n)$ of the group $(U(n)\times U(n))^{\sim}$ is the tensor product of the representations indicated at the end of section 17.8.

17.10

Let now $F=\mathbb{R}$, then $\mathfrak{l}(n)=sp(n)$, $l_{\mathbb{C}}(n)=sp(n,\mathbb{C})$. The embedding $\mathfrak{sp}(n)\to u(2n)$ is defined by the realization of the algebra $\mathfrak{sp}(n)$ (see section 17.2). It gives the embedding $Sp(n)\to U(2n)$ which is lifted into $U(2n)^{\sim}$.

The representation $S_{\mathbb{R}}^{(n)}$ is defined as $S_{\mathbb{C}}^{(n)}|Sp(n)$. The representation $S_{\mathbb{R}}^{(n)\otimes K}$ of the Lie algebra $sp(n,\mathbb{C})$ acts in the space

$$\mathscr{H}(\mathbb{C}^{k,2n})=\mathscr{H}(\mathbb{C}^{k,n}\oplus\mathbb{C}^{k,n}),$$

which is conveniently identified with $\mathscr{H}(\mathbb{C}^{2k,n})$ (compare section 6.11). According to this, we write $M_{\dot{\alpha}i}, D_{\dot{\alpha}i}$ instead of $M_{\alpha i}, D_{\alpha i}$.

The formulae for $S_{\mathbb{R}}^{(n)\otimes K}$ have the following form

$$\begin{bmatrix} -E_{ji} & 0 \\ 0 & E_{ij} \end{bmatrix} \mapsto \left(\sum_{\alpha=1}^{k} (M_{\dot{\alpha}i} D_{\dot{\alpha}j} + M_{\alpha i} D_{\alpha j}) \right) - k\, \delta_{ij} \cdot 1,$$

$$\begin{bmatrix} 0 & E_{ij}+E_{ji} \\ 0 & 0 \end{bmatrix} \mapsto \sum_{\alpha=1}^{k} (D_{\dot{\alpha}i} D_{\alpha j} + D_{\dot{\alpha}j} D_{\alpha i}),$$

$$\begin{bmatrix} 0 & 0 \\ E_{ij}+E_{ji} & 0 \end{bmatrix} \mapsto \sum_{\alpha=1}^{k} (M_{\alpha i} M_{\dot{\alpha}j} + M_{\alpha j} M_{\dot{\alpha}i}).$$

The representation $S_{\mathbb{R}}^{(n)}|M(n)$ of the group $M(n)=U(n)$ is the tensor square of the representation shown in section 17.8.

17.11

We shall define the unitary representation $R_k^{(n)}$ of the group $U(k, F)$ (see Table 7.7) in the space $H(S_F^{(n)\otimes k})$ as follows.

If $F=\mathbb{R}$, then $U(k, F)=Sp(k)$,

$$H(S_{\mathbb{R}}^{(n)\otimes k})=\mathscr{H}(\mathbb{C}^{2k,n})=\mathscr{H}(\mathbb{C}^{2k,1})^{\otimes n}$$

and $R_k^{(n)}$ originates from the identical representation of the group $Sp(k)$ in $\mathbb{C}^{2k,1}$.

If $F=\mathbb{C}$, then $U(k, F)=U(k)$,

$$H(S_{\mathbb{C}}^{(n)\otimes k})=\mathscr{H}(\mathbb{C}^{k,2n})=\mathscr{H}(\mathbb{C}^{k,1}\oplus\mathbb{C}^{k,1})^{\otimes n}$$

and $R_k^{(n)}$ originates from the representation $g\mapsto g\oplus g$ of the group $U(k)$ in $\mathbb{C}^{k,1}\oplus\mathbb{C}^{k,1}$.

If $F=\mathbb{H}$, then $U(k, F)=O(k)$, $H(S_{\mathbb{H}}^{(n)\otimes k})=\mathscr{H}(\mathbb{C}^{k,n})=\mathscr{H}(C^{k,1})^{\otimes n}$, and $R_k^{(n)}$ originates from the identical representation of the group $O(k)$ in $\mathbb{C}^{k,1}$.

17.12

THEOREM OF R. HOWE (compare with theorem 6.14). For any $k=1, 2, \ldots$, the representation $S_F^{(n)\otimes k}$ of the group $L(n)$ from Table 7.7 possesses the symmetry group $U(k, F)$.

For a proof see [9], [30].

17.13

COROLLARY. For any $\sigma\in U(k, F)^\wedge$ occurring in the decomposition of representation $R_k^{(n)}$ of the group $U(k, F)$, an irreducible representation $S_\sigma^{(n)}$ of the group $L(n)$ is defined.

17.14

Thus, we have obtained analogues of all results of §6. After this, all the results of §7 are easily transferred to spinor representations.† We

†Analogues of lemma 7.1 and corollary 7.2 are formulated thus: for any $\sigma\in U(k, F)^\wedge$ an n_0 can be found such that, for $n\geqslant n_0$, the representation σ is contained in $R_k^{(n)}$ and thus $S_\sigma^{(n)}$ is determined. Unlike the boson case, n_0 depends on σ.

observe that the automorphism φ from theorem 7.7 is defined in our situation in precisely the same way and the proof of the theorem does not require changes.

The spinor representation S_F of the group L is determined again as the inductive limit of the representations $S_F^{(n)}$. The representation $S_F^{\otimes k}$ possesses the symmetry group $U(k, F)$ and decomposes into a direct sum of irreducible representations S_σ where σ runs through $U(k, F)^\wedge$. In its turn, S_σ is the inductive limit of the representations $S_\sigma^{(n)}$.

As the representation T_{hol} of the group $U(\infty)$ is taken the canonical representation of this group in $\mathscr{H}(l_2(\mathbb{C}))$. This again is a holomorphic tame representation. It is equivalent to the multiplicity free direct sum of irreducible representations of the type

$$\rho_{(1, \ldots, 1, 0, 0, \ldots)}, \; m \text{ units}, \quad m = 0, 1, 2, \ldots$$

The representation $S_F^{\otimes k}|M$ is again the product of a certain tame holomorphic representation and a one-dimensional representation.

17.15

Remark. For Weil representations, the following propositions are valid:

(A_n) The irreducible representations

$$W_{\pi_1}^{(n)} \quad \text{and} \quad \bar{W}_{\pi_2}^{(n)} \sim W_{\pi_2}^{(n)} \circ \varphi$$

of the group $L(n)$ are not equivalent for any π_1 and π_2. (In fact, let us consider the corresponding irreducible $l_{\mathbb{C}}(n)$-modules. Both of them are infinite-dimensional, but the first is a module with the highest weight and the second with the lowest weight, so that isomorphism between them is not possible.)

(A_∞) The irreducible representations

$$W_{\pi_1} \quad \text{and} \quad \bar{W}_{\pi_2} \sim W_{\pi_2} \circ \varphi$$

of the group L are not equivalent for any π_1 and π_2. (First proof: this follows from the proposition (A_n). Second proof: the representations

$W_{\pi_1}|M$ and $\bar{W}_{\pi_2}|M$ are disjunct because the first is "almost" holomorphic and the second "almost" antiholomorphic; for greater persuasiveness, it is possible to replace M by $[M, M]$.)

The proposition (A_n) does not hold for spinor representations. For example, the spinor representation $S_{\mathbb{H}}^{(n)}$ of the group $L(n) = SO(2n)^{\sim}$ is the sum of two irreducible representations $S_1^{(n)}$ and $S_{-1}^{(n)}$ corresponding to two characters of the group $U(1, \mathbb{H}) = O(1) = \{\pm 1\}$. These representations are conjugate to one another if n is odd and self-conjugate if n is even. This shows that

$$S_F^{(n)} \sim \overline{S_F^{(n)}} \sim S_F^{(n)} \circ \varphi$$

for $F = \mathbb{H}$, and that means, also for $F = \mathbb{C}, \mathbb{R}$.

At the same time the proposition (A_∞), together with the second proof, remains valid.

At first glance this seems improbable. This phenomenon is very clearly manifested in the case $F = \mathbb{R}$, when $L = Sp(\infty)$. In fact, let us consider any irreducible representation of the group $Sp(\infty)$ of the type S_σ, where $\sigma \in Sp(k)^\wedge$. It is an inductive limit of the irreducible representations $S_\sigma^{(n)}$ of the groups $L(n) = Sp(n)$.

We have

$$\overline{S_\sigma^{(n)}} \sim S_\sigma^{(n)} \circ \varphi, \; \bar{S}_\sigma \sim S_\sigma \circ \varphi.$$

The representations $S_\sigma^{(n)}$ and $\overline{S_\sigma^{(n)}}$ are equivalent, since for the group $Sp(n)$ all the irreducible representations are self-conjugate. It is possible also to observe that $\varphi | Sp(n)$ is an inner automorphism. At the same time the representations S_σ and $\bar{S}_\sigma$ are not equivalent.

The explanation consists of the fact that the multiplicity with which $S_\sigma^{(n)}$ enters the decomposition of the representation $S_\sigma^{(n+1)}|Sp(n)$ always exceeds unity. Hence the embedding

$$H(S_\sigma^{(n)}) \to H(S_\sigma^{(n+1)}),$$

which is prescribed by the construction of the representations $S_{\mathbb{R}}^{(n)}$, is not the only one possible. In other words, the diagram

$$\begin{array}{ccc} H(S_\sigma^{(n)}) & \to & H(S_\sigma^{(n+1)}) \\ \downarrow & & \downarrow \\ H(\overline{S_\sigma^{(n)}} \circ \varphi) & \to & H(\overline{S_\sigma^{(n+1)}}), \end{array}$$

where the vertical arrows realize equivalence of the representations, is not commutative.

Thus, the automorphism φ of the infinite-dimensional group L is a "genuinely outer" automorphism in the fermion as in the boson case.

17.16

All results of §8 are also easily transferred to spinor representations. We give the formulae for the spherical function φ_k of the representation $S_F^{\otimes k}$:

If $F=\mathbb{R}$, $g\in Sp(\infty)$,

$$\beta(g)=\begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

then $\varphi_k(g)=(\det a)^k=\overline{(\det d)}^k$.

If $F=\mathbb{C}$, $g\in U(2\infty)^{\sim}$ and

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is the projection of the element g in $U(2\infty)$, then $\varphi_k(g)=(\det a)^k(\det g)^{-k/2}=\overline{(\det d)}^k(\det g)^{k/2}$.

If $F=\mathbb{H}$, $g\in SO(2\infty)^{\sim}$ and

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is the projection of the element g in $SO(2\infty)$, then

$$\varphi_k(g)=(\det a)^{k/2}=(\overline{\det d})^{k/2}.$$

17.17

In the proof of the analogue of theorem 8.3, instead of the group Heis (H) it is necessary to take the real Jordan algebra $J(H)$, which is formed by the bounded self-adjoint operators

$$A_\xi=M_\xi+D_\xi,\quad \xi\in \bar{H},$$

in the space $\mathscr{H}(\bar{H})=\mathscr{F}(H)$ and the identity operator. The complete orthogonal group $\overline{SO}(2\infty)$ is an automorphism group of the Jordan algebra $J(l_2(\mathbb{C}))$.

§18. "Boson" and "fermion" fundamental representations

18.1

For the pairs (G, K) of compact type (see Table 7.2 of Introduction) it will be possible to get an analogue of the second (but not the first) construction of fundamental representations (see **§9**). For pairs of non-compact type we used the Weil representation. For compact pairs, along with the Weil representation, spinor representations are used, as well as certain "intermediate" representations (see **§19**).

For distinguishing the groups L, $L(n)$, $U(k, F)$ from Table 7.3 (**§6**) (boson case) and the corresponding groups from Table 7.7 (**§17**) (fermion case), we shall give the former the index "$+$" and latter the index "$-$".

18.2

The scheme of construction of boson and fermion fundamental representations is as follows. At first three pairs (G, K) of the type $(\mathbb{R})$, $(\mathbb{C})$ and $(\mathbb{H})$ are studied. Boson (respectively fermion) fundamental representations of a pair of type (F) have respectively the form

$$W_{\ell}\circ\tau_s, \quad S_{\ell}\circ\nu_t,$$

where W_{ℓ} is the Weil representation of the corresponding group $L=L^+$, S_{ℓ} the spinor representation of the corresponding group $L=L^-$,

$$\tau_s:(G, K)\to(L^+, M) \quad \text{and} \quad \nu_t:(G, K)\to(L^-, M)$$

are certain embeddings, s and t the parameters.

The group L^+ is selected from Table 7.3 (**§6**) and the group L^- from Table 7.7 (**§17**) in conformity with the type F. The subgroup M is essentially one and the same for L^+ and L^-.

The fundamental representations of the remaining seven pairs of the type (F_i) are obtained by means of the appropriate embedding of the pair of type (F_i) into the pair of type (F) (compare section 16.2).

We shall examine in detail only the case $(\mathbb{R})$. Thus from this point

$$G = U(\infty),\ K = SO(\infty),\ L^+ = Sp(\infty, \mathbb{R})^-,\ L^- = Sp(\infty).$$

18.3

Let us construct a family of embeddings $\{\tau_s\}$ into the group $L^+ = Sp(\infty, \mathbb{R})^-$. The parameter s runs through $\mathbb{R}\backslash\{0\}$. The embeddings τ_s are not single-valued on $U(\infty)$. To make them single-valued, it is necessary temporarily to replace $U(\infty)$ by $U(\infty)^-$. In constructing the fundamental representations, a "gauge multiplier" $\det(\cdot)^{\pm 1/2}$ will be added, making the representation single-valued on the group $U(\infty)$. This small complication does not arise if the group $SU(\infty)$ is studied from the very beginning.

Just as in §9, τ_s is conveniently given at the level of the Lie algebras. Let us put

$$h_s^+ = \begin{bmatrix} \cosh\left(\frac{s}{2}\right) \cdot 1_\infty & \sinh\left(\frac{s}{2}\right) \cdot 1_\infty \\ \sinh\left(\frac{s}{2}\right) \cdot 1_\infty & \cosh\left(\frac{s}{2}\right) \cdot 1_\infty \end{bmatrix}.$$

This is an element of the group $\overline{Sp}(\infty, \mathbb{R})$ (see section 8.2).

For

$$A \in u(\infty) = \bigcup_{n=1}^{\infty} u(n),$$

we put

$$\tau_s(A) = h_s^+ \begin{bmatrix} -A' & 0 \\ 0 & A \end{bmatrix} (h_s^+)^{-1},\ s > 0;$$

$$\tau_s(A) = \tau_{-s}(-A') = \tau_{-s}(\bar{A}),\ s < 0.$$

We shall put $A = X + Y$, where

$$X = \bar{X} = -X' = \frac{A - A'}{2}, \; Y = -\bar{Y} = Y' = \frac{A + A'}{2}.$$

Then

$$\tau_s(X) = \begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix}, \quad s \in \mathbb{R} \setminus \{0\};$$

$$\tau_s(Y) = \begin{bmatrix} -\cosh s \cdot Y & \sinh s \cdot Y \\ -\sinh s \cdot Y & \cosh s \cdot Y \end{bmatrix}, \; s > 0;$$

$$\tau_s(Y) = \begin{bmatrix} \cosh s \cdot Y & \sinh s \cdot Y \\ -\sinh s \cdot Y & -\cosh s \cdot Y \end{bmatrix}, \; s < 0.$$

18.4

Remark. The embedding $\tau_s | SO(\infty)$ does not depend on s and coincides with the canonical embedding

$$SO(\infty) \to SU(\infty) \to U(\infty)^{\sim} \to Sp(\infty, \mathbb{R})^{\sim}.$$

We observe that $\tau_{-s} = \varphi \circ \tau_s$, where φ is the outer automorphism of the group $Sp(\infty, \mathbb{R})^{\sim}$ from section 7.7.

18.5

Remark. Any matrix from $Sp(1, \mathbb{R}) = SU(1, 1)$ may be reduced to the form

$$\begin{bmatrix} \cosh \frac{s}{2} & \sinh \frac{s}{2} \\ \sinh \frac{s}{2} & \cosh \frac{s}{2} \end{bmatrix}, \quad s \geqslant 0,$$

by multiplication on the left and on the right by appropriate elements from the subgroup $U(1) \subset Sp(1, \mathbb{R})$ (compare section 9.15).

18.6

Let us define the *boson fundamental representations* T_s^+, $s\in\mathbb{R}\setminus\{0\}$, of the pair $(U(\infty), SO(\infty))$ as follows:

$$T_s^+(g) = W_{\mathbb{R}}(\tau_s(g))(\det g)^{-\operatorname{sgn}(s)/2};\ g\in U(\infty).$$

Here $\operatorname{sgn}(s)$ is equal to 1 for $s>0$ and -1 for $s<0$.

Both the cofactors on the right-hand side are correctly defined only on $U(\infty)^{\sim}$; however, their product is already a single-valued function on $U(\infty)$.

18.7

Remark. The double-valuedness of the representation $W_{\mathbb{R}}\circ\tau_s$ may be compensated by any "gauge" multiplier of the type

$$\det(\cdot)^a,\ \text{where } a=\pm 1/2,\ \pm 3/2,\ldots$$

Our selection of the "gauge" multiplier is convenient, because as $s\to\pm 0$ the spherical function of the representation T_s^+ tends to the spherical function of the trivial representation (i.e. to the function $f(g)\equiv 1$).

18.8

Remark. The representations T_s^+ and T_{-s}^+ are conjugate to one another.

18.9

Now we construct the family $\{\nu_t\}$ of embeddings of the group $G=U(\infty)$ into the group $L^{\sim}=Sp(\infty)$. The parameter t runs through the open interval $(0,\pi)$.

Let us put

$$h_t^{\sim} = \begin{bmatrix} \cos\left(\frac{t}{2}\right)\cdot 1_\infty & \sin\left(\frac{t}{2}\right)\cdot 1_\infty \\ -\sin\left(\frac{t}{2}\right)\cdot 1_\infty & \cos\left(\frac{t}{2}\right)\cdot 1_\infty \end{bmatrix},\ t\in(0,\pi).$$

This is an element of the group $\overline{Sp}(\infty)$, which is defined by analogy with $\overline{Sp}(\infty, \mathbb{R})$ (see section 8.2).

ν_t is given by the embedding ν_t: $u(\infty) \to sp(\infty)$ of the Lie algebras. If $A \in u(\infty)$, then

$$\nu_t(A) = h_t^- \begin{bmatrix} -A' & 0 \\ 0 & A \end{bmatrix} (h_t^-)^{-1}, 0 < t < \pi.$$

Assuming $A = X + Y$ as in section 18.3, we get

$$\nu_t(X) = \begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix}, 0 < t < \pi;$$

$$\nu_t(Y) = \begin{bmatrix} -\cos t \cdot Y & \sin t \cdot Y \\ \sin t \cdot Y & \cos t \cdot Y \end{bmatrix}, 0 < t < \pi.$$

18.10

Remark. $\nu_t | SO(\infty)$ does not depend on t and coincides with the canonical embedding

$$SO(\infty) \to U(\infty) \to Sp(\infty).$$

We may also repeat remark 18.5.

18.11

Remark. For all $t \in (0, \pi)$, we have

$$\nu_{\pi - t}(A) = \varphi_0(\nu_t(\bar{A})) \quad (A \in U(\infty)),$$

where φ_0 is an automorphism of the group $Sp(\infty)$ given by conjugation by the matrix

$$\begin{bmatrix} i \cdot 1_\infty & 0 \\ 0 & -i \cdot 1_\infty \end{bmatrix} \in \bar{U}(\infty) \subset \overline{Sp}(\infty).$$

We emphasize that φ_0 does not change the equivalence class of the spinor representation S_μ of the group $Sp(\infty)$.

18.12

Remark. For any $t \in (0, \pi)$, we have

$$\nu_{\pi - t} = \varphi \circ \nu_t$$

where φ is the automorphism of the group $Sp(\infty)$ given by the conjugation by the matrix

$$\begin{bmatrix} 0 & i \cdot 1_\infty \\ i \cdot 1_\infty & 0 \end{bmatrix} \in \overline{Sp}(\infty).$$

φ induces an inner automorphism of the subgroup $Sp(n)$ for all $n = 1, 2, \ldots$, yet, nevertheless, φ is a "genuinely outer" automorphism of the group $Sp(\infty)$. In fact, it does not preserve the spherical function of the representation S_μ (see section 17.16) and this means that it does not preserve the equivalence class of the representation S_μ. The representations S_μ and $S_\mu \circ \varphi$ of the group $Sp(\infty)$ are conjugate to one another and are not equivalent (see remark 17.15).

18.13

We define the *fermion fundamental representations* T_t^-, $0 < t < \pi$, of the pair $(U(\infty), SO(\infty))$ as follows:

$$T_t^-(g) = S_\mu(\nu_t(g)) \det(g)^{\operatorname{sgn}(\pi/2 - t)} \quad (g \in U(\infty)).$$

The symbol $\operatorname{sgn}(x)$ is equal to $1, 0, -1$ for $x > 0$, $x = 0$, $x < 0$ respectively. The "gauge multiplier", introduced into the definition of the representation T_t^- is necessary here not for compensation of non-singlevaluedness (because the representation $S_\mu \circ \nu_t$ is correctly defined on $U(\infty)$): it serves the same purpose as in remark 18.7.

18.14

Remark. The representations T_t^- and $T_{\pi - t}^-$ are conjugate to one another. In particular, at the point $t = \pi/2$, the fermion fundamental representation is conjugate to itself.

18.15

In conclusion let us examine, still briefly, a pair of the type ($\mathbb{C}$). In this case,

$$G = U(\infty)\times U(\infty),\ K = U(\infty),\ L^{+} = U(\infty,\infty)^{\sim},\ L^{-} = U(2\infty)^{\sim}.$$

Let us recall that K is the diagonal in G.

The embeddings

$$\tau_s\colon u(\infty)\oplus u(\infty)\to u(\infty,\infty),\ s\in\mathbb{R}\setminus\{0\},$$

are given as follows:

$$\tau_s(A_1\oplus A_2) = h_s^{+}\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}(h_s^{+})^{-1},\ s>0;$$

$$\tau_s(A_1\oplus A_2) = \tau_{-s}(\bar{A}_1\oplus \bar{A}_2),\ s<0.$$

Here $A_1, A_2\in u(\infty)$ and h_s^{+} is the matrix from section 18.3.

The boson fundamental representation T_s^{+}, where $s\in\mathbb{R}\setminus\{0\}$, is given thus:

$$T_s^{+}(g_1, g_2) = W_{\mathbb{C}}(\tau_s(g_1, g_2))\det(g_1 g_2^{-1})^{1/2\,\operatorname{sgn}(s)}$$

$$(g_1, g_2\in U(\infty),\ s\in\mathbb{R}\setminus\{0\}).$$

The representations T_s^{+} and T_{-s}^{+} are conjugate to one another.

The embeddings

$$\nu_t\colon u(\infty)\oplus u(\infty)\to u(2\infty),\ 0<t<\pi,$$

are given as follows:

$$\nu_t(A_1\oplus A_2) = h_t^{-}\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}(h_t^{-})^{-1},\ 0<t<\pi.$$

Here h_t^{-} is the matrix from section 18.9.

The fermion fundamental representation T_t^-, $0<t<\pi$, is given thus

$$T_t^-(g_1, g_2)=S_{\mathbb{C}}(\nu_t(g_1, g_2))\det(g_1g_2^{-1})^{\operatorname{sgn}((\pi/2)-t)}.$$

The representations T_t^- and $T_{\pi-t}^-$ are conjugate to one another. In particular, at the point $t=\pi/2$, we get a self-conjugate representation.

§19. The "intermediate" series of fundamental representations

19.1

Let H be a complex Hilbert space, $U(H)$ the group of all unitary operators in H and Heis (H) the group defined in section 8.4. Let us recall that the elements of this group are pairs $(x, \xi)\in\mathbb{R}\times H$. Group $U(H)$ acts in Heis (H) as automorphisms as follows:

$$u\colon (x, \xi)\mapsto(x, u\xi)\ (u\in U(H)).$$

Let us denote by Heis $(H)\cdot U(H)$ the corresponding semi-direct product.

19.2

Consider the representation V of the group Heis (H) in boson Fock space $\mathscr{F}(\bar{H})$ (see section 8.5) and the canonical representation of the group $U(H)=U(\bar{H})$ in the same space. These two representations give the unitary representation of the group Heis $(H)\cdot U(\bar{H})$ in $\mathscr{F}(H)$ which we shall again denote by V. It is irreducible since its restriction to Heis (H) is irreducible. For $r>0$, we shall denote by V_r the irreducible unitary representation of the group Heis $(H)\cdot U(H)$ obtained from V by the outer automorphism $(x, \xi)\cdot u\to(rx, \sqrt{(r)}\xi)\cdot u$, where $(x, \xi)\in$ Heis (H), $u\in U(H)$. The representation V_r possesses a unique $U(H)$-invariant vector, to within a factor. The corresponding spherical function has the form

$$(x, \xi)\cdot u\mapsto\exp r\left(ix-\frac{\|\xi\|^2}{2}\right).$$

The representation V_r may be characterized as the cyclic representation given by this function.

19.3

Let $\mathscr{G}$ be an abstract group and $x(\cdot)$, $\xi(\cdot)$, $u(\cdot)$ mappings of it into $\mathbb{R}$, H, $U(H)$ respectively.

LEMMA. The mapping

$$g \mapsto (x(g), \xi(g)) \cdot u(g) \; (g \in \mathscr{G}),$$

given by these three mappings is a morphism $\mathscr{G} \to \text{Heis}\,(H) \cdot U(H)$ if and only if the following three conditions are satisfied:

(i) $f \mapsto u(g)$ is a morphism $\mathscr{G} \to U(H)$, i.e., a unitary representation of the group $\mathscr{G}$ in H;

(ii) $g \mapsto \xi(g)$ is a 1-cocycle on $\mathscr{G}$ with values in H, i.e.,

$$\xi(g_1 g_2) = \xi(g_1) + u(g_1)\xi(g_2) \; (g_1, g_2 \in \mathscr{G});$$

(iii) the real function $x(g)$ satisfies the relation

$$x(g_1 g_2) = x(g_1) + x(g_2) - \text{Im}(\xi(g_1), u(g_1)\xi(g_2))_H.$$

The proof is trivial.

19.4

We shall describe now one method of constructing morphisms $\mathscr{G} \to \text{Heis}\,(H) \cdot U(H)$ giving non-trivial results for infinite-dimensional spaces H.

Let a unitary representation $g \to u(g)$ of the group G in H be given. Let us assume that in H there is a G-invariant dense subspace $\Phi \subset H$. Let us denote by Φ' the space of all antilinear functionals on Φ (without any continuity conditions). Then it is possible to write

$$\Phi \subset H \subset \Phi'.$$

The canonical sesquilinear pairing between Φ and Φ' will be denoted just like the scalar product in H. We shall denote the obvious action of the group $\mathscr{G}$ in Φ' just like the action in H.

Let us assume now that in Φ' there exists a vector η such that

$$u(g)\eta-\eta\in\Phi \ (g\in\mathscr{G}).$$

Let us then put

$$\xi(g)=u(g)\eta-\eta, \quad x(g)=\operatorname{Im}(\eta, \xi(g)).$$

LEMMA. The mappings $u(\cdot)$, $\xi(\cdot)$, $x(\cdot)$ satisfy the conditions (i)–(iii) of lemma 19.3 and hence they give a morphism

$$\mathscr{G}\to \operatorname{Heis}(H)\cdot U(H).$$

The proof is trivial.

19.5

Remark. The function $\xi(g)$ from section 19.4 is a trivial 1-cocycle with values in Φ'. However, as a cocycle with values in H, it may be non-trivial (see below). The corresponding morphism is also then non-trivial in the sense that it is not conjugate to any morphism into the subgroup $U(H)$.

19.6

We shall carry out the construction of the intermediate series of fundamental representations on the example of the pair ($\mathbb{R}$), i.e.

$$G=U(\infty),\ K=SO(\infty).$$

Let us put

$$H(n)=\{\xi\in\mathbb{C}^{n,n}: \xi=\xi'\},\ M(n)=U(n).$$

The group $M(n)$ acts in $H(n)$ as follows:

$$u\cdot\xi=\bar{u}\xi\bar{u}', \text{ where } u\in M(n)=U(n),\ \xi\in H(n).$$

Let us introduce into $H(n)$ the scalar product

$$(\xi, \eta) = \operatorname{tr} \xi\eta^* = \operatorname{tr} \xi\bar{\eta}.$$

Let us denote by $L^0(n)$ the semi-direct product of the group Heis $(H(n))$ by $M(n)$. The group $L^0(n)$ is a subgroup in the group Heis $(H(n)) \cdot U(H(n))$. Let us denote by $V_r^{(n)}$ the restriction of the canonical representation V_r of this group (see section 19.2) to $L^0(n)$.

$V_r^{(n)}$ is an irreducible representation of the group $L^0(n)$ in the space $\mathscr{H}(H(n))$. From the definition of the action of the group $M(n) = U(n)$ in $H(n)$, it is seen that $V_r^{(n)} | U(n)$ decomposes into irreducible representations with non-negative highest weights. We call here the weight $(\lambda_1, \ldots, \lambda_n)$ non-negative if $\lambda_n \geqslant 0$.

We shall give here an explicit realization of the representation $V_r^{(n)}$:

$$V_r^{(n)}(x, \xi) f(z) = \exp(ir\,x - r\|\xi\|^2/2 - \sqrt{r}\,(z, \xi))\, f(z + \sqrt{r}\xi),$$

$$V_r^{(n)}(u) f(z) = f(u'zu).$$

Here $f \in \mathscr{H}(H(n))$, $x \in \mathbb{R}$, $z, \xi \in H(n)$, $u \in U(n)$.

19.7

Let us put

$$L^0 = \bigcup_{n=1}^{\infty} L^0(n).$$

The group L^0 is a subgroup of the group Heis $(H) \cdot U(H)$, where H denotes the completion of the space

$$H(\infty) = \bigcup_{n=1}^{\infty} H(n):$$

$$H = \{\xi \in \mathbb{C}^{\infty,\infty} : \xi = \xi', \operatorname{tr} \xi\xi^* = \operatorname{tr} \xi\bar{\xi} < +\infty\}.$$

The restriction of the standard representation V_r of the group Heis $(H) \cdot U(H)$ to L^0 is again denoted by V_r. This representation is the inductive limit of the representations $V_r^{(n)}$. It is irreducible.

The subgroup

$$M=\bigcup_{n=1}^{\infty} M(n)$$

in $\bar{L}^0$ is isomorphic to $U(\infty)$ and $V_r|M$ is a tame holomorphic representation of it.

The representation V_r of the group L^0 acts in $\mathscr{H}(H)$ precisely as described in section 19.6; it need only be assumed that ξ is a finite matrix and that f is a cylindrical function.

19.8

We shall now construct the embeddings

$$\mu_1, \mu_{-1}: (U(\infty), SO(\infty))\to(L^0, M),$$

using the procedure given in 19.4.

The space H is defined in section 19.7. We shall take $H(\infty)$ (see section 19.7) as Φ. Then Φ' may be identified with

$$\{\xi\in\mathbb{C}^{\infty,\infty}: \xi=\xi'\}.$$

Let us take as $\eta\in\Phi'$ the vector 1_∞. Then for $g\in U(\infty)$ we have:

$$u(g)=g,$$

$$\xi(g)=\hat{g}\cdot 1_\infty\cdot\bar{g}'-1_\infty=(g')^{-1}g^{-1}-1_\infty,$$

$$x(g)=\operatorname{Im}\operatorname{tr}(gg'-1).$$

Let us denote the corresponding embedding $U(\infty)\to L^0$ by μ_1. It is evident that

$$\mu_1(G(n))\subset L^0(n),\ \mu_1(K(n)\subset M(n),\ \mu_1(K)\subset M.$$

Further, we put

$$\mu_{-1}(g)=\mu_1(\bar{g})\ (g\in G=U(\infty)).$$

It is clear that $\mu_{\pm 1}|K$ coincides with the canonical embedding

$$K = SO(\infty) \to U(\infty) = M \subset L^0.$$

19.9

For $r > 0$ we define the intermediate fundamental representations T_r^0, T_{-r}^0 of the pair $(U(\infty), SO(\infty))$ as $V_r \circ \mu_{\pm 1}$. We shall give explicit formulae for them.

$$T_r^0(g) f(z) = \exp[r \operatorname{tr}(gg' - 1) - \sqrt{r} \operatorname{tr}(z(gg' - 1))]$$

$$\times f(g'zg - \sqrt{r}(g'g - 1)),$$

$$T_{-r}^0(g) = T_r^0(\bar{g}).$$

Here $g \in U(\infty)$, $z = z' \in \mathbb{C}^{\infty,\infty}$, f is a cylindrical function.

The representations T_r^0, T_{-r}^0 are conjugate to one another.

19.10

The following fact will be used in §**20**.

Lemma. Let $r_1 > 0$, $r_2 > 0$. Then

$$T_{r_1}^0 \otimes T_{r_2}^0 \sim T_{r_1 + r_2}^0 \otimes T,$$

where T is a certain tame representation of the group $U(\infty)$.

Proof. $T_{r_1}^0 \otimes T_{r_2}^0$ acts in the space $\mathscr{H}(H) \otimes \mathscr{H}(H) = \mathscr{H}(H \oplus H)$ as follows:

$$(T_{r_1}^0 \otimes T_{r_2}^0)(g) f(z_1, z_2) =$$

$$= \exp[(r_1 + r_2)\operatorname{tr}(gg' - 1) - \operatorname{tr}((\sqrt{r_1} z_1 + \sqrt{r_2} z_2)(gg' - 1))]$$

$$\times f(g' z_1 g - \sqrt{r_1}(g'g - 1), g' z_2 g - \sqrt{r_2}(g'g - 1)).$$

We consider here f as a cylindrical function. Let us replace the variables

$$\frac{\sqrt{r_1}}{\sqrt{r_1+r_2}} z_1 + \frac{\sqrt{r_2}}{\sqrt{r_1+r_2}} = z$$

$$\frac{\sqrt{r_2}}{\sqrt{r_1+r_2}} z_1 - \frac{\sqrt{r_1}}{\sqrt{r_1+r_2}} = \zeta.$$

We observe that this transformation preserves the Gaussian measure on $\Phi' \times \Phi'$. In the new variables our representation appears thus:

$$(T_{r_1}^{0} \otimes T_{r_2}^{0})(g) f(z, \xi) = \exp[(r_1+r_2)\operatorname{tr}(gg'-1) - \sqrt{r_1+r_2}\operatorname{tr}(z(gg'-1))]$$

$$\times f(g'zg - \sqrt{r_1+r_2}(g'g-1), g'\zeta g).$$

Now our proposition becomes evident: it is necessary to take for T the canonical representation of the group $U(H)$ in $\mathscr{H}(H)$, restricted to $U(\infty)$.

19.11

Remark. The group $L^0(n)$ is very similar to the groups $L^+(n) = Sp(n, \mathbb{R})^\sim$ and $L^-(n) = Sp(n)$.

In fact, let $\mathfrak{l}^0(n)$ be the Lie algebra of the group $L^0(n)$ and $\mathfrak{l}_C^0(n)$ its complexification. The algebra $\mathfrak{l}_C^0(n)$ possesses the decomposition

$$\mathfrak{l}_C^0(n) = \mathfrak{l}_{-1}^0(n) \oplus \mathfrak{l}_0^0(n) \oplus \mathfrak{l}_1^0(n)$$

of the same sort as the algebra $\mathfrak{l}_C(n) = sp(n, \mathbb{C})$. Here $\mathfrak{l}_0^0(n)$ is the complexification of subalgebra

$$\mathbb{R} \oplus u(n) \subset \mathfrak{l}^0(n)$$

and $\mathfrak{l}_{-1}^0(n) \oplus \mathfrak{l}_1^0(n)$ is the complexification of the subspace $H(n) \subset \mathfrak{l}^0(n)$.

The analogy becomes more complete if we replace $\mathfrak{l}^0(n)$ by $[\mathfrak{l}^0(n), \mathfrak{l}^0(n)]$: this leads to the fact that $\mathbb{R} \oplus u(n)$ *is* replaced by $\mathbb{R} \oplus su(n)$.

We observe that the Lie algebra $[\mathfrak{l}_C^0(n), \mathfrak{l}_C^0(n)]$ may be obtained from $\mathfrak{l}_C(n)$ by a process of "contraction" in the sense of Inönü and Wigner.

Finally, we observe that the triality

$$\{L^+(n), L^-(n), L^0(n)\}$$

is fully analogous to the triality in the theory of symmetric spaces (spaces of negative curvature, positive curvature and zero curvature).

19.12

The construction given in this paragraph can be transferred to all pairs (G, K) of compact type. In all cases the cocycle $\xi(\cdot)$ is constructed with the help of an involution that distinguishes the subgroup K from G.

Let us consider the pair ($\mathbb{C}$). In this case

$$G = U(\infty) \times U(\infty),\ K = U(\infty),\ G(n) = U(n) \times U(n),\ K(n) = U(n),$$

$$H(n) = \mathbb{C}^{n,n},\ M(n) = U(n) \times U(n),\ L^0(n) = \mathrm{Heis}(H(n)) \cdot M(n).$$

The action of $M(n)$ on $H(n)$ has the following form:

$$(u_1, u_2)\colon \xi \mapsto \bar{u}_1 \xi u_2^{-1} ((u_1, u_2) \in M(n),\ \xi \in H(n)).$$

The morphism $u(\cdot)$ from $G(n)$ into $M(n)$ has the form

$$(g_1, g_2) \mapsto (\bar{g}_1, g_2).$$

As η, let us again take the matrix $1_\infty \in \mathbb{C}^{\infty,\infty}$.

Let us consider further the pair ($\mathbb{H}$). In this case

$$G = U(2\infty),\ K = Sp(\infty),\ G(n) = U(2n),\ K(n) = Sp(n),$$

$$H(n) = \{\xi \in \mathbb{C}^{2n,2n}\colon \xi = -\xi'\},\ M(n) = U(2n),$$

$$L^0(n) = \mathrm{Heis}(H(n)), \cdot M(n).$$

$M(n)$ acts on $H(n)$ as follows:

$$u\colon \xi \mapsto \bar{u} \xi \bar{u}' (u \in M(n),\ \xi \in H(n)).$$

The morphism $u(\cdot)$ from $G(n)$ into $M(n)$ is the identical mapping. As the vector η, we take the matrix

$$\begin{bmatrix} 0 & 1_\infty \\ -1_\infty & 0 \end{bmatrix}.$$

For pairs (G, K) of type (F_i), embeddings into pairs of type (F) are again used in the construction of the intermediate fundamental representations.

§20. Construction of irreducible admissible representations

20.1

In this section we shall describe the structure of the ring generated by the fundamental representations which were constructed in §§**18** and **19**. For the pairs (G, K) of compact type the definition of admissible representations given in section 9.5 remains in force. All fundamental representations are admissible. Hence all the irreducible representations obtained as a result of decomposition of their tensor products will also be admissible representations.

Let us recall that we provide the groups L, $L(n)$, $U(k, F)$ with the signs "+" or "−" (see section 18.1).

We shall consider in detail only the pair $(\mathbb{R})$.

20.2

Let $k = 1, 2, \ldots$. Let us recall that $U^+(k, \mathbb{R}) = O(k)$. Let $\pi \in O(k)^\wedge$ and W_π be the corresponding irreducible representation of the group $L^+ = Sp(\infty, \mathbb{R})^\sim$. Let

$$\tau_s \colon U(\infty)^\sim \to Sp(\infty, \mathbb{R})^\sim \ (s \in \mathbb{R} \setminus \{0\})$$

be the embedding constructed in section 18.3. Let us put

$$T^+_{\pi, s}(g) = W_\pi(\tau_s(g))(\det g)^{-1/2\, k\, \mathrm{sgn}(s)} \ (g \in U(\infty)).$$

This is a correctly defined admissible representation of the pair $(U(\infty), SO(\infty))$.

THEOREM. The representation $T^+_{\pi,s}$ of the group $U(\infty)$ is irreducible for all $k=1, 2, \ldots, \pi \in O(k)^\wedge$, $s \in \mathbb{R} \backslash \{0\}$. Moreover, its restriction to $SU(\infty)$ is irreducible. Thus,

$$(T^+_s)^{\otimes k} \sim \bigoplus_{\pi \in O(k)^\wedge} (\dim \pi) \cdot T^+_{\pi,s}.$$

The proof is fully analogous to the proof of theorem 11.1. Consider the tame representation ρ of the group $K = SO(\infty)$:

$$\rho = (T^+_s)^{\otimes k} | SO(\infty) = (W_{\mathbb{R}}{}^{\otimes k} \circ \tau_s) | SO(\infty)$$

and its holomorphic extension ρ^* to the group $K^* = U(\infty)$. The representation ρ^* coincides, to within a one-dimensional factor, with the representation

$$W_{\mathbb{R}}{}^{\otimes k} | M$$

(let us recall that the subgroup $M \subset Sp(\infty, \mathbb{R})^\sim$ is $U(\infty)^\sim$).

Now we check that, for any $n \geqslant 3$, the groups

$$\tau_s(SU(n)) \quad \text{and} \quad [M(n), M(n)] = SU(n)$$

jointly generate $Sp(n, \mathbb{R})^\sim$.

20.3

Let $l = 1, 2, \ldots$. Let us recall that $U^-(l, \mathbb{R}) = Sp(l)$. Let $\sigma \in Sp(l)^\wedge$ and S_σ be the corresponding irreducible representation of the group $L^- = Sp(\infty)$.

Let

$$\nu_t \colon U(\infty) \to Sp(\infty) \; (0 < t < \pi)$$

be the embedding constructed in section 18.9. Let us put

$$T^-_{\sigma,t}(g) = S_\sigma(\nu_t(g)(\det g)^{l\,\mathrm{sgn}((\pi/2)-t)} \; (g \in U(\infty)).$$

THEOREM. The representation $T^-_{\sigma,t}$ of the group $U(\infty)$ is irreducible for all $l=1, 2, \ldots, \sigma\in Sp(l)^\wedge$ and $t\in(0, \pi)$. Moreover, its restriction to $SU(\infty)$ is irreducible. Thus,

$$(T_t^-)^{\otimes l} \sim \bigoplus_{\sigma\in Sp(l)^\wedge} (\dim \sigma)\cdot T^-_{\sigma,t}.$$

The proof is the same as for theorem 20.2.

20.4

Let $r>0$; T_r^0 be the representation of the group $U(\infty)$ constructed in section 19.9; ρ an irreducible arbitrary tame holomorphic representation of the group $U(\infty)$.

THEOREM. The representation $T_r^0\otimes\rho$ of the group $U(\infty)$ is irreducible. Moreover, its restriction to $SU(\infty)$ is irreducible.

The proof again follows the same idea. We shall examine the group L^0 and its representation V_r (sections 19.6–19.7). Then

$$(T_r^0\otimes\rho)(g)=(V_r\otimes\rho)(\mu_1(g)), \quad g\in U(\infty)$$

(here ρ is considered simultaneously as the representation of the group L^0 that is trivial on the Heisenberg subgroup).

We show first that the representation $V_r\otimes\rho$ is irreducible. Let

$$A\in(V\otimes\rho)(L^0)'.$$

Let us represent $H(V_r\otimes\rho)$ as $H(V_r)\otimes H(\rho)$. Since V_r is an irreducible representation of the Heisenberg subgroup, we have

$$A=1\otimes B,$$

where B is a certain operator in $H(\rho)$. But B must commute with the representation ρ, from which it follows that B is scalar. This means that $V_r\otimes\rho$ is irreducible. We observe that in this discussion it is possible to replace L^0 by $[L^0, L^0]$ (also, the subgroup $M=U(\infty)\subset L^0$ is replaced by $[M, M]=SU(\infty)$).

We now observe that the representation $(V_r\otimes\rho)|M$ of the subgroup $M=U(\infty)$ is holomorphic (this follows from the holomorphy of the representation ρ). Thus this representation coincides with the holomorphic extension of the representation $(T_r^0 \otimes \rho)|SO(\infty)$.

It now remains to prove that the subgroups $\mu_1(SU(n))$ and $[M(n), M(n)] = SU(n)$ generate the whole group $[L^0(n), L^0(n)] = \mathrm{Heis}(H(n)) \cdot SU(n)$, but this does not cause any difficulties.

20.5

Remark. The irreducible representations of the group L^0 of the type $V_r \otimes \rho$, where $\rho = \rho_\lambda$, play the same role in our construction as the irreducible representations W_π and S_σ of the groups L^+ and L^- respectively.

We observe that $V_r \otimes \rho$ is the inductive limit of the irreducible representations $V_r^{(n)} \otimes \rho^{(n)}$ of the groups $L^0(n)$, where $\rho^{(n)}$ denotes the irreducible representation of the group $U(n)$ with the highest weight $(\lambda_1, \ldots, \lambda_n)$. We observe further that $V_r^{(n)} \otimes \rho^{(n)}$ remains irreducible after restriction to the subgroup

$$[L^0(n), L^0(n)] = \mathrm{Heis}(H(n)) \cdot SU(n).$$

20.6

THEOREM. The following representations of the group $U(\infty)$ are pairwise conjugate to one another:

$$T^+_{\pi,s} \text{ and } T^+_{\pi,-s} (\pi \in O(k)^\wedge,\ s \in \mathbb{R} \backslash \{0\}),$$

$$T^-_{\sigma,t} \text{ and } T^-_{\sigma,\pi-t} (\sigma \in Sp(l)^\wedge,\ t \in (0, \pi)),$$

$$T^0_r \otimes \rho_\lambda \text{ and } T^0_{-r} \otimes \bar{\rho}_\lambda (r > 0,\ \lambda \in \Lambda).$$

The proof is analogous to the proof of theorem 11.6 given in section 11.20.

20.7

Let

$$p, q = 1, 2, \ldots;\ k_1, \ldots, k_p,\ l_1, \ldots, l_q = 1, 2, \ldots,$$

$$\pi_i \in O(k_i)^\wedge \quad (i=1, \ldots, p); \quad \sigma_j \in Sp(l_j)^\wedge \quad (j=1, \ldots, q),$$

$$s_1, \ldots, s_p \in \mathbb{R}\backslash\{0\};\ t_1, \ldots, t_q \in (0, \pi);\ r_1 > 0,\ r_2 > 0.$$

Let $\rho = \rho_{\lambda,\mu}$ be an irreducible tame representation of the group $U(\infty)$; let us recall that $\rho = \rho_\lambda \otimes \bar{\rho}_\mu$.

We shall assume that the numbers $s_1, \ldots, s_p$ are pairwise distinct and that $t_1, \ldots, t_q$ are also pairwise distinct.

Let us put

$$T = \left(\bigotimes_{i=1}^{p} T^{+}_{\pi_i, s_i} \right) \otimes \left(\bigotimes_{j=1}^{q} T^{-}_{\sigma_j, t_j} \right) \otimes T^{0}_{r_1} \otimes T^{0}_{-r_2} \otimes \rho. \tag{1}$$

It is evident that T is an admissible representation of the pair $(U(\infty), SO(\infty))$.

20.8

THEOREM. Any representation T from section 20.7 is irreducible. Moreover, its restriction to $SU(\infty)$ is irreducible (of course, the theorem continues to hold if some of the factors are omitted in (1)).

The proof differs only slightly from the proof of theorem 11.3. However, we shall give certain details so as to demonstrate how well the boson, fermion and intermediate representations supplement one another.

Let us put

$$A = (L^+)^p \times (L^-)^q \times [L^0, L^0]^2,$$

$$\tau = \tau_{s_1} \times \ldots \times \tau_{s_p} \times \nu_{t_1} \times \ldots \times \nu_{t_q} \times \mu_1 \times \mu_{-1} : SU(\infty) \to A,$$

$d: SU(\infty) \to A$ be the diagonal embedding.

Here the upper index p, q or 2 indicates the product of p, q or 2 copies of the corresponding group.

We observe that T is equivalent to the restriction to $\tau(SU(\infty))$ of the irreducible representation

$$\left(\bigotimes_{i=1}^{p} W_{\pi_i} \right) \otimes \left(\bigotimes_{j=1}^{q} S_{\sigma_j} \right) \otimes (V_{r_1} \otimes \rho_\lambda) \otimes (V_{r_2} \otimes \rho_\mu)$$

of the group A. Our usual argument shows that it is sufficient to check that the subgroups $\tau(SU(n))$ and $d(SU(n))$ generate the whole group

$$A(n) = L^+(n)^p \times L^-(n)^q \times [L^0(n), L^0(n)]^2$$

$$= Sp(n, \mathbb{R})^p \times Sp(n)^q \times (\mathrm{Heis}(H(n)) \cdot SU(n))^2$$

for all $n \geqslant 3$.

We shall study the complexified Lie algebra of the group $A(n)$ which is the direct sum of $p+q$ copies of the algebra $\mathfrak{l}_{\mathbb{C}}(n) = sp(n, \mathbb{C})$ and two copies of the algebra $[\mathfrak{l}_{\mathbb{C}}{}^0(n), \mathfrak{l}_{\mathbb{C}}{}^0(n)]$. Let $\mathfrak{b}(n)$ be the smallest complex subalgebra containing $\tau(su(n))$ and $d(su(n))$; we have to prove that $\mathfrak{b}(n) = \mathfrak{a}(n)$.

We cannot immediately use lemma 11.15 for the algebra $\mathfrak{a}(n)$ as it is not semisimple. Hence we slightly change the argument.

Let us examine the decomposition

$$\mathfrak{a}(n) = \mathfrak{a}_{-1}(n) \oplus \mathfrak{a}_0(n) \oplus \mathfrak{a}_1(n),$$

which originates from the decompositions of the algebras $sp(n, \mathbb{C})$ and $[\mathfrak{l}_{\mathbb{C}}{}^0(n), \mathfrak{l}_{\mathbb{C}}{}^0(n)$ (see section 19.11). As usual, let $\mathscr{H}_{-1}$, $\mathscr{H}_0$ and $\mathscr{H}_1$ be the corresponding projections.

The $d(su(n))$-modules $\mathfrak{a}_{-1}(n), \mathfrak{a}_0(n), \mathfrak{a}_1(n)$ are pairwise disjunct. Hence $b(n)$ is stable with respect to the projections $\mathscr{H}_{-1}, \mathscr{H}_0, \mathscr{H}_1$.

We now show that the subalgebras

$$\mathscr{H}_0(\tau(su(n))) \text{ and } d(su(n))$$

generate $[\mathfrak{a}_0(n), \mathfrak{a}_0(n)]$, from which the equality $b(n) = \mathfrak{a}(n)$ that we used easily follows.

The algebra $[\mathfrak{a}_0(n), \mathfrak{a}_0(n)]$ is semisimple: it is the direct sum of $p+q+2$ copies of the algebra $sl(n, \mathbb{C})$. Hence we may use lemma 11.15 for it.

We shall write an explicit form for the element

$$(\mathscr{H}_0 \circ \tau)(A) \in [\mathfrak{a}_0(n), \mathfrak{a}_0(n)], \text{ where } A \in su(n).$$

It will be sufficient for us to take $A = Y = Y'$. We have

$$(\mathcal{H}_0 \circ \tau)(Y) = (a_1 Y) \oplus \ldots \oplus (a_p Y) \oplus (b_1 Y) \oplus \ldots \oplus$$

$$(b_q Y) \oplus (C_1 Y) \oplus (C_2 Y) \in sl(n, \mathbb{C}) \oplus \ldots \oplus sl(n, \mathbb{C}) (p+q+2 \text{ times}),$$

where

$$a_i = \begin{cases} \cosh s_i, & \text{if} \quad s_i > 0, \\ -\cosh s_i, & \text{if} \quad s_i < 0; \end{cases}$$

$$b_j = \cos t_j;\ C_1 = 1,\ C_2 = -1.$$

We observe that all these numbers are pairwise distinct, since

$$|a_i| > 1,\ |b_j| < 1,\ a_{i_1} \neq a_{i_2},\ b_{j_1} \neq b_{j_2}.$$

Hence we may conclude the proof just as in section 11.17.

20.9

THEOREM. For the irreducible representations from section 20.7, the natural analogue of theorem 11.5 on pairwise non-equivalence is true.

Proof is precisely the same as in §**12**.

20.10

Remark. The totality of the irreducible admissible representations of the pair $(U(\infty), SO(\infty))$ constructed by us is closed with respect to the operations of conjugation and taking of tensor products (with subsequent decomposition into irreducible components). This quickly follows from our construction (see also lemma 19.10).

20.11

Remark. The restriction of any irreducible representation from section 20.7 to the subgroup $K = SO(\infty)$ has a finitely-multiple spectrum. In §**21**, we shall see that there exist irreducible representations for which this is not so.

20.12

It is convenient to unite the three families of embeddings $\{\tau_s\}$, $\{\nu_t\}$, $\{\mu_{\pm 1}\}$, where $s \in \mathbb{R} \setminus \{0\}$, $t \in (0, \pi)$, into one family $\{\tau_x\}$ where $x \in \mathbb{R}$. Let us write for this

$$x^+(s) = \operatorname{sgn}(s) \cosh s, \quad x^-(t) = \cos t, \quad x^0(\pm 1) = \pm 1.$$

Then

$$\{x^+(s)\} \cup \{x^-(t)\} \cup \{x^0(\pm 1)\} = \mathbb{R}.$$

For $x \neq \pm 1$, we put $a(x) = \sqrt{1 + x/2}$, $b(x) = \sqrt{1 - x/2}$, where for $|x| > 1$ the branch of the root is selected arbitrarily. Let

$$\tau_x(g) = \begin{bmatrix} a(x)\cdot 1 & b(x)\cdot 1 \\ -b(x)\cdot 1 & a(x)\cdot 1 \end{bmatrix} \begin{bmatrix} \bar{g} & 0 \\ 0 & g \end{bmatrix} \begin{bmatrix} a(x)\cdot 1 & -b(x)\cdot 1 \\ b(x)\cdot 1 & a(x)\cdot 1 \end{bmatrix}$$

$$= \begin{bmatrix} \left(\dfrac{1+x}{2}\right)\bar{g} + \left(\dfrac{1-x}{2}\right)g & \dfrac{1}{2}\sqrt{1-x^2}\,(g-\bar{g}) \\ \dfrac{1}{2}\sqrt{1-x^2}\,(g-\bar{g}) & \left(\dfrac{1+x}{2}\right)g + \left(\dfrac{1-x}{2}\right)\bar{g} \end{bmatrix}, \quad g \in U(\infty).$$

If $|x| < 1$, then $\tau_x(g) \in L^- = Sp(\infty)$ while $\tau_x = \nu_t$ where $x = x^-(t)$. If $|x| > 1$, then $\tau_x(g) \in Sp(\infty, \mathbb{R})$. Considering g as an element of the group $U(\infty)^\sim$ as in section 18.3, we may lift $\tau_x(g)$ into $L^+ = Sp(\infty, \mathbb{R})^\sim$.

For $|x| > 1$, our new mapping τ_x differs from the old mapping τ_s where $x = x^+(s)$ by an automorphism of the group L^+ that does not change the Weil representation.

Finally, at the points $x = \pm 1$, we put $\tau_{\pm 1} = \mu_{\pm 1}$.

20.13

Let us denote by $G(n)$ the group of (all) functions $f(x)$ taking values in the groups $L^+(n)$, $L^-(n)$ and $L^0(n)$ for $|x| > 1$, $|x| < 1$ and $x = \pm 1$ respectively. Let $G^* = \cup\, G(n)$.

Let us define the embedding τ of the group $G = U(\infty)$ (or, more precisely, of the group $U(\infty)^{-}$) into the group G^* by putting $\tau(g)(x) = \tau_x(g)$, where τ_x was defined in section 20.12. Let us observe that τ maps the subgroup $K = SO(\infty)$ into the subgroup of constant functions.

The group G^* possesses a family of irreducible representations of the following type:

$$U_{x_1,\ldots,x_r}(f) = \bigotimes_{m=1}^{r} U_{x_m}(f(x_m)),\ x_1 < \ldots < x_r,\ f \in G^*.$$

Here U_{x_m} has the form $W_\pi, S_\sigma, V_r \otimes \rho_\lambda$ respectively for $|x_m| > 1$, $|x_m| < 1$, $x_m = \pm 1$, where π, σ, r, λ depend on m. The representations $U_{x_1,\ldots,x_r}$ will be called *holomorphic*.

It is clear that the irreducible representations T of the pair $(U(\infty), SO(\infty))$ constructed in section 20.7 are precisely the representations of the type $\chi \otimes (U \circ \tau)$ where U is an irreducible holomorphic representation of the group G^* and χ the corresponding "gauge multiplier".

The given construction is an analogue of the construction from section 12.5.

20.14

Consider the Lie algebra $\mathfrak{g}^*(n)$ of (all) functions on $\mathbb{R}\setminus\{\pm 1\}$ taking values in the Lie algebra $l^+(n) = sp(n, \mathbb{R})$ for $|x| > 1$ and in the Lie algebra $l^-(n) = sp(n)$ for $|x| < 1$. We shall define the embedding τ of the Lie algebra $\mathfrak{g}(n) = u(n)$ into the Lie algebra $g^*(n)$ as the differential at the identity of the embedding of the groups τ: $G(n) \to G^*(n)$.

THEOREM (compare with theorem 12.6). For $n \geq 3$, the subalgebra in $\mathfrak{g}^*(n)$ generated by the algebra $\tau(su(n))$ and the algebra $d(su(n))$ of constant $su(n)$-valued functions, consists of all functions of the type

$$x \mapsto \begin{bmatrix} -i(1-x^2)\alpha(x)\cdot 1_n + \overline{A(x)} & \sqrt{1-x^2}\,B(x) \\ -\sqrt{1-x^2}\,\overline{B(x)} & i(1-x^2)\alpha(x)\cdot 1_n + A(x) \end{bmatrix},$$

where

$$\alpha(\cdot) \in \mathbb{R}[x],\ A(\cdot) \in \mathbb{R}[x] \otimes su(n),\ B(\cdot) \equiv B(\cdot)^t \in \mathbb{R}[x] \otimes \mathbb{C}^{n,n}.$$

The proof is obtained by some modification of the arguments from section 20.8.

20.15

Remark. The matrix functions described in theorem 20.14, being extended to the whole real line in an obvious manner, take, at the points $x=\pm 1$, values from the subalgebra $su(n)$. We observe, however, that the triplet of functions $\{\alpha(x), B(x), A(x)\}$ gives at $x=\pm 1$ a morphism of our Lie algebra into $[l^0(n), l^0(n)]$. This effect agrees with remark 19.11. It agrees also with the above definition of the group $G^*(n)$, where we have assumed that, at the points $x=\pm 1$, our functions take values in the group $L^0(n)$.

20.16

Remark. The results of this section transfer to all other (G, K)-pairs of compact type. For the pairs $(\mathbb{C})$ and $(\mathbb{H})$, nothing changes essentially. For pairs (F_i), just as in the non-compact case, two additional effects arise.

Let (G, K) be one of the seven pairs (F_i) and (G', K') the corresponding pair (F). By definition, the fundamental representations of the pair (G', K') are restrictions of fundamental representations of the pair (G', K').

The first effect consists of the fact that $T|G \sim \bar{T}|G$ for any fundamental representation T of the pair (G', K'), which is clearly seen at the level of spherical functions (compare theorem 16.5).

The second effect consists of the fact that if T is the unique self-conjugate fermion fundamental representation of the group G', then the symmetry group of the representation $T^{\otimes k}|$ is wider than the symmetry group $U^-(k, F)$ of the representation $T^{\otimes k}$. Here the analogue of theorem 16.8 holds. The corresponding group $L=L^-$ is a "compact form" of the group $L=L^+$ from Table 7.6 (see section 16.7).

For the pairs $(\mathbb{R}_2)$, $(\mathbb{C}_3)$ and $(\mathbb{H}_2)$ it is possible to repeat remark 16.11 after replacing W_t by S_t.

20.17

In conclusion, we shall prove one result mentioned in the Introduction.

THEOREM. Let (G^1, K^1) and (G^2, K^2) be two different pairs of compact type for which G^1 and G^2 are isomorphic (see Table 7.2). If the unitary representation T of the group $G = G^1 = G^2$ is admissible for both the pairs, then it is a tame representation of the group G (the converse is obvious).

PROOF. Changing, if necessary, the numbering of the subgroups, it may be assumed that $G^1(n) = G^2(n) = G(n)$. Let us denote by G_n, K_n^1 and K_n^2 the subgroups "complementary" to $G(n)$, $K^1(n)$ and $K^2(n)$ respectively (see section 2.4). Let H_n, H_n^1 and H_n^2 be subspaces of vectors in $H = H(T)$ which are invariant with respect to G_n, K_n^1 and K_n^2 respectively.

By agreement,

$$\bigcup_n H_n^1 \quad \text{and} \quad \bigcup_n H_n^2$$

are dense in H and we have to prove that

$$\bigcup_n H_n$$

is dense in H. The key observation is that $K^1K^2K^1 = G$ and similarly $K_n^1K_n^2K_n^1 = G_n$ for all n.

Let $\xi \in H$ and $\varepsilon > 0$ be arbitrary. If n is sufficiently large, then $\|T(g)\xi - \xi\| \leqslant \varepsilon$ for all $g \in G_n$. In fact, for large n it is possible to find vectors in H_n^1 and in H_n^2 close to ξ. But then all the elements from K_n^1 and from K_n^2 only slightly displace ξ; this means that the same is true also for $G_n = K_n^1K_n^2K_n^1$.

Let A be the weak closure of the convex hull of the G_n-orbit of the point ξ. The function $\eta \mapsto \|\eta\|$ is weakly lower semicontinuous on A and hence achieves its minimum at a certain point $\eta_0 \in A$. If $T(g)\eta_0 \neq \eta_0$ for a certain $g \in G_n$, then $\|\eta_1\| < \|\eta_0\|$ for $\eta_1 = (\eta_0 + T(g)\eta_0)/2$ and that is impossible because $\eta_1 \in A$. Thus, $T(g)\eta_0 = \eta_0$ for all $g \in G_n$, i.e., $\eta_0 \in H_n$. But A lies in the sphere of radius ε with its centre at ξ, and that means that $\|\eta_0 - \xi\| \leqslant \varepsilon$.

We have proved that

$$\bigcup_n H_n$$

is dense in H, i.e., T is a tame representation.

§21. Spherical functions. Properties of continuity. Infinite tensor products

21.1

We observe that each of the fundamental representations of the pair $(U(\infty), SO(\infty))$ possesses a unique $SO(\infty)$–invariant vector, to within a factor.

THEOREM (Compare with Theorem 13.3). Normalized spherical functions φ_s^+ φ_t^- $\varphi_{\pm r}^0$ of the fundamental representations T_s^+, T_t^-, $T_{\pm r}^0$, where $s \in \mathbb{R}\setminus\{0\}$, $t \in (0, \pi)$, $r > 0$, are given by the following formulae (below $g \in U(\infty)$):

$$\varphi_s^+(g) = \det\left(\cosh^2\left(\frac{s}{2}\right)\cdot 1 - \sinh^2\left(\frac{s}{2}\right)\cdot gg'\right)^{-1/2}$$

$$\text{and } \varphi_{-s}^+(g) = \varphi_s^+(\bar{g}) = \overline{\varphi_s^+(g)} \quad \text{for } s > 0;$$

$$\varphi_t^-(g) = \det\left(\cos^2\left(\frac{t}{2}\right)\cdot 1 + \sin^2\left(\frac{t}{2}\right)\cdot gg'\right)$$

$$\text{and } \varphi_{\pi-t}^-(g) = \varphi_t^-(\bar{g}) = \overline{\varphi_t^-(g)} \text{ for } t \in \left(0, \frac{\pi}{2}\right);$$

$$\varphi_{\pi/2}^-(g) = \det\left(\frac{g+\bar{g}}{2}\right) = \det\left(1 + \frac{1}{2}(gg' - 1)\right)\det \bar{g}$$

$$= \det\left(1 + \frac{1}{2}\overline{(gg' - 1)}\right)\det g;$$

$$\varphi_r^0(g) = \exp(r\,\mathrm{tr}(gg' - 1)) \text{ and } \varphi_{-r}^0(g) = \varphi_r^0(\bar{g})$$

$$= \overline{\varphi_r^0(g)} \text{ for } r > 0.$$

The proof follows the second method of proof of theorem 13.3: it is based on the explicit formulae for spherical functions of the representations $W_{\mathbb{R}}$, $S_{\mathbb{R}}$, V_r (see sections 8.1, 17.6, 19.16).

21.2

THEOREM (compare theorem 13.4 and lemma 13.2). The representation T of the pair $(U(\infty), SO(\infty))$, given by formula (1) from section 20.7, possesses a non-zero $SO(\infty)$–invariant vector if and only if all representations π_i, σ_j, ρ are trivial representations of the corresponding groups. If this condition is satisfied, then the normalized spherical function of the representation T is equal to

$$\prod_{i=1}^{p} (\varphi^{+}_{s_i})^{k_i} \prod_{j=1}^{q} (\varphi^{-}_{t_j})^{l_j} \cdot \varphi^{0}_{r_1}\, \varphi^{0}_{-r_2}.$$

The proof is obvious.

21.3

Consider the group $\bar{U}(\infty)$ of all unitary operators in the Hilbert space $l^2(\mathbb{C})$ and distinguish in it the subset $U_2(\infty) = \{g \in \bar{U}(\infty)$: $gg' - 1$ is a Hilbert–Schmidt operator$\}$. (Do not confuse this with the notation in section 2.4!) It is easy to check that $U_2(\infty)$ is a subgroup. It contains the complete orthogonal group $\overline{SO}(\infty)$. Let us recall that we topologize $\overline{SO}(\infty)$ by the weak = strong operator topology.

LEMMA (COMPARE WITH LEMMA 13.7). $U_2(\infty)$ possesses the structure of a topological group for which the fundamental system of neighbourhoods of the identity is formed by the sets of the type AB_ε, where A is an arbitrary neighbourhood of the identity in the topological group $\overline{SO}(\infty)$, and

$$B_\varepsilon = \{a \in \bar{U}(\infty)\colon a = a',\ \|a - 1\|_2 < \varepsilon\},$$

where $\|\cdot\|_2$ indicates the Hilbert–Schmidt norm.

The simple proof is omitted.

21.4

Consider on the group $U_2(\infty)$ the function

$$\psi(g) = \det\left(h \exp\left(\frac{1}{2}(h^{-1} - h)\right)\right), \text{ where } g \in U_2(\infty),\ h = gg'.$$

It is correctly defined, since under the sign of the determinant stands an operator of the type $1+$ (nuclear operator). The function ψ is continuous on $U_2(\infty)$ and maps it into the circle $|z|=1$.

It is easy to check that the degree of the mapping

$$z \mapsto \psi\left(\begin{bmatrix} z & 0 \\ 0 & 1_\infty \end{bmatrix}\right)$$

is equal to 1. This shows that $U_2(\infty)$ is not simply connected. Let us denote by $\tilde{U}_2(\infty)$ the minimal covering over $U_2(\infty)$ on which the function $\ln \psi$ becomes single-valued. The fibre of this covering is the group $\mathbb{Z}$. $\tilde{U}_2(\infty)$ is a topological group containing $\overline{SO}(\infty)$. It is possible to show that $\tilde{U}_2(\infty)$ is the universal covering for $U_2(\infty)$.

The fundamental principle of treating the group $\tilde{U}_2(\infty)$ consists of the fact that all calculations may be done in the matrix group $U_2(\infty)$; but in addition it is permitted to consider functions of the type $\psi(g)^a$, where $a \in \mathbb{R}$.

Let us define also the group $\tilde{U}(\infty)$ (do not confuse this with $U(\infty)^\sim$!) as the universal ($\mathbb{Z}-$) covering over $U(\infty)$; the function $\ln \det(gg')$ becomes single-valued. $\tilde{U}(\infty)$ is a dense subgroup in $\tilde{U}_2(\infty)$.

21.5

Theorem (Compare with Theorem 13.8). Let us regard the fundamental representations T_s^+, T_t^-, $T_{\pm r}^0$ of the pair $(U(\infty), SO(\infty))$ as representations of the group $\tilde{U}(\infty)$. Then they may be multiplied by appropriate "gauge multipliers" of the type $\det(gg')^a$, where $a \in \mathbb{R}$, as a result of which we get representations that admit a continuous extension to the group $\tilde{U}_2(\infty)$. By the same means, an analogous result holds also for all the representations constructed in section 20.7.

Proof. Let us use the notations of sections 20.12–20.13 and define modified fundamental representations of the pair $(\tilde{U}(\infty), SO(\infty))$ as follows:

$$\tilde{T}_x(g) = W_{\mathbb{R}}(\tau_x(g)) \det(gg')^{-x/4}, \quad x \in \mathbb{R},\ |x| > 1;$$

$$\tilde{T}_x(g) = S_{\mathbb{R}}(\tau_x(g)) \det(gg')^{x/2}, \quad x \in \mathbb{R},\ |x| < 1;$$

$\tilde{T}^{0}_{\pm r}(g) = V_r(\mu_{\pm 1}(g)) \det(gg')^{+r}, \; r>0.$

Their spherical functions are as follows:

$$\tilde{\varphi}_x(g) = \det\left(\left(\frac{1+x}{2}\right)\bar{g} + \left(\frac{1-x}{2}\right)g\right)^{-1/2} \det(gg')^{-x/4}, \; |x|>1;$$

$$\tilde{\varphi}_x(g) = \det\left(\left(\frac{1+x}{2}\right)\bar{g} + \left(\frac{1-x}{2}\right)g\right) \det(gg')^{x/2}, \; |x|<1;$$

$$\tilde{\varphi}^{0}_{\pm r}(g) = \exp(r\,\mathrm{tr}((gg')^{\pm 1} - 1)) \det(gg')^{\mp r}, \; r>0.$$

After simple transformations we get (below $h = gg'$):

$$\tilde{\varphi}_x(g) = \det\left[\left(1 + \frac{1-x}{2}(h-1)\right) \exp\left(\frac{1-x}{4}(h^{-1}-h)\right)\right]^{-1/2}$$

$$\times\, \psi(g)^{(1-x)/4}, \; |x|>1;$$

$$\tilde{\varphi}_x(g) = \det\left[\left(1 + \frac{1-x}{2}(h-1)\right) \exp\left(\frac{1-x}{4}(h^{-1}-h)\right)\right]$$

$$\times\, \psi(g)^{(x-1)/2}, \; |x|<1;$$

$$\tilde{\varphi}^{0}_{\pm r}(g) = \exp\left(r\,\mathrm{tr}\left(\frac{h^{-1}+h}{2} - 1\right)\right] \psi(g)^{\mp r}, \; r>0.$$

These expressions are already correctly defined on the group $\tilde{U}_2(\infty)$, that proves the assertion of the theorem in the particular case of spherical representations. In the general case, we use the same device as in sections 13.11 and 17.17.

21.6

We observe that the spherical functions of the representations T^+_s, T^-_t tend to unity as $s \to 0$, $t \to 0$ or $t \to \pi$. This fact allows one to generalize the construction of the representations T from section 20.7 as follows.

We shall examine a set of data of the same type as in section 20.7, with, however, the difference that now the indices i and j will run through a countable set. We shall assume that the representations $\pi_i \in O(k_i)^\wedge$ and $\sigma_j \in Sp(l_j)^\wedge$ are different from the trivial representations only for a finite number of indices. We shall assume also that

$$\sum_{i=1}^{\infty} k_i s_i^2 < +\infty, \sum_{j=1}^{\infty} l_j t_j^2 (\pi - t_j)^2 < +\infty. \tag{1}$$

LEMMA. Under these conditions there exists a representation of the group $U(\infty)$

$$T = \left(\bigotimes_{i=1}^{\infty} T^+_{\pi_i, s_i} \right) \otimes \left(\bigotimes_{j=1}^{\infty} T^-_{\sigma_j, t_j} \right) \otimes T^0_{r_1} \otimes T^0_{-r_2} \otimes \rho_{\lambda,\mu}, \tag{2}$$

where the infinite tensor product is understood in the sense of von Neumann [16], while in the Hilbert spaces being multiplied the $SO(\infty)$-invariant vectors are taken as distinguished vectors.

PROOF. It is sufficient to establish the existence of an infinite tensor product of spherical representations; that leads to the convergence of the infinite product of the corresponding spherical functions. The latter is checked with the help of (1) and the formulae of section 21.2.

21.7

THEOREM. The representations (2), constructed in lemma 21.6, are irreducible admissible representations of the pair $(U(\infty), SO(\infty))$ (of course, in (2) it is possible to omit some of the factors; it is also possible to replace one of the two infinite tensor products by a finite one).

PROOF. The admissibility of the representation T is checked trivially. We shall prove that $T|SU(\infty)$ is irreducible. The basic idea here remains the same as that in the proof of theorems 11.3 and 20.8; however, some technical complications arise which we shall discuss now.

In the notation of section 20.12, we shall put

$$M = \{x^+(s_i)\} \cup \{x^-(t_j)\} \cup \{\pm 1\}.$$

M is a bounded countable set in $\mathbb{R}$. The condition (1) shows that

$$\sum_{x \in M} |x^2 - 1| < +\infty.$$

In particular, the only points of accumulation of the set M are ± 1.

For $n = 1, 2, \ldots$, we denote by $\mathfrak{a}(n)$ the Lie algebra whose elements are the matrix functions from section 20.14, but which are now regarded as functions on $M \subset \mathbb{R}$. Besides this, we additionally assume that these functions are continuous on M, i.e. continuous at ± 1. Let us provide $\mathfrak{a}(n)$ with the topology of uniform convergence on M of the functions $\alpha(x)$, $A(x)$, $B(x)$. This topology may be given by a norm with respect to which $\mathfrak{a}(n)$ becomes a Banach Lie algebra.

Let us now consider the functions on M with values in the groups $Sp(n, \mathbb{R})^{\sim}$, $Sp(n)$, $SU(n)$ respectively for $|x| > 1$, $|x| < 1$, $x = \pm 1$, having close to the points ± 1 the form

$$f(x) = \begin{bmatrix} \exp(-i(1-x^2)\alpha(x) \cdot 1_n + \overline{A(x)}) & 0 \\ 0 & \exp(i(1-x^2)\alpha(x) \cdot 1_n + A(x)) \end{bmatrix}$$
$$\times \exp\begin{bmatrix} 0 & \sqrt{1-x^2}\, B(x) \\ -\sqrt{1-x^2}\, \overline{B(x)} & 0 \end{bmatrix},$$

where α, A, B, are as before assumed continuous at $x = \pm 1$ (the branch of the root for $|x| > 1$ may be selected arbitrarily). Such functions form the group $A(n)$, which admits the structure of a Banach Lie group with Lie algebra $\mathfrak{a}(n)$ (the letter A is used here with two different meanings!).

The mappings $f \to f(\pm 1)$ give morphisms of the group $A(n)$ in $SU(n)$, which may be lifted to the group $[L^0(n), L^0(n)]$ (see remark 20.15). Thus, we may assume that $f(\pm 1)$ lies in the group $[L^0(n), L^0(n)]$.

For $x \in M$, we define the irreducible representation U_x as one of the following representations: W_{π_i} if $x = x^+(s_i)$; S_{σ_j} if $x = x^-(t_j)$; $V_{r_1} \otimes \rho_\lambda$, if $x = 1$, $V_{r_2} \otimes \rho_\mu$ if $x = -1$.

From (1) follows the existence of the representation U, where

$$U(f) = \bigotimes_{x \in M} U_x(f(x)), f \in A = \cup A(n).$$

It is easy to check that U is irreducible.

Let us define the embedding τ of the group $[G, G]=SU(\infty)$ into the group A just as in section 20.13. Then $T=U\circ\tau$. We observe that, for $n\geqslant 3$, the group $\tau([G(n), G(n)])$, together with the group $d([M(n), M(n)])$ of constant functions generate a dense subgroup in $A(n)$. In fact, it is sufficient to check this assertion at the level of Lie algebra and then it follows from theorem 20.14.

Now the proof can be completed just as in section 11.13.

21.8

THEOREM. For the representations T from section 21.5, the analogues of theorems 11.5 and 20.9 on pairwise non-equivalence are true.

The proof is obtained by analogy with the proof of theorem 11.5 and taking into consideration the construction of section 21.7.

21.9

Remark. As is seen from lemma 19.10, there is no need to study infinite tensor products of representations of the type $T^0_{\pm r}$.

21.10

Remark. Theorem 21.5 obviously carries over to infinite tensor products.

21.11

Remark. The results of this paragraph can be transferred to all other pairs (G, K) of compact type. For example, in theorem 21.5, in the case of the pair $(U(\infty)\times U(\infty), U(\infty))$, it is necessary to consider the universal $(\mathbb{Z}-)$ covering over the group $\{(g_1, g_2)\in \bar{U}(\infty)\times \bar{U}(\infty)\colon g_1g_2^{-1}-1$ is a Hilbert-Schmidt operator$\}$. This covering is constructed with the help of the function

$$\psi(g_1, g_2)=\det(h\exp(h^{-1}-h)/2),\text{ where } h=g_1g_2^{-1}.$$

In the case of pairs of the type (F_i), the difficulties connected with the selection of the "gauge multipliers" do not arise at all, since, when embedding such a pair (G, K) into the pair (G', K') of the type (F) (see 18.2), the image of the group G lies in $[G', G']$.

PART V

SUPPLEMENTS: "ABSTRACT" THEOREMS

§22. Approximation of irreducible representations for general inductive limits

22.1

In this paragraph it is assumed that

$$G = \bigcup_{n=1}^{\infty} G(n),$$

where the $G(n)$ are arbitrary locally compact separable groups, while $G(n)$ is a closed subgroup in $G(n+1)$. The group G is topologized by the topology of the inductive limit.

22.2

LEMMA. Any compact set in G is completely contained in $G(m)$, where m is sufficiently large.

The proof is trivial.

22.3

Below, T will denote a certain unitary representation of the group G (continuous in the topology of the inductive limit) and $\{T_n\}$ a certain sequence of unitary representations of the groups $G(n)$; T_n is a representation of the group $G(n)$;

Let us assume that

$$\Xi = \{\xi_1, \ldots, \xi_s\} \subset H(T);$$

$$\Xi_n = \{\xi_{1n}, \ldots, \xi_{sn}\} \subset H(T_n),\ n = 1, 2, \ldots$$

We shall write $(T_n, \Xi_n) \to (T, \Xi)$ if

$$(T_n(g)\xi_{in}, \xi_{jn}) \to (T(g)\xi_i, \xi_j)\ (1 \leqslant i, j \leqslant s,\ g \in G)$$

uniformly on compact sets in G.

We observe that the matrix elements on the left-hand side have meaning for any $g \in G$ as soon as n is sufficiently large. The expression "uniformly on compact sets" makes sense in view of lemma 22.2.

22.4

Definition. We shall say that the sequence $\{T_n\}$ of unitary representations of the group $G(n)$ *approximates* the unitary representation T of the group G if, for any finite subset $\Xi \subset H(T)$, it is possible to select finite subsets $\Xi_n \subset H(T_n)$ of the same cardinality so that

$$(T_n, \Xi_n) \to (T, \Xi) \text{ as } n \to \infty.$$

in the sense indicated in section 22.3.

If this condition is satisfied, we shall write $T_n \to T$.

22.5

Examples. If T is arbitrary and $T_n = T|G(n)$, then $T_n \to T$. If T is an inductive limit of the representations T_n, then $T_n \to T$.

These examples are trivial. A non-trivial example of approximation is provided by theorem 14.2.

22.6

Lemma. Let us assume that the condition of definition 22.4 is satisfied for all Ξ from a certain total subset in $H(T)$. Then $T_n \to T$.

Let us recall that a subset is called *total* if its linear span is dense.

The proof is trivial.

22.7

Lemma. Let us assume that T possesses a cyclic vector ξ (for example, T is irreducible) and that the condition of definition 22.4 is satisfied for $\Xi = \{\xi\}$. Then $T_n \to T$.

Proof. By the condition of the lemma, there exist vectors $\xi_n \in H(T_n)$ such that $(T_n(g)\xi_n, \xi_n) \to (T(g)\xi, \xi)$ as $n \to \infty$ uniformly on compact sets in G.

Let us fix $g_1, \ldots, g_s \in G$ and put

$$\eta_i = T(g_i)\xi,\ \eta_{in} = T_n(g_i)\xi_n,\ 1 \leqslant i \leqslant s.$$

This notation makes sense for all sufficiently large n. It is obvious that

$$(T_n, \{\eta_{1n}, \ldots, \eta_{sn}\}) \to (T, \{\eta_1, \ldots, \eta_s\})$$

in the sense of section 22.3. It now remains to use lemma 22.6.

22.8

Let φ, φ_1, $\varphi_2, \ldots$ be normalized continuous positive definite functions on the groups G, $G(1)$, $G(2), \ldots$ respectively and let T, T_1, $T_2, \ldots$ be the corresponding cyclic unitary representations.

LEMMA. If $\varphi_n \to \varphi$ uniformly on the compact sets, then $T_n \to T$.

This quickly follows from lemma 22.7.

22.9

THEOREM.† With the assumptions of section 22.1 we have: for any irreducible unitary representation T of the group G there is a sequence $\{T_n\}$ of irreducible unitary representations of the groups $G(n)$ approximating T in the sense of definition 22.4.

By virtue of lemma 22.8, this theorem is equivalent to the following theorem.

22.10

THEOREM. For any continuous, normalized, indecomposable, positive definite function φ on G, there is a sequence $\{\varphi_n\}$, where φ_n is a continuous normalized indecomposable positive definite function on $G(n)$, such that $\varphi_n \to \varphi$ uniformly on compact sets.

We shall prove theorem 22.10 below.

†In the setting of admissible representations of (G, K)-pairs, a somewhat stronger version of this theorem is given in [44].

22.11

If A is a convex subset in a certain real vector space L, then we shall denote by ex(A) the set of all extreme points in A.

LEMMA. If L_1, L_2 are vector spaces, $B_1 \subset L_1$ and $B_2 \subset L_2$ convex subsets, then

$$\mathrm{ex}(B_1 \times B_2) = \mathrm{ex}(B_1) \times \mathrm{ex}(B_2).$$

The proof is trivial.

22.12

Let L be a locally convex real vector space, L' the space of all continuous linear functionals on L, $A \subset L$ a convex compact set. For $\xi \in L'$ and $\alpha \in \mathbb{R}$, we put

$$U(\xi, \alpha) = \{y \in A \colon \xi(y) > \alpha\}, \; V(\xi, \alpha) = \{y \in A \colon \xi(y) \geqslant \alpha\}.$$

LEMMA. Let $x \in \mathrm{ex}(A)$ and V be a neighbourhood of x in L. Then $\xi \in L', \alpha \in \mathbb{R}$ can be found so that

$$x \in U(\xi, \alpha), \quad V(\xi, \alpha) \subseteq V.$$

PROOF. The intersection of all sets $V(\xi, \alpha)$ for which $x \in U(\xi, \alpha)$ coincides with $\{x\}$ by the Hahn–Banach theorem. Since $A \setminus V$ is compact, it is sufficient to check the following assertion:

If $\xi_1, \xi_2, \alpha_1, \alpha_2$ are such that

$$x \in U(\xi_1, \alpha_1) \cap U(\xi_2, \alpha_2),$$

then ξ_3 and α_3 can be found such that

$$x \in U(\xi_3, \alpha_3), \quad V(\xi_3, \alpha_3) \subseteq V(\xi_1, \alpha_1) \cap V(\xi_2, \alpha_2).$$

Consider the complements of $U(\xi_1, \alpha_1)$ and $U(\xi_2, \alpha_2)$ in A. These are two convex compact sets not containing the point x. Their convex hull B does not contain X since x is an extreme point. But then ξ_3 and α_3 can be found such that

$$\xi_3(y) < \alpha_3 \text{ for all } y \in B,\ \xi_3(x) > \alpha_3.$$

This proves our proposition.

The given argument is taken from [3], Appendix B, No. B14.

22.13

Lemma. Let us assume that L is a locally convex space, that $A_1, A_2, \ldots$ are convex compact sets in L satisfying the first axiom of countability (for example, metrizable compact sets), and that

$$A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots,\ A = A_1 \cap A_2 \cap \ldots$$

Then for any $x \in \mathrm{ex}(A)$ a sequence of points $x_n \in \mathrm{ex}(A_n)$ can be found such that $x_n \to x$.

Proof. It is sufficient to prove that for any neighbourhood V of the point x in L there is a sequence of points $y_n \in \mathrm{ex}(A_n)$ lying in V for all sufficiently large n.

By lemma 22.12, $\xi \in L'$ and $\alpha \in \mathbb{R}$ can be found such that

$$x \in V(\xi, \alpha) \subseteq V.$$

For any n, the set

$$\{Z \in A_n : \xi(z) \geqslant \alpha\}$$

is non-empty and hence, it contains a point $y_n \in \mathrm{ex}(A_n)$. Let y be an arbitrary limit point for the sequence $\{y_n\}$. It is clear that $y \in A$ and that $\xi(y) \geqslant \alpha$. Hence $y \in V$. Our proposition easily follows from here.

22.14

Let us return to the group G and introduce the following notations:

L_n is the space $L^\infty(G(n))$ (with respect to Haar measure) endowed with the weak-star topology as the space dual to $L^1(G(n))$;

Q_n is the set of all continuous positive definite functions on $G(n)$ whose values at the identity do not exceed 1;

$P_n \subset Q_n$ is the subset of normalized indecomposable functions;

Q and P are the analogous sets for the group G.

We observe that

$$\mathrm{ex}(Q_n) = P_n \cup \{0\}, \ \mathrm{ex}(Q) = P \cup \{0\}.$$

We may regard Q_n as a convex compact set in L_n. Since the groups $G(n)$ are assumed separable the compact sets Q_n are metrizable.

22.15

We shall denote by Res the restriction operator from $G(n+1)$ to $G(n)$. The value of n will always be clear from the context.

LEMMA. We have Res $Q_{n+1} \subseteq Q_n$, and Q may be identified with the projective limit of the sets Q_n.

This is obvious.

22.16

To illustrate the basic idea of the proof of theorem 22.10 we shall prove it now assuming the discreteness of the groups $G(n)$. In this case Res gives a continuous mapping from L_{n+1} to L_n.

Let us examine the space $L = L_1 \times L_2 \times \ldots$ with the product topology. This is a locally convex space and

$$\dot{Q} = Q_1 \times Q_2 \times \ldots \subset L$$

is a convex metrizable compact set. We shall put

$$A_n = \{f = (f_1, f_2, \ldots) \in \dot{Q} : f_i = \mathrm{Res}\, f_{i+1}, \ i = 1, \ldots, n-1\}.$$

A_n is a convex compact set, which is isomorphic to $Q_n \times Q_{n+1} \times \ldots$.

We observe that

$$f = (f_1, f_2, \ldots) \in \mathrm{ex}(A_n) \Rightarrow f_n \in \mathrm{ex}(Q_n).$$

In fact, this follows from lemma 22.11, if we take

$$B_1 = Q_n,\ B_2 = Q_{n+1} \times Q_{n+2} \times \dots$$

We may identify Q with $A = A_1 \cap A_2 \cap \dots$. In this identification, $\mathrm{ex}(A)$ is identified with $\mathrm{ex}(Q) = P \cup \{0\}$.

Let $\varphi \in P$. We shall regard φ as an element from $\mathrm{ex}(A)$ and denote it by x. According to lemma 22.13, a sequence of points $x_n \in \mathrm{ex}(A_n)$ can be found such that $x_n \to x$ in the topology of space L.

We shall denote by φ_n the n-th component of the point x_n. Then $\varphi_n \in \mathrm{ex}(Q_n)$. From the definition of the topology in L, it follows that, for fixed m, we have:

$$\varphi_n | G(m) \to \varphi | G(m)$$

in the topology of space L_m, i.e. pointwise.

If $\varphi_n \equiv 0$ for an infinite set of indices n, then obviously $\varphi \equiv 0$, which is impossible. Thus, $\varphi_n \in P_n$. Since the groups $G(n)$ are discrete, the convergence on the compact sets in G coincides with the pointwise convergence.

This proves the theorem in the particular case being studied.

22.17

In the general case, the mapping Res cannot be considered as a mapping from L_n to L_{n-1}, hence the construction of the compact sets A_n becomes more complicated.

If f and g are two functions on a certain group, we shall write $f \gg g$ if $f - g$ is positive definite.

We put

$$B_n = \{f = (f_1, \dots, f_n) \in Q_1 \times \dots \times Q_n : f_i \gg \mathrm{Res}\, f_{i+1} \text{ for } i = 1, \dots, n-1\},$$

$$L = L_1 \times L_2 \times \dots,$$

$$A_n = B_n \times Q_{n+1} \times Q_{n+1} \times \dots \subset Q_1 \times Q_2 \times \dots \subset L.$$

It is evident that A_n is convex.

22.18

LEMMA. A_{ll} is closed in the topology of space L and hence this is a metrizable compact set.

PROOF. It is sufficient to check that, for $i=1,2,\ldots$, the set

$$\{(f_i, f_{i+1}) \in Q_i \times Q_{i+1}: f_i \geqslant \operatorname{Res} f_{i+1}\} \subset L_i \times L_{i+1}$$

is closed.

We shall regard the elements of the space $L^1(G(i))$ as measures that are absolutely continuous with respect to Haar measure. Any measure on $G(i)$ may be considered simultaneously as a measure on $G(i+1)$. We shall denote by $\mu \mapsto \check{\mu}$ the canonical involution of the Banach algebra $L^1(G(i))$.

We have now

$$f_i \geqslant \operatorname{Res} f_{i+1} \Leftrightarrow \langle f_i, \check{\mu} * \mu\rangle \geqslant \langle \operatorname{Res} f_{i+1}, \check{\mu} * \mu\rangle \forall \mu \in L^1(G(i))$$

$$\Leftrightarrow \langle f_i, \check{\mu} * \mu\rangle \geqslant \langle f_{i+1}, \check{\mu} * \mu\rangle \forall \mu \in L^1(G(i))$$

$$\Leftrightarrow \langle f_i, \check{\mu} * \mu\rangle \geqslant (\mu * f_{i+1} * \check{\mu})(e) \forall \mu \in L^1(G(i)).$$

We observe that $\mu * f_{i+1} * \check{\mu}$ is a continuous positive definite function on $G(i+1)$. Hence

$$(\mu * f_{i+1} * \check{\mu})(e) = \sup_{\nu} \langle \mu * f_{i+1} * \check{\mu}, \nu\rangle$$

$$= \sup_{\nu} \langle f_{i+1}, \check{\mu} * \nu * \mu\rangle,$$

where ν runs through the set of all probability measures from $L^1(G(i+1))$.

Thus

$$f_i \geqslant \operatorname{Res} f_{i+1} \Leftrightarrow \langle f_i, \check{\mu} * \mu\rangle \geqslant \langle f_{i+1}, \check{\mu} * \nu * \mu\rangle$$

for all μ and ν. This proves our proposition.

22.19

For $m \leqslant n$, we put

$$Q_{nm} = \{(f_1, \ldots, f_n) \in Q_1 \times \ldots \times Q_n : f_i = \mathrm{Res}^{m-i} f_m$$

$$\text{for } i = 1, \ldots, m; \quad f_i = 0 \quad \text{for} \quad i = m+1, \ldots, n\}.$$

It is evident that the mapping $f \mapsto f_m$ establishes an isomorphism between Q_{nm} and Q_m. For any $g \in Q_m$, we shall denote by $g^{(n)}$ the corresponding element from Q_{nm}.

22.20

LEMMA. The set B_n defined in 22.17 is the convex hull of its subsets $Q_{n1}, \ldots, Q_{nn}$.

PROOF. Let

$$f = (f_1, \ldots, f_n) \in B_n \subset Q_1 \times \ldots \times Q_n.$$

For $i = 1, \ldots, n-1$, we define $g_i \in L_i$ by the condition

$$f_i = \mathrm{Res}\, f_{i+1} + g_i.$$

It follows from the definition of B_n that $g_i \geqslant 0$. Further,

$$f_i(e) = f_{i+1}(e) + g_i(e), \ f_{i+1}(e) \geqslant 0.$$

Hence, in particular, $g_i(e) \leqslant 1$. Thus $g_i \in Q_i$.

It is obvious that

$$f_n^{(n)} + g_{n-1}^{(n)} + \ldots + g_1^{(n)} = f,$$

$$f_n(e) + g_{n-1}(e) + \ldots + g_1(e) = f_1(e).$$

Hence f is a convex combination of the elements

$$(a_n f_n)^{(n)}, (a_{n-1}\, g_{n-1})^{(n)}, \ldots, (a_1\, g_1)^{(n)},$$

where the numbers $a_n, \ldots, a_1$ are as follows:

$$a_n = \frac{f_1(e)}{f_n(e)}, \; a_{n-1} = \frac{f_1(e)}{g_{n-1}(e)}, \ldots, a_1 = \frac{f_1(e)}{g_1(e)}$$

(if $g_i(e) = 0$ for certain numbers i, then $g_i \equiv 0$ and the corresponding elements $(a_i g_i)^{(n)}$ must be omitted).

We observe that

$$a_n f_n(e) = a_{n-1} g_{n-1}(e) = \ldots = a_1 g_1(e) = f_1(e) \leqslant 1.$$

Hence

$$a_n f_n \in Q_n, \; a_{n-1} g_{n-1} \in Q_{n-1}, \ldots, a_1 g_1 \in Q_1.$$

So our elements lie respectively in

$$Q_{nn}, \; Q_{n,n-1}, \ldots, Q_{n1},$$

which proves the lemma.

22.21

LEMMA.

$$\mathrm{ex}(B_n) = \mathrm{ex}(Q_{n1}) \cup \mathrm{ex}(Q_{n2}) \cup \ldots \cup \mathrm{ex}(Q_{nn}).$$

PROOF. It follows from lemma 22.20 that

$$\mathrm{ex}(B_n) \subseteq \mathrm{ex}(Q_{n1}) \cup \ldots \cup \mathrm{ex}(Q_{nn}).$$

The reverse inclusion is obvious.

22.22

Consider the sets $A_1, A_2, \ldots$ from section 22.17 and put $A = A_1 \cap A_2 \cap \ldots$. It is clear that

$$A = \{f = (f_1, f_2, \ldots) \in Q_1 \times Q_2 \times \ldots : f_i \gg \mathrm{Res}\, f_{i+1} \quad \text{for} \quad i = 1, 2, \ldots\}.$$

We may regard Q as a subset in A by virtue of lemma 22.15.

LEMMA. $\operatorname{ex}(Q) \subseteq \operatorname{ex}(A)$.

PROOF. Let $f=(f_1, f_2, \ldots)$ be a point from A. Then

$$f_1(e) \geqslant f_2(e) \geqslant \ldots$$

It is clear that Q may be distinguished from A by the condition

$$f_1(e) = f_2(e) = \ldots,$$

which proves the lemma.

22.23

Remark. It is easy to prove that

$$\operatorname{ex}(A) = \operatorname{ex}(Q) \cup \operatorname{ex}(Q_{\infty 1}) \cup \operatorname{ex}(Q_{\infty 2}) \cup \ldots,$$

where

$$Q_{\infty m} = \{f=(f_1, f_2, \ldots) \in Q_1 \times Q_2 \times \ldots : f_i = \operatorname{Res}^{m-i} f_m$$

$$\text{for } i=1, \ldots, m: f_i = 0 \text{ for } i = m+1, m+2, \ldots\}.$$

In other words $\operatorname{ex}(A)$ may be identified with the union of the sets

$$\operatorname{ex}(Q), \operatorname{ex}(Q_1), \operatorname{ex}(Q_2), \ldots$$

(which intersect pairwise only in $\{0\}$).

22.24

Completion of the proof of theorem 22.10 (compare with section 22.16). We shall prove that, for any $\varphi \in P$, one may find a sequence of numbers $n_1 < n_2 < \ldots$ and also elements $\varphi_{n_j} \in \operatorname{ex}(P_{n_j})$ such that

$$\varphi_{n_j} \to \varphi \quad \text{as} \quad j \to \infty$$

uniformly on the compact sets in G.

This assertion seems weaker than that of theorem 22.10, but as is easily seen, is equivalent to it.

We shall identify φ with a point $x \in A$. By virtue of lemma 22.22, $x \in \mathrm{ex}(A)$. Thanks to lemma 22.18, we can apply lemma 22.13 and obtain a sequence of points $x_n \in \mathrm{ex}(A_n)$ converging to x in the topology of the space L.

If we write

$$x_n = (f_{n1}, f_{n2}, \ldots), \text{ where } f_{ni} \in Q_i,$$

then we find that, for any fixed m and as $n \to \infty$,

$$f_{nm} \to \varphi \mid G(m)$$

in the topology of the space $L(m)$.

We observe that $\varphi \not\equiv 0$. Hence $f_{nm} \not\equiv 0$ for all n that are sufficiently large in comparison with m.

From the definition of A_n (section 22.17) and the condition $x_n \in \mathrm{ex}(A_n)$, it follows that

$$(f_{n1}, \ldots, f_{nn}) \in \mathrm{ex}(B_n).$$

Hence

$$(f_{n1}, \ldots, f_{nn}) \in \mathrm{ex}(Q_{nk}) \setminus \{0\}.$$

for a certain $k = k(n) \leqslant n$ (see lemma 22.21).

This means, in particular, that

$$f_{nk} \in P_{k(n)}, \quad f_{ni} \equiv 0 \quad \text{for} \quad i > k.$$

It is obvious that $k(n) \to \infty$ as $n \to \infty$ and that

$$f_{nk(n)} \mid G(m) \to \varphi \mid G(m)$$

in the topology of the spaces L_m as $n \to \infty$.

It is well known that, for normalized positive definite functions on $G(m)$, the convergence in the topology of L_m coincides with uniform convergence on compact sets. This is a general fact that is valid for an arbitrary locally compact group (see [3], 13.5.2).

Now changing, if necessary, the parametrization of the functions $f_{nk(n)} \in P_{k(n)}$, we get the formulated proposition.

§23. Spherical pairs and spherical representations. The functional equation and the theorem of multiplicativity

23.1

Let G be a topological group and K a closed subgroup of G.

DEFINITION. We shall say that (G, K) is a *spherical pair* if, for any irreducible unitary representation T of the group G, the subspace $H(T)^K$ of all K-invariant vectors in $H(T)$ has dimension $\leqslant 1$.

If G is locally compact and K compact, then this condition is equivalent to the commutativity of the algebra of K-biinvariant functions from $L^1(G)$ (or the algebra of K-biinvariant finite measures).

23.2

DEFINITION. Let (G, K) be a spherical pair. The unitary representation T of the group is called a spherical representation of the pair (G, K) if T is irreducible and $H(T)^K$ is one-dimensional. The function φ on G is called *spherical* if it has the form

$$\varphi(g) = (T(g)\xi, \xi), \text{ where } g \in G, \ \xi \in H(T)^K, \ \|\xi\| = 1,$$

for a certain spherical representation T.

23.3

LEMMA. φ is a spherical function if and only if the following three conditions are satisfied:

(a) φ is positive definite and $\varphi(e) = 1$;

(b) φ is continuous on G and K-biinvariant;

(c) φ is an extreme point in the set of all functions on G satisfying the conditions (a) and (b) (nothing is changed if condition (b) is discarded here).

The proof is standard.

23.4

Let us assume that (G, K) is a spherical pair, where G is locally compact and K is compact. It is well known that any spherical function φ on G satisfies the functional equation

$$\int_K \varphi(g_1 k g_2) dk = \varphi(g_1)\varphi(g_2), \tag{1}$$

where dk is the normalized Haar measure on K.

We observe further that condition (c) in lemma 23.3 may be replaced by (1).

23.5

From this point we shall assume that

$$G = \bigcup_{n=1}^{\infty} G(n),\ K = \bigcup_{n=1}^{\infty} K(n),$$

$G(n)$ is locally compact, $K(n)$ is compact, $G(n)$ is closed in $G(n+1)$ and $K(n)$ coincides with $G(n) \cap K$. We shall assume also that $(G(n), K(n))$ are spherical pairs for all n. The groups G and K are topologized by the topology of the inductive limit.

23.6

THEOREM. With the assumptions of section 23.5, we have

(i) (G, K) is a spherical pair;

(ii) The function φ on G is spherical if and only if it satisfies the conditions (a) and (b) of lemma 23.3 and also the following equation (2), which is an analogue of equation (1):

$$\lim_{n \to \infty} \int_{K(n)} \varphi(g_1 k g_2) dk = \varphi(g_1)\varphi(g_2) \quad (g_1, g_2 \in G). \tag{2}$$

PROOF.

(i) Let T be an arbitrary unitary representation of the group G and P the projector onto the subspace $H(T)^K$. It is obvious that

$P \in T(G)''$. We shall show that the algebra $PT(G)''P$ is always commutative. Obviously, (i) will follow from this.

It is sufficient to check that

$$PT(g_1)PT(g_2)P = PT(g_2)PT(g_1)P \quad (g_1, g_2 \in G). \tag{3}$$

Let P_n be the projector onto the subspace of all $K(n)$-invariant vectors in $H(T)$. Since $(G(n), K(n))$ is a spherical pair, we have

$$P_n T(g_1) P_n T(g_2) P_n = P_n T(g_2) P_n T(g_1) P_n \quad (g_1, g_2 \in G), \tag{4}$$

if n is so large that $g_1, g_2 \in G(n)$.

But as $n \to \infty$ the projectors P_n strongly converge to P. Hence, passing to the limit in (4), we get (3).

(ii) Let us assume that φ is a spherical function of a certain spherical representation T. The validity of the conditions (a) and (b) is obvious. Let us prove (2).

Let $\xi \in H(T)^K$, $\|\xi\| = 1$. Then

$$PT(g)P\xi = \varphi(g)\xi \quad (g \in G).$$

From this,

$$PT(g_1)PT(g_2)P\xi = \varphi(g_1)\varphi(g_2)\xi \quad (g_1, g_2 \in G)$$

and this means that

$$(T(g_1)PT(g_2)\xi, \xi) = \varphi(g_1)\varphi(g_2) \quad (g_1, g_2 \in G). \tag{5}$$

The left side in (5) coincides with

$$\lim_{n \to \infty} (T(g_1)P_n T(g_2)\xi, \xi) = \lim_{n \to \infty} \int_{K(n)} \varphi(g_1 k g_2) dk.$$

This proves (2).

Conversely, let us assume that φ satisfies the conditions (a), (b), (2) and prove that φ is a spherical function. Consider the unitary representation T of the group G with cyclic vector ξ, $\|\xi\| = 1$, generated by the function φ. It is continuous, because φ is continuous. It is clear that $\xi \in H(T)^K$. It remains to prove that T is irreducible, and that reduces to the fact that $H(T)^K$ coincides with $\mathbb{C}\xi$.

For this, it is sufficient to check that

$$PT(g)\xi \in \mathbb{C}\xi \quad (g \in G).$$

We shall show that

$$PT(g)\xi = \varphi(g)\xi.$$

For this, it is sufficient to check that

$$(PT(g)\xi, T(h)\xi) = \varphi(g)(\xi, T(h)\xi) \quad (g, h \in G).$$

Let us rewrite the last equation thus:

$$(T(h^{-1})PT(g)\xi, \xi) = \varphi(h^{-1})\varphi(g) \quad (g, h \in G). \tag{6}$$

Since

$$(T(h^{-1})PT(g)\xi, \xi) = \lim_{n \to \infty} \int_{K(n)} \varphi(h^{-1}kg)dk,$$

the conditions (6) and (2) are equivalent.

23.7

Corollary. All pairs (G, K) enumerated in Tables 7.1 and 7.2 from the Introduction are spherical pairs.

In fact, it follows from the assertion (i) of theorem 23.6, since the corresponding pairs $(G(n), K(n))$ are spherical.

23.8

Let (G, K) be any of the 20 pairs of non-compact or compact type (see Table 7.2 from the Introduction). We write $\Gamma = K\backslash G/K$ and observe that any K-biinvariant function on G may be considered as a function on Γ.

THEOREM.

(i) The set Γ possesses a natural structure of a commutative semigroup with the neutral element.

(ii) Let φ be a function on G satisfying conditions (a) and (b) of lemma 23.3. φ is spherical if and only if φ is multiplicative as a function on Γ.

For a proof of (ii), we shall show that the property of multiplicativity is equivalent to the "functional equation" (2), after which our assertion will follow from theorem 22.6(ii). In checking the equivalence we do not use the positive definiteness. It is sufficient to know that φ satisfies (b).

The proof of the theorem is set forth in sections 23.10–23.15. For the sake of simplicity, we examine only pairs ($\mathbb{R}$), ($\mathbb{C}$) and ($\mathbb{H}$) of non-compact type. The transference to the remaining pairs does not pose any difficulty.

23.9

COROLLARY. For any pair (G, K) from Table 7.2 of the Introduction, we have: the product of two spherical functions is again a spherical function.

23.10

The semigroup structure in Γ is introduced as follows. Let

$$a_1 = Kg_1K,\ a_2 = Kg_2K \in \Gamma, \text{ where } g_1, g_2 \in G(m) \subset G.$$

The $a_1 a_2 = Kg_3K$, where

$$g_3 = \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix} \in G(2m) \subset G.$$

It is easy to check the correctness of this definition and also that the introduced multiplication is commutative and associative.

23.11

In the case when (G, K) is one of the three pairs

$$(GL^{+}(\infty, \mathbb{R}), SO(\infty)), (GL(\infty, \mathbb{C}), U(\infty)), (GL(\infty, \mathbb{H}), Sp(\infty)),$$

to which we restrict ourselves now, the operation of multiplication in Γ may be described as follows.

Let us recall (see section 13.1) that the elements of the set Γ may be parametrized by the unordered sets $(t_1, t_2, \ldots)$ of real numbers, among which only a finite number are different from 0. In these terms, multiplication in Γ reduces to the union of two sets.

The property of multiplicativity now takes the following form:

$$\varphi(g_{t_1 t_2} \ldots) = \prod_{i=1}^{\infty} \Phi(t_i),$$

where

$$\Phi(t) = \varphi(g_{t00\ldots}), \quad t \in \mathbb{R}.$$

In §**13**, we have seen that the spherical functions known to us possess such a property.

23.12

We shall prove the following proposition. For any continuous K-biinvariant function φ on G and any $g_1, g_2 \in G(m)$, $m = 1, 2, \ldots$, we have

$$\lim_{n\to\infty} \int_{K(n)} \varphi(g_1 u g_2) du = \varphi(g_3), \text{ where } g_3 = \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix}.$$

The equivalence of the property of multiplicativity and equation (2) will immediately follow from this.

Let us introduce the matrix

$$w = \begin{bmatrix} 0 & 1_m \\ -1_m & 0 \end{bmatrix} \in K(2m) \subset K.$$

It is obvious that

$$\varphi(g_1 w g_2) = \varphi(g_1 w g_2 w^{-1}) = \varphi(g_3).$$

Let us recall that we are denoting by $K_m(n)$ the subgroup in $K(n)$ consisting of matrices of the type

$$\begin{bmatrix} 1_m & 0 \\ 0 & * \end{bmatrix}.$$

It is evident that the function

$$u \mapsto \varphi(g_1 u g_2)$$

is biinvariant with respect to $K_m(n)$.

Now we can formulate the main idea of the proof: it turns out that, as $n \to \infty$, "almost all" double cosets $K_m(n) \cdot u \cdot K_m(n)$ are concentrated close to the class $K_m(n) \cdot w \cdot K_m(n)$.

23.13

Let us examine the mapping

$$\theta\colon K(n) \to F^{m,m}, \quad n > m,$$

which associates with a matrix from $K(n)$ its upper left corner of size $m \times m$. This mapping is constant on double cosets mod $K_m(n)$.

LEMMA. Let $n > 2m$. Then

(i) any double coset mod $K_m(n)$ in $K(n)$ intersects $K(2m) \subset K(n)$;

(ii) if $u \in K(n)$ and $\|\theta(u)\| \leqslant \varepsilon$, then the coset containing u contains a matrix $x(u) \in K(2m)$ such that

$$\|x(u) - w\| < \delta, \quad \text{where } \delta = O(\varepsilon).$$

This evaluation is uniform with respect to n.

PROOF.

(i) Let

$$u = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in K(n) \text{ (then } a = \theta(u)).$$

Using a transformation of the type

$$c \mapsto v_1 c, \quad b \mapsto b v_2, \quad \text{where} \quad v_1, v_2 \in K(n-m),$$

we can make all rows in the matrices b' and c, starting with the $(m+1)$-th, zero. After this d will take the form

$$d = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}, \text{ where } d_1 \in F^{m,m},\ d_2 \in F^{n-2m,n-2m}.$$

After multiplying u on the left or right by the matrix

$$\begin{bmatrix} 1_{2m-1} & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & d_2 \end{bmatrix}^{-1},$$

where $a = 1$ for $F \neq \mathbb{R}$, $a = \det d_2 = \pm 1$ for $F = \mathbb{R}$, we get the required proposition.

(ii) By virtue of (i), it is possible to suppose that

$$u = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, a, b, c, d \in F^{m,m}.$$

We observe that a, b, c, d satisfy the relations

$$aa^* + bb^* = a^*a + c^*c = cc^* + dd^* = b^*b + d^*d = 1_m.$$

Hence if the matrix a is small, then the matrices b and c are close to unitary ones and the matrix d is also small. From this, (ii) is obtained very easily.

23.14

For $\varepsilon > 0$, we put

$$K^{\varepsilon}(n) = \{u \in K(n) : \|\theta(u)\| \leqslant \varepsilon\}.$$

Let $v(\varepsilon, n)$ denote the volume of the set $K^{\varepsilon}(n)$ with respect to the normalized Haar measure on $K(n)$.

LEMMA. If $\varepsilon > 0$ is arbitrary, but fixed, then $v(\varepsilon, n) \to 1$ as $n \to \infty$.
This follows at once from section 5.3.

23.15

We can now prove the proposition formulated at the beginning of section 23.12. We have:

$$\int_{K(n)} \varphi(g_1 u g_2) du = \int_{K^\varepsilon(n)} \varphi(g_1 u g_2) du + \int_{K(n)\setminus K^\varepsilon(n)} \varphi(g_1 u g_2) du.$$

From assertion (i) of lemma 23.13, it follows that

$$\theta(g_1 K g_2) = \theta(g_1 K(2m) g_2).$$

Hence the function φ is bounded on the set $g_1 K g_2 \subset G$. It follows from here and from lemma 23.14 that, for large n, the integral over $K(n)\setminus K^\varepsilon(n)$ is close to 0 together with ε.

On the other hand, by virtue of lemma 23.13 (ii)

$$\varphi(g_1 u\, g_2) = \varphi(g_1 x(u) g_2) = \varphi(g_1 w g_2) + O(\varepsilon),$$

from which it is evident that the integral over $K^\varepsilon(n)$ is close to $\varphi(g_1 w g_2)$ together with ε.

(We also make use of the fact that φ is uniformly continuous on any compact set lying in $G(2m)$.)

This proves our proposition.

Theorem 23.8 is fully proved.

§24. The link with the theory of factor representations

24.1

Let K be an arbitrary topological group and π its continuous unitary representation in a separable Hilbert space.

DEFINITION. π is called a *factor representation* if the von Neumann algebra $\pi(K)''$ is a factor.

Factor representations π_1 and π_2 of the group K are called *quasiequivalent* if there exists an isomorphism

$$\varphi\colon \pi_1(K)'' \to \pi_2(K)''$$

such that $\varphi \circ \pi_1 = \pi_2$.

Perhaps it is more natural to regard any factor representation as a continuous morphism of the group K into the group of unitary elements of a certain factor; the latter is considered in either of the two coinciding topologies: ultra-weak or ultra-strong.

24.2

Let T be an irreducible unitary representation of the group $K \times K$ and $\pi = T|(K \times \{e\})$.

LEMMA. π is a factor representation. Any factor representation is obtained by such a method to within quasiequivalence.

PROOF. The first assertion is obvious. To verify the second, it is necessary to select a standard form for the factor $\pi(K)''$ and define T by the condition

$$T|(K \times \{e\}) = \pi, \; T|(\{e\} \times K) = J\pi J,$$

where J is the involution in Hilbert space $H(\pi)$ connected with the given standard form and realizing the antilinear isomorphism of the factor $\pi(K)''$ onto its commutant $\pi(K)'$.

24.3

Let

$$d\colon K \to K \times K$$

denote the diagonal embedding. Let T be an irreducible unitary representation of the group $K \times K$ possessing a $d(K)$-invariant vector ξ, $\|\xi\| = 1$. Let us put $\pi = T|(K \times \{e\})$.

LEMMA. $\pi(K)''$ is a factor of the type II_1 or I_n.

PROOF. The function

$$\varphi: x \mapsto (x\xi, \xi), \ x \in \pi(K)'',$$

is a finite trace on the factor $\pi(K)''$.

24.4

COROLLARY. $(K \times K, d(K))$ is always a spherical pair in the sense of definition 23.1.

PROOF. Let T be an irreducible unitary representation of the group $K \times K$ possessing a $d(K)$-invariant vector ξ with $\|\xi\| = 1$. Then, for all $u_1, \ u_2 \in K, \ (T(u_1, u_2)\xi, \xi) = (\pi(u_2^{-1}u_1)\xi, \xi) = \varphi(u_2^{-1}u_1)$. Since the trace on the factor is defined uniquely, the left side does not depend on the selection ξ. This means that ξ is uniquely defined to within a number factor.

24.5

THEOREM. Let K be an arbitrary topological group. The functor $T \mapsto \pi$ from section 24.3 establishes a bijection between the set of equivalence classes of spherical representations T of the pair $(K \times K, d(K))$ and the set of classes of quasiequivalence of factor representations of the group K such that $\pi(K)''$ as is a factor of the type II_1 or I_n.

PROOF. We construct the reverse functor $\pi \mapsto T$. Let π be a factor representation of the type II_1 or I_n, $\mathrm{A} = \pi(K)''$ and φ the normalized trace on A. Let us endow A with the structure of a pre-Hilbert space with scalar product

$$(x, y) = \varphi(y^* x).$$

Then A becomes a Hilbert algebra. The group $K \times K$ acts in A by the rule:

$$(u_1, u_2, x) \mapsto u_1 x u_2^{-1} \quad (u_1, u_2 \in K, \quad x \in A).$$

This action determines the unitary representation T of the group $K \times K$ in the completion of the space A. Its irreducibility follows from the theorem on the commutant for Hilbert algebras. T is a spherical representation, because the vector $x_0 = 1 \in A$ is invariant with respect to $d(K)$.

24.6

Remark. Let K be one of the groups $SO(\infty)$, $U(\infty)$, $Sp(\infty)$. Any spherical representation of the pair $(K \times K, d(K))$ is then an admissible representation. Thus, theorem 24.5 shows that the irreducible admissible spherical representations of the pair $(K \times K, d(K))$ and the factor representations of the type II_1 for K are essentially the same.

24.7

Let K be one of the groups $SO(\infty)$, $U(\infty)$, $Sp(\infty)$ and π a factor representation of K. We assume that $\pi(K)''$ has type I_∞ or II_∞ and denote by φ a semifinite trace on $\pi(K)''$.

Theorem. If the trace φ is not trivial on the subalgebra

$$\bigcup_{n=1}^{\infty} \pi(K(n))'' \subset \pi(K)'',$$

then π is obtained from a certain irreducible admissible representation T of the pair $(K \times K, d(K))$.

Proof. Let A be the subspace in $\pi(K)''$ consisting of "Hilbert–Schmidt elements", i e.,

$$A = \{x \in \pi(K)'' : \varphi(x^* x) < +\infty\}.$$

Repeating the argument of section 24.5, we construct an irreducible unitary representation T of the group $K \times K$ in the completion of the space A. It remains to prove that T is admissible.

By assumption the element

$$x \in \pi(K(n))'' \cap A, \quad x \neq 0$$

can be found for sufficiently large n. It is invariant with respect to the subgroup $K_n \subset K \simeq d(K)$. This means that the subspace $H_\infty(T \mid d(K))$ is non-trivial. It is dense in $H(T)$, since T is irreducible.

24.8

The factor representations π of type I_∞ satisfying the condition of theorem 24.7 are precisely the irreducible tame representations. (If $K=U(\infty)$, then the product of an irreducible tame representation by a one-dimensional one may be taken as well.) This follows from the following theorem, which we shall not prove here (it can be obtained with the help of theorem 2.23).

THEOREM. Let K be one of the groups $SO(\infty)$, $U(\infty)$, $Sp(\infty)$ and π be an irreducible unitary representation of K.

(i) If π is tame, then, for any sufficiently large n, an irreducible unitary representation of the group $K(n)$ can be found which enters $\pi|K(n)$ with finite multiplicity.

(ii) Conversely, if for any n an irreducible unitary representation of the group $K(n)$ entering $\pi|K(n)$ with finite multiplicity can be found, then π is tame (to within a factor of the form $\det(\cdot)^m$, $m\in\mathbb{Z}$, if $K=U(\infty)$).

24.9

Remark. Examples of factor representations π of type II_∞ satisfying the condition of theorem 24.7 are provided by factor representations of the form $\pi_1\otimes\pi_2$, where π_1 is a factor representation of type II_1 and π_2 is an irreducible tame representation.

COMMENTS

TO THE INTRODUCTION

The idea of the main method ("passage into the complex region along the subgroup K") was explained in the author's note [22]. The formalism of (G, K)-pairs and their admissible representations was proposed by the author [24]. Yu. A. Neretin found very interesting applications of this formalism for the construction of representations of the group of diffeomorphisms of the circle (see his article in this book).

P. de la Harpe [7] studied other infinite-dimensional generalizations of the classical groups, namely certain groups connected with factors.

TO §2

The theory of tame representations was initiated by A. A. Kirillov, whose note [11] contains several very important ideas. The representations ρ_λ of the full unitary group $\bar{U}(\infty)$ had earlier been characterized by I. E. Segal [26]. For more details on tame representations, see the author's articles [20] and [25]; they are studied from other points of view in [1], [12], [14], [17], [18], [31].

TO §3

The material of this paragraph is standard. The realization of the space $\mathscr{H}(\mathbb{C}^\infty)$ in the coordinate-free form is given in [27].

TO §4

The integral transform $I_\mathbb{R}$ is well known. Corollary 4.7 and theorem 4.19 (for $F=\mathbb{R}$ and $F=\mathbb{C}$) were obtained also in [14], [17] and [18]. Other methods and language are used in these works. The quaternion case has apparently not been studied earlier; however, more significant is the systematic use of the language of holomorphic extensions.

TO §5

The key result—lemma 5.3—is well known [41], [42]. The idea of the proof given in section 5.3 was mentioned in [20], section 4.8. In the case $F=\mathbb{R}$, $k=1$, the assertion of the lemma is classical and goes back to Maxwell.

TO §6

For a more detailed description of the theory of unitarizable highest weight modules, see [10]. Weil's representation of the group $Sp(n, \mathbb{R})^\sim$ has been thoroughly studied from various points of view (see, for example, [13]). Theorem 6.14 is due to R. Howe [9] (see also [30]). M. Kashiwara and M. Vergne [10] found explicit formulae for the highest weights and highest vectors of the representations $W_\pi^{(n)}$ of the group $Sp(n, \mathbb{R})^\sim$ and $U(n, n)^\sim$. Their results may be transferred to the group $SO^*(2n)$ also.

TO §7

Lemma 7.1 and theorem 7.6 may also be deduced from the explicit formulae in [10]. Weil's representation $W_{\mathbb{R}}$ of the infinite-dimensional symplectic group was studied by many authors (see, for example, [28], [33]).

TO §8

This section is written actually for the sake of corollary 8.14. Theorem 8.3 is well known. The extension of the group for the proof of continuity of the representation (section 8.9) is a very useful method; I drew this idea from [16a].

TO §§9–12

A detailed description of results obtained by the author and announced in [22] is given here.

TO §13

Theorem 13.8 was formulated by the author in [22]. In the case $F=\mathbb{R}$, $s=0$, it was proved earlier by D. Shale [28] by another method. N. I. Nessonov [15] succeeded in proving the completeness of the list of spherical representations of the group $GL(\infty, \mathbb{C})$.

TO §§14–15

The main result (theorem 14.2) was formulated briefly, due to the paucity of space, at the end of the author's note [22]. N. I. Nessonov [15] arrived at similar results in the case of spherical representations of the group $GL(\infty, \mathbb{C})$ using explicit formulae for spherical functions.

TO §16

The results of this section were announced by the author [22]. For further results concerning $SO_0(\infty,\infty)$ and allied groups $U(\infty,\infty)$ and $Sp(\infty,\infty)$, see [44].

TO §17

Many works are devoted to spinor representations of the groups $SO(2n)^{\sim}$ and $SO(2\infty)^{\sim}$ (see, for example, the survey of A. M. Vershik [35] and his article in this book). Theorem 17.2 was proved by R. Howe (see [9] and [30]). It would be possible to derive explicit formulae (à la Kashiwara–Vergne [10]) for the highest weights and highest vectors of the representations $S_\sigma^{(n)}$. I did not do this because these formulae are not used here.

TO §18–21

The results of these sections have been announced in [24]. I got them by following the analogy with the non-compact case. In the context of II_1-representations of the group $U(\infty)$, the fundamental representations were earlier constructed by D. Voiculescu [39] and also by A. M. Vershik and S. V. Kerov [37]. Lemma 19.3 is a reformulation of a well-known construction [6]. In respect to the theory of factor representations, the really new results of **§18–21** are theorems on irreducibility 20.8 and 21.7. In particular, they allow us to make more precise the realization of II_1-representations in "degenerate" cases, i.e., when, among the fundamental representations being multiplied, equivalent ones are found, cf. corollary at the end of [37].

The central result of the theory of II_1-representations is their complete classification (for the group $U(\infty)$). As shown in [2] and [37], it is implicitly contained in one old paper by A. Edrey, which, at first glance, seems to be unconnected with our topic. A. M. Vershik and S. V. Kerov have outlined in [37] yet another, very remarkable proof. When the classification theorem is translated into our language, a description of the spherical representations of the pair $(U(\infty)\times U(\infty),\ U(\infty))$ is obtained.

A few words on theorem 21.5. Earlier D. Voiculeseu [39] showed that all II_1-representations of the group $U(\infty)$ are continuous in the nuclear topology. Then R. P. Boyer [1], quoting J. Rosenberg, observed that some of them (but not all), are continuous also in Hilbert–Schmidt topology. Theorem 21.5, showing that it is always possible to take this topology if we pass to the universal covering, was the result of attempts to get a complete analogy with the noncompact case.

TO §22

Definition 22.4 and theorem 22.9 were formulated by the author [24]. For a more flexible concept of approximation and further results, see [44]. Working with characters, A. M. Vershik and S. V. Kerov [37] prove an approximation theorem in another way, with the help of the "ergodic method" of A. M. Vershik [34].

It is interesting to compare the construction of the compact set A from section 22.22 with the construction of the enveloping C^*-algebra for the inductive limit of compact groups, given in the book [31]. In both the constructions, are necessarily arise "superfluous elements": in [31] they are the states on pre-limit group algebras, and in our case extreme points of the compact sets $Q_{\infty m}$ (see remark 22.23); i.e. they are essentially the same. In fact, both constructions are equivalent: our set A may be identified with the set of positive functionals, with norm not exceeding 1, on the enveloping C^*-algebra. Notice also that the construction of the enveloping C^*-algebra from [31] is valid for locally compact pre-limit groups as well as for compact ones.

A more general proposition than lemma 22.13 is proved in [29].

TO §23

Theorem 23.6 was announced by the author in [23]. Theorem 23.8 and corollary 23.9 in the context of the theory of characters was obtained earlier by D. Voiculescu [38], [39] using another method; see also his work [40]. The approach to the multiplicativity theorem based on the asymptotics of Haar measure was proposed by the author in [20]. There it was applied to pairs (G, K) of finite rank.

Other approachels to multiplicativity theorems have been proposed by R. S. Ismagilov (see [21]), by N. I. Nessonov [15] and by A. M. Vershik and S. V. Kerov (see their article in this volume).

TO §24.

Theorems 24.5 and 24.7 were formulated by the author [24]. In the technical sense they are almost trivial. However, the observation underlying them is very important, since it links our theory with factor representation theory. In connection with remark 24.9, see the works of R. P. Boyer [2] and G. Ya. Gitel'son [5].

In conclusion a few words about Weil and spinor representations. They may be correctly defined on one-dimensional central extensions of the groups $Sp(\infty, \mathbb{R})$ and $SO(2\infty)$ respectively (instead of taking a 2-covering). This point of view, which goes back (in the case of the symplectic group) to A. Weil, is propagated by A. M. Vershik [35]. As has now become clear, the groups $L^{\pm}$ are preferably defined in terms of one-dimensional central extensions. The fact is that then K^* as a whole (and not only $[K^*, K^*]$) may be embedded in them, which leads to simplification and a better understanding of the constructions of our article (see [45]).

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